Mathematics

for the international student

9 MYP 4

Pamela Vollmar

Michael Haese

Robert Haese

Sandra Haese

Mark Humphries

for use with
IB Middle Years
Programme

MATHEMATICS FOR THE INTERNATIONAL STUDENT 9 (MYP 4)

Pamela Vollmar   B.Sc.(Hons.), PGCE.
Michael Haese   B.Sc.(Hons.), Ph.D.
Robert Haese   B.Sc.
Sandra Haese   B.Sc.
Mark Humphries   B.Sc.(Hons.)

The textbook and its accompanying CD have been developed independently of the International Baccalaureate Organization (IBO). The textbook and CD are in no way connected with, or endorsed by, the IBO.

FOREWORD

This book may be used as a general textbook at about 9th Grade (or Year 9) level in classes where students are expected to complete a rigorous course in Mathematics. It is the fourth book in our Middle Years series 'Mathematics for the International Student'.

In terms of the IB Middle Years Programme (MYP), our series does not pretend to be a definitive course. In response to requests from teachers who use 'Mathematics for the International Student' at IB Diploma level, we have endeavoured to interpret their requirements, as expressed to us, for a series that would prepare students for the Mathematics courses at Diploma level. We have developed the series independently of the International Baccalaureate Organization (IBO) in consultation with experienced teachers of IB Mathematics. Neither the series nor this text is endorsed by the IBO.

In regard to this book, it is not our intention that each chapter be worked through in full. Time constraints will not allow for this. Teachers must select exercises carefully, according to the abilities and prior knowledge of their students, to make the most efficient use of time and give as thorough coverage of content as possible.

To avoid producing a book that would be too bulky for students, we have presented some chapters on the CD, as printable pages:

> Chapter 26: Variation
> Chapter 27: Two variable analysis
> Chapter 26: Logic

The above were selected because the content could be regarded as extension material for most 9th Grade (or Year 9) students.

We understand the emphasis that the IB MYP places on the five Areas of Interaction and in response there are links on the CD to printable pages which offer ideas for projects and investigations to help busy teachers (see p. 5).

Frequent use of the interactive features on the CD should nurture a much deeper understanding and appreciation of mathematical concepts. The inclusion of our new **Self Tutor** software (see p. 4) is intended to help students who have been absent from classes or who experience difficulty understanding the material.

The book contains many problems to cater for a range of student abilities and interests, and efforts have been made to contextualise problems so that students can see the practical applications of the mathematics they are studying.

We welcome your feedback. Email: info@haeseandharris.com.au
Web: www.haeseandharris.com.au

PV, PMH, RCH, SHH, MH

USING THE INTERACTIVE CD

The interactive CD is ideal for independent study.

Students can revisit concepts taught in class and undertake their own revision and practice. The CD also has the text of the book, allowing students to leave the textbook at school and keep the CD at home.

By clicking on the relevant icon, a range of new interactive features can be accessed:

♦ Self Tutor
♦ Areas of Interaction links to printable pages
♦ Printable Chapters
♦ Interactive Links – to spreadsheets, video clips, graphing and geometry software, computer demonstrations and simulations
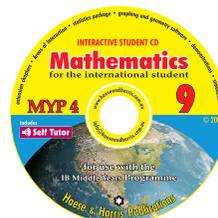
INTERACTIVE LINK

SELF TUTOR is a new exciting feature of this book.

The ◀) **Self Tutor** icon on each worked example denotes an active link on the CD.

Simply 'click' on the ◀) **Self Tutor** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Ideal for students who have missed lessons or need extra help.

Example 2   ◀) **Self Tutor**

Simplify by collecting like terms:

a $-a - 1 + 3a + 4$ **b** $5a - b^2 + 2a - 3b^2$

a $-a - 1 + 3a + 4$

$= -a + 3a - 1 + 4$

$= 2a + 3$

$\{-a$ and $3a$ are like terms

-1 and 4 are like terms$\}$

b $5a - b^2 + 2a - 3b^2$

$= 5a + 2a - b^2 - 3b^2$

$= 7a - 4b^2$

$\{5a$ and $2a$ are like terms

$-b^2$ and $-3b^2$ are like terms$\}$

See **Chapter 3**, **Algebraic expansion and simplification**, p. 73

AREAS OF INTERACTION

The International Baccalaureate Middle Years Programme focuses teaching and learning through five Areas of Interaction:

- ♦ Approaches to learning
- ♦ Community and service
- ♦ Human ingenuity

- ♦ Environments
- ♦ Health and social education

The Areas of Interaction are intended as a focus for developing connections between different subject areas in the curriculum and to promote an understanding of the interrelatedness of different branches of knowledge and the coherence of knowledge as a whole.

> Click on the heading to access a printable 'pop-up' version of the link.

In an effort to assist busy teachers, we offer the following printable pages of ideas for projects and investigations:

LINKS
click here

CHESS BOARD CALCULATIONS

Areas of interaction:
Approaches to learning/Human ingenuity

Links to printable pages of ideas for projects and investigations

TABLE OF CONTENTS

Graphics calculator instructions

Contents:

In this course it is assumed that you have a **graphics calculator**. If you learn how to operate your calculator successfully, you should experience little difficulty with future arithmetic calculations.

There are many different brands (and types) of calculators. Different calculators do not have exactly the same keys. It is therefore important that you have an instruction booklet for your calculator, and use it whenever you need to.

However, to help get you started, we have included here some basic instructions for the **Texas Instruments TI-83** and the **Casio fx-9860G** calculators. Note that instructions given may need to be modified slightly for other models.

GETTING STARTED

Texas Instruments TI-83

The screen which appears when the calculator is turned on is the **home screen**. This is where most basic calculations are performed.

You can return to this screen from any menu by pressing 2nd MODE .

When you are on this screen you can type in an expression and evaluate it using the ENTER key.

Casio fx-9860g

Press MENU to access the Main Menu, and select **RUN·MAT**.

This is where most of the basic calculations are performed.

When you are on this screen you can type in an expression and evaluate it using the EXE key.

A   BASIC CALCULATIONS

Most modern calculators have the rules for **Order of Operations** built into them. This order is sometimes referred to as BEDMAS.

This section explains how to enter different types of numbers such as negative numbers and fractions, and how to perform calculations using grouping symbols (brackets), powers, and square roots. It also explains how to round off using your calculator.

NEGATIVE NUMBERS

To enter negative numbers we use the **sign change** key. On both the **TI-83** and **Casio** this looks like (−) .

Simply press the sign change key and then type in the number.

For example, to enter −7, press (−) 7.

FRACTIONS

On most scientific calculators and also the **Casio** graphics calculator there is a special key for entering fractions. No such key exists for the **TI-83**, so we use a different method.

Texas Instruments TI-83

To enter common fractions, we enter the fraction as a division.

For example, we enter $\frac{3}{4}$ by typing   3 $\boxed{\div}$ 4.   If the fraction is part of a larger calculation, it is generally wise to place this division in brackets, i.e., $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$.

To enter mixed numbers, either convert the mixed number to an improper fraction and enter as a common fraction *or* enter the fraction as a sum.

For example, we can enter $2\frac{3}{4}$ as $\boxed{(}$ 11 $\boxed{\div}$ 4 $\boxed{)}$ *or* $\boxed{(}$ 2 $\boxed{+}$ 3 $\boxed{\div}$ 4 $\boxed{)}$.

Casio fx-9860g

To enter fractions we use the **fraction** key $\boxed{\text{a b/c}}$.

For example, we enter $\frac{3}{4}$ by typing   3 $\boxed{\text{a b/c}}$ 4   and $2\frac{3}{4}$ by typing   2 $\boxed{\text{a b/c}}$ 3 $\boxed{\text{a b/c}}$ 4. Press $\boxed{\text{SHIFT}}$ $\boxed{\text{a b/c}}$ $(a\frac{b}{c} \leftrightarrow \frac{d}{c})$ to convert between mixed numbers and improper fractions.

SIMPLIFYING FRACTIONS & RATIOS

Graphics calculators can *sometimes* be used to express fractions and ratios in simplest form.

Texas Instruments TI-83

To express the fraction $\frac{35}{56}$ in simplest form, press   35 $\boxed{\div}$ 56 $\boxed{\text{MATH}}$ **1** $\boxed{\text{ENTER}}$.   The result is $\frac{5}{8}$.

To express the ratio $\frac{2}{3} : 1\frac{1}{4}$ in simplest form, press $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 1 $\boxed{+}$ 1 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{MATH}}$ **1** $\boxed{\text{ENTER}}$.

The ratio is   8 : 15.
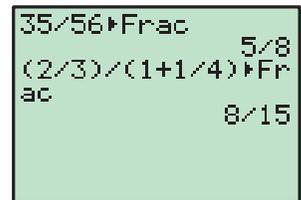
Casio fx-9860g

To express the fraction $\frac{35}{56}$ in simplest form, press   35 $\boxed{\text{a b/c}}$ 56 $\boxed{\text{EXE}}$.   The result is $\frac{5}{8}$.

To express the ratio $\frac{2}{3} : 1\frac{1}{4}$ in simplest form, press   2 $\boxed{\text{a b/c}}$ 3 $\boxed{\div}$ 1 $\boxed{\text{a b/c}}$ 1 $\boxed{\text{a b/c}}$ 4 $\boxed{\text{EXE}}$.   The ratio is 8 : 15.
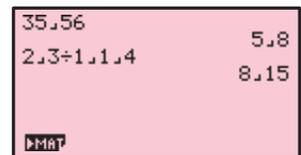
ENTERING TIMES

In questions involving time, it is often necessary to be able to express time in terms of hours, minutes and seconds.

Texas Instruments TI-83

To enter 2 hours 27 minutes, press   2 2nd MATRX **(ANGLE)**
1:^{o} 27 2nd MATRX **2:$'$**.   This is equivalent to 2.45 hours.

To express 8.17 hours in terms of hours, minutes and seconds,
press   8.17 2nd MATRX **4:▶DMS** ENTER .
This is equivalent to 8 hours, 10 minutes and 12 seconds.
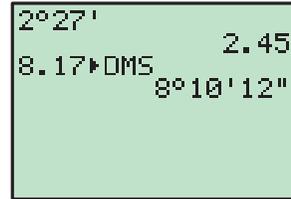
Casio fx-9860g

To enter 2 hours 27 minutes, press   2 OPTN F6 F5 **(ANGL)**
F4 (°''') 27 F4 (°''') EXE .   This is equivalent to 2.45 hours.

To express 8.17 hours in terms of hours, minutes and seconds,
press   8.17 OPTN F6 F5 **(ANGL)** F6 F3 **(▶DMS)** EXE .
This is equivalent to 8 hours, 10 minutes and 12 seconds.
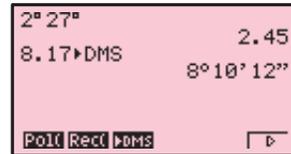
B   BASIC FUNCTIONS

GROUPING SYMBOLS (BRACKETS)

Both the **TI-83** and **Casio** have bracket keys that look like [(and)] .

Brackets are regularly used in mathematics to indicate an expression which needs to be evaluated before other operations are carried out.

For example, to enter $2 \times (4 + 1)$ we type   2 × [(4 + 1)] .

We also use brackets to make sure the calculator understands the expression we are typing in.

For example, to enter $\frac{2}{4+1}$ we type   2 ÷ [(4 + 1)] .   If we typed   2 ÷ 4 + 1
the calculator would think we meant $\frac{2}{4} + 1$.

In general, it is a good idea to place brackets around any complicated expressions which need to be evaluated separately.

POWER KEYS

Both the **TI-83** and **Casio** also have power keys that look like [∧] . We type the base first, press the power key, then enter the index or exponent.

For example, to enter 25^3 we type   25 [∧] 3.

Note that there are special keys which allow us to quickly evaluate squares.

Numbers can be squared on both **TI-83** and **Casio** using the special key x^2 .

For example, to enter 25^2 we type   25 x^2 .

SQUARE ROOTS

To enter square roots on either calculator we need to use a secondary function (see the **Secondary Function and Alpha Keys**).

Texas Instruments TI-83

The **TI-83** uses a secondary function key 2nd .

To enter $\sqrt{36}$ we press   2nd x^2 36) .

The end bracket is used to tell the calculator we have finished entering terms under the square root sign.

Casio fx-9860g

The Casio uses a shift key SHIFT to get to its second functions.

To enter $\sqrt{36}$ we press   SHIFT x^2 36.

If there is a more complicated expression under the square root sign you should enter it in brackets.

For example, to enter $\sqrt{18 \div 2}$ we press   SHIFT x^2 (18 ÷ 2) .

ROUNDING OFF

You can use your calculator to round off answers to a fixed number of decimal places.

Texas Instruments TI-83

To round to 2 decimal places, press MODE then ▼ to scroll down to Float.

Use the ▶ button to move the cursor over the 2 and press ENTER . Press 2nd MODE to return to the home screen.

If you want to unfix the number of decimal places, press MODE ▼ ENTER to highlight Float.

Casio fx-9860g

To round to 2 decimal places, select **RUN·MAT** from the Main Menu, and press SHIFT MENU to enter the setup screen. Scroll down to Display, and press F1 (**Fix**). Press 2 EXE to select the number of decimal places. Press EXIT to return to the home screen.

To unfix the number of decimal places, press SHIFT MENU to return to the setup screen, scroll down to Display, and press F3 (**Norm**).

INVERSE TRIGONOMETRIC FUNCTIONS

To enter inverse trigonometric functions, you will need to use a secondary function (see the **Secondary Function and Alpha Keys**).

Texas Instruments TI-83

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of SIN , COS and TAN respectively. They are accessed by using the secondary function key 2nd .

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

To calculate this, press   2nd COS 3 ÷ 5) ENTER .

Casio fx-9860g

The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are the secondary functions of sin , cos and tan respectively. They are accessed by using the secondary function key SHIFT .

For example, if $\cos x = \frac{3}{5}$, then $x = \cos^{-1}\left(\frac{3}{5}\right)$.

To calculate this, press   SHIFT cos (3 ÷ 5) EXE .

SCIENTIFIC NOTATION

If a number is too large or too small to be displayed neatly on the screen, it will be expressed in scientific notation, that is, in the form $a \times 10^k$ where $1 \leqslant a \leqslant 10$ and k is an integer.

Texas Instruments TI-83

To evaluate 2300^3, press   2300 ^ 3 ENTER .   The answer displayed is   1.2167E10,   which means 1.2167×10^{10}.

To evaluate $\frac{3}{20\,000}$, press   3 ÷ 20 000 ENTER .   The answer displayed is   1.5E−4,   which means 1.5×10^{-4}.

You can enter values in scientific notation using the EE function, which is accessed by pressing 2nd , .

For example, to evaluate $\frac{2.6 \times 10^{14}}{13}$, press   2.6 2nd , 14 ÷ 13 ENTER .   The answer is 2×10^{13}.

Casio fx-9860g

To evaluate 2300^3, press $\boxed{2300}$ $\boxed{\wedge}$ $\boxed{3}$ $\boxed{\text{EXE}}$. The answer displayed is $1.2167\text{E}+10$, which means 1.2167×10^{10}.
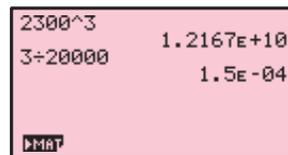
To evaluate $\frac{3}{20\,000}$, press $\boxed{3}$ $\boxed{\div}$ $\boxed{20\,000}$ $\boxed{\text{EXE}}$. The answer displayed is $1.5\text{E}-04$, which means 1.5×10^{-4}.

You can enter values in scientific notation using the $\boxed{\text{EXP}}$ key. For example, to evaluate $\frac{2.6\times10^{14}}{13}$, press $\boxed{2.6}$ $\boxed{\text{EXP}}$ $\boxed{14}$ $\boxed{\div}$ $\boxed{13}$ $\boxed{\text{EXE}}$. The answer is 2×10^{13}.

```
2300^3          1.2167E+10
3÷20000            1.5E-04


▶MAT
```

```
2.6E14÷13          2E+13



▶MAT
```
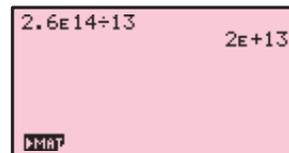
C SECONDARY FUNCTION AND ALPHA KEYS

Texas Instruments TI-83

The **secondary function** of each key is displayed in yellow above the key. It is accessed by pressing the $\boxed{\text{2nd}}$ key, followed by the key corresponding to the desired secondary function. For example, to calculate $\sqrt{36}$, press $\boxed{\text{2nd}}$ $\boxed{x^2}$ $\boxed{36}$ $\boxed{)}$ $\boxed{\text{ENTER}}$.

The **alpha function** of each key is displayed in green above the key. It is accessed by pressing the $\boxed{\text{ALPHA}}$ key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values into memory which can be recalled later. Refer to the **Memory** section.

Casio fx-9860g

The **shift function** of each key is displayed in yellow above the key. It is accessed by pressing the $\boxed{\text{SHIFT}}$ key followed by the key corresponding to the desired shift function.

For example, to calculate $\sqrt{36}$, press $\boxed{\text{SHIFT}}$ $\boxed{x^2}$ $\boxed{36}$ $\boxed{\text{EXE}}$.

The **alpha function** of each key is displayed in red above the key. It is accessed by pressing the $\boxed{\text{ALPHA}}$ key followed by the key corresponding to the desired letter. The main purpose of the alpha keys is to store values which can be recalled later.

D MEMORY

Utilising the memory features of your calculator allows you to recall calculations you have performed previously. This not only saves time, but also enables you to maintain accuracy in your calculations.

SPECIFIC STORAGE TO MEMORY

Values can be stored into the variable letters A, B, ..., Z using either calculator. Storing a value in memory is useful if you need that value multiple times.

Texas Instruments TI-83

Suppose we wish to store the number 15.4829 for use in a number of calculations. Type in the number then press STO► ALPHA MATH **(A)** ENTER .

```
15.4829→A
          15.4829
A+10
           25.4829
A^3
      3711.563767
```

We can now add 10 to this value by pressing ALPHA MATH + 10 ENTER , or cube this value by pressing ALPHA MATH ^ 3 ENTER .

Casio fx-9860g

Suppose we wish to store the number 15.4829 for use in a number of calculations. Type in the number then press → ALPHA X,θ,T **(A)** EXE .

```
15.4829→A
          15.4829
A+10
          25.4829
A^3
      3711.563767
▶MAT
```

We can now add 10 to this value by pressing ALPHA X,θ,T + 10 EXE , or cube this value by pressing ALPHA X,θ,T ^ 3 EXE .

ANS VARIABLE

Texas Instruments TI-83

The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing 2nd (−) .

```
3*4
               12
17−Ans
                5
```

For example, suppose you evaluate 3×4, and then wish to subtract this from 17. This can be done by pressing 17 − 2nd (−) ENTER .

If you start an expression with an operator such as + , − , etc, the previous answer **Ans** is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing ÷ 2 ENTER .

```
               12
17−Ans
                5
Ans/2
              2.5
Ans▶Frac
              5/2
```

If you wish to view the answer in fractional form, press MATH **1** ENTER .

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The variable **Ans** holds the most recent evaluated expression, and can be used in calculations by pressing $\boxed{\text{SHIFT}}$ $\boxed{(-)}$. For example, suppose you evaluate 3×4, and then wish to subtract this from 17. This can be done by pressing 17 $\boxed{-}$ $\boxed{\text{SHIFT}}$ $\boxed{(-)}$ $\boxed{\text{EXE}}$.

If you start an expression with an operator such as $\boxed{+}$, $\boxed{-}$, etc, the previous answer Ans is automatically inserted ahead of the operator. For example, the previous answer can be halved simply by pressing $\boxed{\div}$ 2 $\boxed{\text{EXE}}$.

If you wish to view the answer in fractional form, press $\boxed{\text{F}\leftrightarrow\text{D}}$.

RECALLING PREVIOUS EXPRESSIONS

Texas Instruments TI-83

The **ENTRY** function recalls previously evaluated expressions, and is used by pressing $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$.

This function is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$. If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$.

The change can then be made by moving the cursor over the 3 and changing it to a 4, then pressing $\boxed{\text{ENTER}}$.

If you have made an error in your original calculation, and intended to calculate $1500 + \sqrt{132}$, again you can recall the previous command by pressing $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$.

Move the cursor to the first 0.

You can insert the digit 5, rather than overwriting the 0, by pressing $\boxed{\text{2nd}}$ $\boxed{\text{DEL}}$ 5 $\boxed{\text{ENTER}}$.

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Pressing the left cursor key allows you to edit the most recently evaluated expression, and is useful if you wish to repeat a calculation with a minor change, or if you have made an error in typing.

Suppose you have evaluated $100 + \sqrt{132}$.

If you now want to evaluate $100 + \sqrt{142}$, instead of retyping the command, it can be recalled by pressing the left cursor key.

Move the cursor between the 3 and the 2, then press $\boxed{\text{DEL}}$ 4 to remove the 3 and change it to a 4. Press $\boxed{\text{EXE}}$ to re-evaluate the expression.

E   LISTS

Lists are used for a number of purposes on the calculator. They enable us to enter sets of numbers, and we use them to generate number sequences using algebraic rules.

CREATING A LIST

Texas Instruments TI-83

Press $\boxed{\text{STAT}}$ 1 to take you to the **list editor** screen.

To enter the data $\{2, 5, 1, 6, 0, 8\}$ into **List1**, start by moving the cursor to the first entry of **L1**. Press 2 $\boxed{\text{ENTER}}$ 5 $\boxed{\text{ENTER}}$ ...... and so on until all the data is entered.
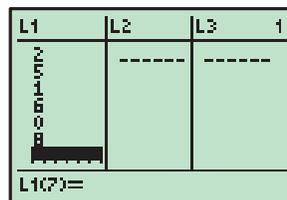
Casio fx-9860g

Selecting **STAT** from the Main Menu takes you to the **list editor** screen.

To enter the data $\{2, 5, 1, 6, 0, 8\}$ into **List 1**, start by moving the cursor to the first entry of **List 1**. Press 2 $\boxed{\text{EXE}}$ 5 $\boxed{\text{EXE}}$ ...... and so on until all the data is entered.
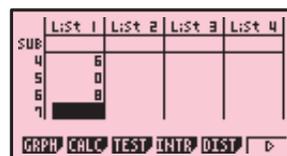
DELETING LIST DATA

Texas Instruments TI-83

Pressing $\boxed{\text{STAT}}$ **1** takes you to the **list editor** screen.

Move the cursor to the heading of the list you want to delete then press $\boxed{\text{CLEAR}}$ $\boxed{\text{ENTER}}$.

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Selecting **STAT** from the Main Menu takes you to the **list editor** screen.

Move the cursor to anywhere on the list you wish to delete, then press $\boxed{\text{F6}}$ **(▷)** $\boxed{\text{F4}}$ **(DEL-A)** $\boxed{\text{F1}}$ **(Yes)**.

REFERENCING LISTS

Texas Instruments TI-83

Lists can be referenced by using the secondary functions of the keypad numbers 1–6.

For example, suppose you want to add 2 to each element of **List1** and display the results in **List2**. To do this, move the cursor to the heading of **L2** and press $\boxed{\text{2nd}}$ 1 $\boxed{+}$ 2 $\boxed{\text{ENTER}}$.

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Lists can be referenced using the List function, which is accessed by pressing [SHIFT] **1**.

For example, if you want to add 2 to each element of **List 1** and display the results in **List 2**, move the cursor to the heading of **List 2** and press [SHIFT] 1 **(List)** 1 [+] 2 [EXE] .

Casio models without the List function can do this by pressing [OPTN] [F1] **(LIST)** [F1] **(List)** 1 [+] 2 [EXE] .

NUMBER SEQUENCES

Texas Instruments TI-83

You can create a sequence of numbers defined by a certain rule using the *seq* command.

This command is accessed by pressing [2nd] [STAT] [▶] to enter the **OPS** section of the List menu, then selecting **5:seq**.

For example, to store the sequence of even numbers from 2 to 8 in **List3**, move the cursor to the heading of **L3**, then press [2nd] [STAT] [▶] **5** to enter the *seq* command, followed by 2 [X,T,θ,n] [,] [X,T,θ,n] [,] 1 [,] 4 [)] [ENTER] .

This evaluates $2x$ for every value of x from 1 to 4.
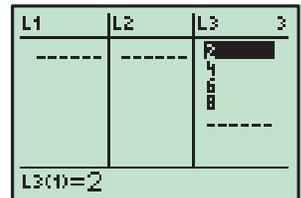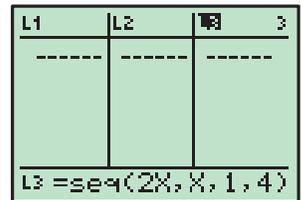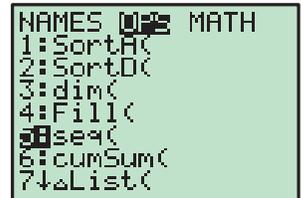
Casio fx-9860g

You can create a sequence of numbers defined by a certain rule using the *seq* command.

This command is accessed by pressing [OPTN] [F1] **(LIST)** [F5] **(Seq)**.

For example, to store the sequence of even numbers from 2 to 8 in **List 3**, move the cursor to the heading of **List 3**, then press [OPTN] [F1] [F5] to enter a sequence, followed by 2 [X,θ,T] [,] [X,θ,T] [,] 1 [,] 4 [,] 1 [)] [EXE] .

This evaluates $2x$ for every value of x from 1 to 4 with an increment of 1.
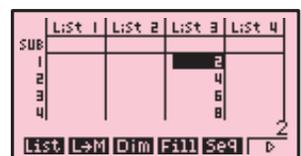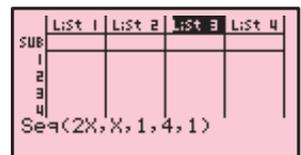
F   STATISTICAL GRAPHS

STATISTICS

Your graphics calculator is a useful tool for analysing data and creating statistical graphs.

In this section we will produce descriptive statistics and graphs for the data set   5 2 3 3 6 4 5 3 7 5 7 1 8 9 5.

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Enter the data set into **List1** using the instructions on page **18**. To obtain descriptive statistics of the data set, press   STAT ▶ **1:1-Var Stats** 2nd **1 (L₁)** ENTER .
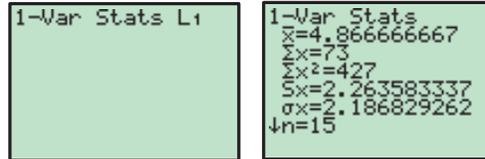
To obtain a boxplot of the data, press 2nd Y= **(STAT PLOT) 1** and set up **Statplot1** as shown. Press ZOOM **9:ZoomStat** to graph the boxplot with an appropriate window.
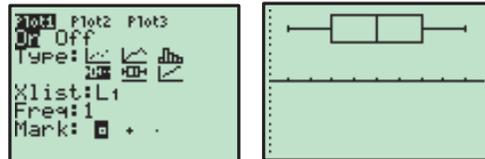
To obtain a vertical bar chart of the data, press 2nd Y= **1**, and change the type of graph to a vertical bar chart as shown. Press ZOOM **9:ZoomStat** to draw the bar chart. Press WINDOW and set the **Xscl** to 1, then GRAPH to redraw the bar chart.
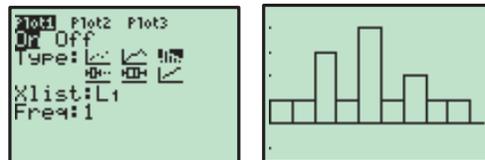
We will now enter a second set of data, and compare it to the first.

Enter the data set 9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4 into **List2**, press 2nd Y= **1**, and change the type of graph back to a boxplot as shown. Move the cursor to the top of the screen and select **Plot2**. Set up **Statplot2** in the same manner, except set the **XList** to **L₂**. Press ZOOM **9:ZoomStat** to draw the side-by-side boxplots.
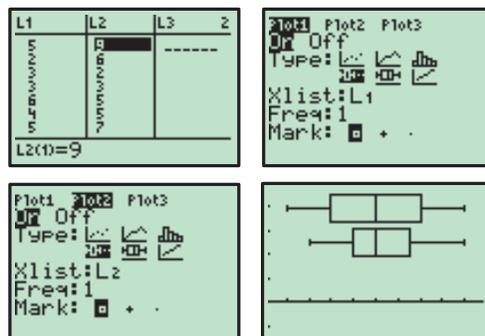
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Enter the data into **List 1** using the instructions on page **18**. To obtain the descriptive statistics, press F6 **(▷)** until the **GRPH** icon is in the bottom left corner of the screen, then press F2 **(CALC)** F1 **(1VAR)**.
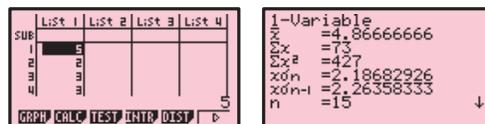
To obtain a boxplot of the data, press EXIT EXIT F1 (GRPH) F6 (SET), and set up **StatGraph 1** as shown. Press EXIT F1 (GPH1) to draw the boxplot.
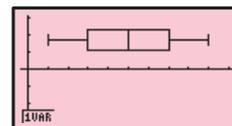
To obtain a vertical bar chart of the data, press EXIT F6 (SET) F2 (GPH 2), and set up **StatGraph 2** as shown. Press EXIT F2 (GPH 2) to draw the bar chart (set Start to 0, and Width to 1).

We will now enter a second set of data, and compare it to the first.

Enter the data set   9 6 2 3 5 5 7 5 6 7 6 3 4 4 5 8 4   into **List 2**, then press F6 (SET) F2 (GPH2) and set up **StatGraph 2** to draw a boxplot of this data set as shown. Press EXIT F4 (SEL), and turn on both **StatGraph 1** and **StatGraph 2**. Press F6 (DRAW) to draw the side-by-side boxplots.
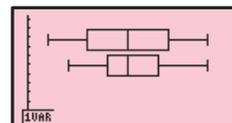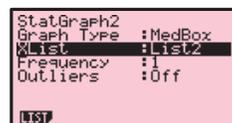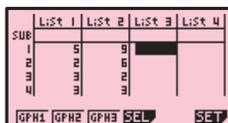
G   WORKING WITH FUNCTIONS

GRAPHING FUNCTIONS

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Pressing Y= selects the **Y=** editor, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing CLEAR .
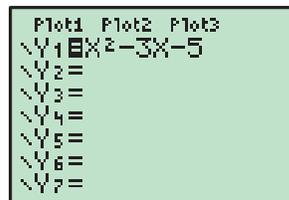
To graph the function $y = x^2 - 3x - 5$,   move the cursor to **Y₁**, and press X,T,θ,n x² − 3 X,T,θ,n − 5 ENTER .   This stores the function into **Y₁**. Press GRAPH to draw a graph of the function.
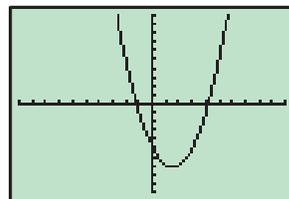
To view a table of values for the function, press 2nd GRAPH (**TABLE**). The starting point and interval of the table values can be adjusted by pressing 2nd WINDOW (**TBLSET**).

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Selecting **GRAPH** from the Main Menu takes you to the Graph Function screen, where you can store functions to graph. Delete any unwanted functions by scrolling down to the function and pressing DEL F1 **(Yes)**.

To graph the function $y = x^2 - 3x - 5$, move the cursor to **Y1** and press X,θ,T x^2 − 3 X,θ,T − 5 EXE . This stores the function into **Y1**. Press F6 **(DRAW)** to draw a graph of the function.

To view a table of values for the function, press MENU and select **TABLE**. The function is stored in **Y1**, but not selected. Press F1 **(SEL)** to select the function, and F6 **(TABL)** to view the table. You can adjust the table settings by pressing EXIT and then F5 **(SET)** from the Table Function screen.
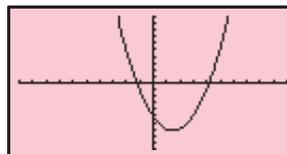
FINDING POINTS OF INTERSECTION

It is often useful to find the points of intersection of two graphs, for instance, when you are trying to solve simultaneous equations.

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We can solve $y = 11 - 3x$ and $y = \dfrac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines. Press Y= , then store $11 - 3x$ into **Y1** and $\dfrac{12 - x}{2}$ into **Y2**. Press GRAPH to draw a graph of the functions.

To find their point of intersection, press 2nd TRACE **(CALC)** **5**, which selects **5:intersect**. Press ENTER twice to specify the functions **Y1** and **Y2** as the functions you want to find the intersection of, then use the arrow keys to move the cursor close to the point of intersection and press ENTER once more.

The solution $x = 2$, $y = 5$ is given.
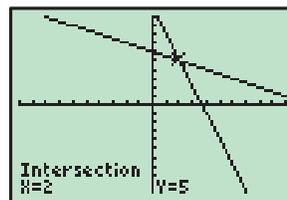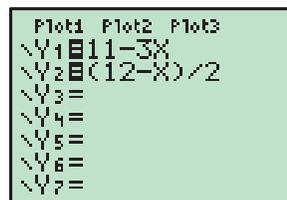
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We can solve $y = 11 - 3x$ and $y = \dfrac{12 - x}{2}$ simultaneously by finding the point of intersection of these two lines. Select **GRAPH** from the Main Menu, then store $11 - 3x$ into **Y1** and $\dfrac{12 - x}{2}$ into **Y2**. Press F6 **(DRAW)** to draw a graph of the functions.

To find their point of intersection, press $\boxed{\text{F5}}$ **(G-Solv)** $\boxed{\text{F5}}$ **(ISCT)**. The solution $x = 2$, $y = 5$ is given.
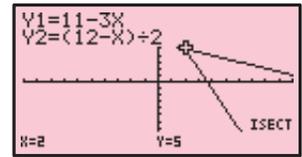
Note: If there is more than one point of intersection, the remaining points of intersection can be found by pressing $\boxed{\blacktriangleright}$.

SOLVING $f(x) = 0$

In the special case when you wish to solve an equation of the form $f(x) = 0$, this can be done by graphing $y = f(x)$ and then finding when this graph cuts the x-axis.

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To solve $x^3 - 3x^2 + x + 1 = 0$, press $\boxed{\text{Y=}}$ and store $x^3 - 3x^2 + x + 1$ into **Y₁**. Press $\boxed{\text{GRAPH}}$ to draw the graph.
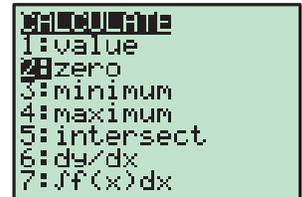
To find where this function first cuts the x-axis, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ **(CALC) 2**, which selects **2:zero**. Move the cursor to the left of the first zero and press $\boxed{\text{ENTER}}$, then move the cursor to the right of the first zero and press $\boxed{\text{ENTER}}$. Finally, move the cursor close to the first zero and press $\boxed{\text{ENTER}}$ once more. The solution $x \approx -0.414$ is given.
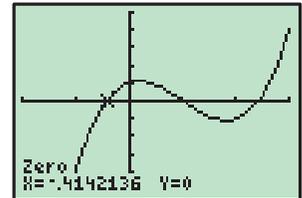
Repeat this process to find the remaining solutions $x = 1$ and $x \approx 2.41$.

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To solve $x^3 - 3x^2 + x + 1 = 0$, select **GRAPH** from the Main Menu and store $x^3 - 3x^2 + x + 1$ into **Y1**. Press $\boxed{\text{F6}}$ **(DRAW)** to draw the graph.
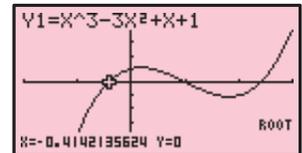
To find where this function cuts the x-axis, press $\boxed{\text{F5}}$ **(G-Solv)** $\boxed{\text{F1}}$ **(ROOT)**. The first solution $x \approx -0.414$ is given.

Press $\boxed{\blacktriangleright}$ to find the remaining solutions $x = 1$ and $x \approx 2.41$.

TURNING POINTS

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To find the turning point (vertex) of $y = -x^2 + 2x + 3$, press $\boxed{\text{Y=}}$ and store $-x^2 + 2x + 3$ into **Y₁**. Press $\boxed{\text{GRAPH}}$ to draw the graph.
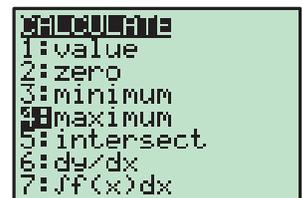
From the graph, it is clear that the vertex is a maximum, so press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ **(CALC) 4** to select **4:maximum**.

Move the cursor to the left of the vertex and press ENTER , then move the cursor to the right of the vertex and press ENTER . Finally, move the cursor close to the vertex and press ENTER once more. The vertex is $(1, 4)$.
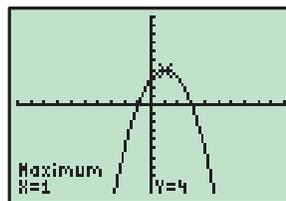
Casio fx-9860g

To find the turning point (vertex) of $y = -x^2 + 2x + 3$, select **GRAPH** from the Main Menu and store $-x^2 + 2x + 3$ into **Y1**. Press F6 **(DRAW)** to draw the graph.

From the graph, it is clear that the vertex is a maximum, so to find the vertex press F5 **(G-Solv)** F2 **(MAX)**.

The vertex is $(1, 4)$.
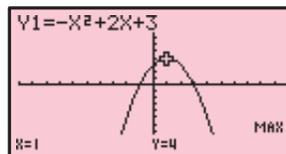
ADJUSTING THE VIEWING WINDOW

When graphing functions it is important that you are able to view all the important features of the graph. As a general rule it is best to start with a large viewing window to make sure all the features of the graph are visible. You can then make the window smaller if necessary.

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Some useful commands for adjusting the viewing window include:

ZOOM **0:ZoomFit** :   This command scales the y-axis to fit the minimum and maximum values of the displayed graph within the current x-axis range.

ZOOM **6:ZStandard** :   This command returns the viewing window to the default setting of $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.

If neither of these commands are helpful, the viewing window can be adjusted manually by pressing WINDOW and setting the minimum and maximum values for the x and y axes.
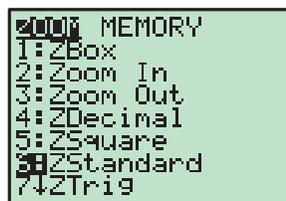
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The viewing window can be adjusted by pressing SHIFT F3 **(V-Window)**. You can manually set the minimum and maximum values of the x and y axes, or press F3 **(STD)** to obtain the standard viewing window $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$.

H   MATRICES

Matrices are easily stored in a graphics calculator. This is particularly valuable if we need to perform a number of operations with the same matrices.

STORING MATRICES

The matrix $\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$ can be stored using these instructions:

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Press $\boxed{\text{MATRX}}$ to display the matrices screen, and use $\boxed{\blacktriangleright}$ to select the **EDIT** menu. This is where you define matrices and enter their elements.

Press **1** to select **1:[A]**. Press 3 $\boxed{\text{ENTER}}$ 2 $\boxed{\text{ENTER}}$ to define matrix **A** as a 3×2 matrix.

Enter the elements of the matrix, pressing $\boxed{\text{ENTER}}$ after each entry.

Press $\boxed{\text{SHIFT}}$ $\boxed{\text{MODE}}$ **(QUIT)** when you are done.
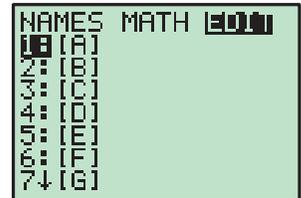
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Select **Run·Mat** from the main menu, and press $\boxed{\text{F1}}$ **(▶MAT)**. This is where you define matrices and enter their elements.

To define matrix **A**, make sure **Mat A** is highlighted, and press $\boxed{\text{F3}}$ **(DIM)** 3 $\boxed{\text{EXE}}$ 2 $\boxed{\text{EXE}}$ $\boxed{\text{EXE}}$.

Enter the elements of the matrix, pressing $\boxed{\text{EXE}}$ after each entry.

Press $\boxed{\text{EXIT}}$ twice to return to the home screen when you are done.
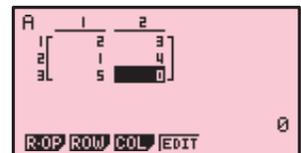
MATRIX ADDITION AND SUBTRACTION

Consider $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$.

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Define matrices **A**, **B** and **C**.

To find **A** + **B**, press MATRX 1 to enter matrix **A**, then + ,
then MATRX 2 to enter matrix **B**.

Press ENTER to display the results.

Attempting to find **A** + **C** will produce an error message, as **A**
and **C** have different orders.

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Define matrices **A**, **B** and **C**.

To find **A** + **B**, press OPTN F2 (**MAT**) F1 (**Mat**) ALPHA
A to enter matrix **A**, then + , then F1 (**Mat**) ALPHA B
to enter matrix **B**.

Press EXE to display the result.

Attempting to find **A** + **C** will produce an error message, as **A**
and **C** have different orders.

Operations of subtraction and scalar multipliciation can be performed in a similar manner.

MATRIX MULTIPLICIATION

Consider finding $\begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 4 & 7 \end{pmatrix}$.

Texas Instruments TI-83

Define matrices $\mathbf{A} = \begin{pmatrix} 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 4 & 7 \end{pmatrix}$.

To find **AB**, press MATRX 1 to enter matrix **A**, then × , then
MATRX 2 to enter matrix **B**.

Press ENTER to display the result.

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Define matrices $\mathbf{A} = \begin{pmatrix} 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 4 & 7 \end{pmatrix}$.

To find **AB**, press OPTN F2 (**MAT**) F1 (**Mat**) ALPHA A
to enter matrix **A**, then × , then F1 (**Mat**) ALPHA B to
enter matrix **B**.

Press EXE to display the result.

I TWO VARIABLE ANALYSIS

We can use our graphics calculator to find the line of best fit connecting two variables. We can also find the values of Pearson's correlation coefficient r and the coefficient of determination r^2, which measure the strength of the linear correlation between the two variables.

We will examine the relationship between the variables x and y for the data:

x	1	2	3	4	5	6	7
y	5	8	10	13	16	18	20

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Enter the x values into **List1** and the y values into **List2** using the instructions given on page **18**.

To produce a scatter diagram of the data, press 2nd Y= (**STAT PLOT**) **1**, and set up **Statplot 1** as shown.

Press ZOOM **9 : ZoomStat** to draw the scatter diagram.

We will now find the line of best fit. Press STAT ▶ **4:LinReg(ax+b)** to select the linear regression option from the CALC menu.

Press 2nd **1 (L₁)** , 2nd **2 (L₂)** , VARS ▶ **1 1 (Y₁)**. This specifies the lists **L₁** and **L₂** as the lists which hold the data, and the line of best fit will be pasted into the function **Y₁**. Press ENTER to view the results.

The line of best fit is given as $y \approx 2.54x + 2.71$. If the r and r^2 values are not shown, you need to turn on the Diagnostic by pressing 2nd **0** (**CATALOG**) and selecting **DiagnosticOn**.

Press GRAPH to view the line of best fit.
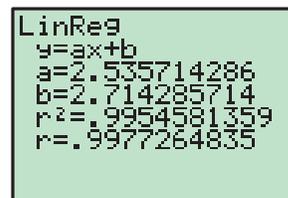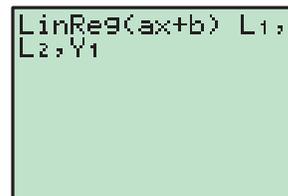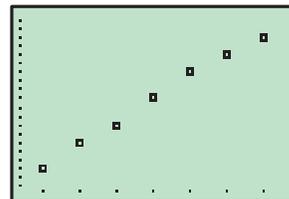
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Enter the x values into **List 1** and the y values into **List 2** using the instructions given on page **18**.

To produce a scatter diagram for the data, press F1 (**GRPH**) F6 (**SET**), and set up **StatGraph 1** as shown. Press EXIT F1 (**GPH1**) to draw the scatter diagram.
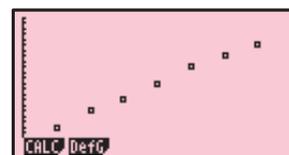
To find the line of best fit, press ☐F1 (**CALC**) ☐F2 (**X**).

We can see that the line of best fit is given as $y \approx 2.54x + 2.71$, and we can view the r and r^2 values.
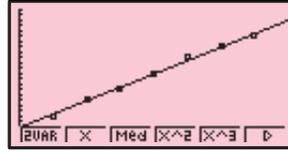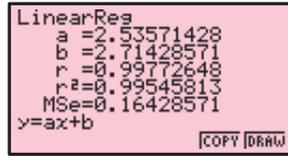
Press ☐F6 (**DRAW**) to view the line of best fit.

```
LinearReg
  a =2.53571428
  b =2.71428571
  r =0.99772648
  r²=0.99545813
  MSe=0.16428571
y=ax+b
                 COPY DRAW
```


CHALLENGE SETS

Click on the icon to access printable Challenge Sets.

CHALLENGE SETS

CHALLENGE SET 1

1 Show that every number of the form $2^n - 1$ where $n \geqslant 2$ cannot be a perfect square.
   Hint: Consider $2^n - 1 = (2a)^2$ and $2^n - 1 = (2a+1)^2$.
   Show that both of these cases are not possible.
2 Solve simultaneously: $a + b - 1$, $b + c - 2$ and c
3 A pie can be cut into portions (not necessarily of equal size). cuts across it. Determine the maximum number of portions
4 $100a + 10b + c = 99a + 9b + (a + b + c)$. Use this to "a 3-digit number is divisible by 3 only when the sum of the Can you extend your argument to 4-digit numbers?
5 If $\dfrac{n}{b} = 3$, find the value of $\dfrac{a^2 + 4ab}{b^2}$.
6 The shaded region in the square has area x cm². The two unshaded regions each have area y cm². Find $x + 2y$ and $x + y$. Hence find the area of the shaded region.
7 A 2-digit number is four times as large as the sum of its Show that there are four possible numbers with this prop

CHALLENGE SET 2

1 The '**Three Dice Trick**':
   Adele says to Georgie: "*Throw three dice and observe the numbers which result. Double the first number, add 5, multiply by 5, add the second number, multiply by 10, then add the third number. Now tell me your answer.*"
   Adele then subtracts 250 from Georgie's answer and imm thrown on the dice.
   For example, if Georgie throws a 4, a 2 and a 3, her calc
   $$4 \times 2 = 8$$
   $$8 + 5 = 13$$
   $$13 \times 5 = 65$$
   $$65 + 2 = 67$$
   $$67 \times 10 = 670$$
   $$670 + 3 = 673$$
   Adele's calculation is $673 - 250 = 423$, from which s 4, a 2 and a 3.
   Explain why this calculation always produces the correct

CHALLENGE SET 3

1 Explain why $2679\,430\,077\,712\,313$ cannot be a perfect square.
2 Find the area of the shaded region:
3 A rectangular box has adjacent faces with areas 20 cm², Find the lengths of its sides.
4 Find the smallest positive integer which leaves a remain and 5.

CHALLENGE SET 4

1 Cecil Pitt makes cylindrical tanks. The dimensions of his tanks are such that:
   • the radius and height are a whole number of metres
   • the surface area in m² and volume in m³ have the same value.
   Show that there are exactly 3 sizes of tanks Cecil could produce at his factory.
2 7, a, b, c, and 28 are equally spaced on the number line. $a + b + c + d =$
3 A square sheet of card has three lines drawn from the midpoints of its sides as shown alongside. Draw the figure on paper or card and cut out the four regions. Re-align the regions to form an isosceles triangle.
4 The current flow in a river is 5 km h⁻¹. When a boat travels at maximum speed against the current it reaches its destination in 2 hours. On the return trip it takes just 1½ hours.

CHALLENGE SET 5

1 Store A reduces its prices by 10%. One week later it increases its prices by $x\%$ so that its prices are back to what they were originally. Find x.
2 The diagram alongside shows overhead plan views of tennis balls in piles.
   a How many balls are in pile 4?
   b How many balls are in pile 8?
3 When Janice numbers the pages of her algebra book she uses pages are there in Janice's algebra book?
4 A, B and C are the midpoints of the three sides of a triangle. Is it always possible to bend along [AB], [BC] and [CA] to form a tetrahedron? Explain your answer.
5 A ship is sailing at 15 km h⁻¹ directly towards a lighthouse. from the bow or front of the ship, to the stern or rear. At wh relative to an observer in the lighthouse?
6 Two numbers differ by 2 and their reciprocals differ by $\frac{1}{24}$.
   Hint: $\frac{1}{a} - \frac{1}{b} = ?$
7 Prove that if the product of 3 consecutive integers is increas then the result will always be a perfect cube.

CHALLENGE SET 6

1 An apple, a banana and an orange cost 60 pence. A banana, an orange and a grapefruit cost 70 pence. An orange, a grapefruit and an apple cost 90 pence. A grapefruit, an apple and a banana cost 80 pence. Find the total cost of one of each fruit mentioned.
2 At 6 pm the hands of a clock are 180° apart. How long will it be before the hands of the clock are again 180° apart?
3 If $\dfrac{a}{a+3b} = 2$, find the value of $\dfrac{a-4b}{a+b}$.
4 A two litre can is an litre can. Explain why ratio $1 : \sqrt[3]{2}$.
5 Pete takes 3 days to dig a well working at a constant rate. M same sized well, in the same soil, also working at a constant If they work together to dig the well, each working at the sam will they take to dig the well?
6 How many 6-digit numbers such as 612543 can be written u 5 and 6 once only in any order?
7 a calculator, find the smaller of 2^{173} and 5^{75}.
   visible by 3 and $2N - 1$ is divisible by 7, prov

CHALLENGE SET 7

1 If 36 blocks are placed together as shown, and the outer surfaces are painted, how many blocks have:
   a 3 faces painted   **b** 2 faces painted
   c 1 face painted   **d** 0 faces painted?
2 Suppose the 36 blocks of question 1 are glued together as show glue on them?
3 For a 3-day Mathematics excursion, a tray of fruit buns was pu a half of the buns were eaten. On day 2 a half of the remaind 3 a quarter of the remainder were consumed. 9 buns remained originally purchased?
4 How many squares are contained in the figure alongside? Do not forget 2×2 and 3×3 squares.
5 $1 = 2^1 - 1$
   $3 = 2^2 - 1$
   $7 = 2^3 - 1$
   a Check that the pattern along the next 3 lines.
   b Assuming that the pattern the sum of: $1 + 2 + 2^2 + 2$

CHALLENGE SET 8

1 Find the next 3 digits of the number pattern $2\,5\,8\,1\,1\,1\,4\,1\,7\,2\,0 \ldots$.
2 List all possible orderings of the letters A, A, B, C. Do this in a systematic way so that no possibilities are left out. You should find 12 of them.
3 are 3 different views of the same die. What numbers are opposite each other?
4 In Australian Rules football, a 'goal' is worth 6 points and a 'behind' is worth 1 point. So, if a team scores 3 'goals' and 4 'behinds' it has $3 \times 6 + 4 = 22$ points. On the scoreboard at the MCG, Collingwood has a total of 95 points. However, if the 'goals' and 'behinds' were reversed, the total points would be 80. How many 'goals' and 'behinds' has Collingwood scored?
5 The following facts are known about numbers a, b, c and d:
   • b is larger than a by 4   • a is 3 less than c • d is 8 less than c.
   What is the difference between b and d?
6 In a particular test, all of the questions are of equal mark value. Peter answered 9 of the first 10 questions correctly and got $\frac{1}{3}$ of the remaining questions correct. If Peter scored 50% for the test, how many questions did it contain?
7 A and B are two cyclists who intend to travel the same journey. A leaves at 12 noon and averages 20 km h⁻¹. B leaves 45 minutes later and averages 25 km h⁻¹. How far does B have to travel to catch up to A?
8 Two spheres of diameter 6 cm and 4 cm touch each other at A as they rest on a horizontal table. How high is A above the table?

CHALLENGE SET 9

1 A car tyre has outer radius 30 cm. How many revolutions of the tyre are necessary to travel a distance of 10 km?
2 A rectangular pris C as shown. The the area of face face C if the vol
3 are the 6 different ways in which 3 postage stamps may different ways may 4 postage stamps be purchased?
4 $\dfrac{A}{x-1} + \dfrac{B}{x+2} = \dfrac{4x-1}{(x-1)(x+2)}$ for all $x \neq 1$ or 2.
5 **a** How many different handshakes are possible betwee 1   2 people   3 people
   b Each person on a committee shakes hands with eve are 36 handshakes. How many people are on the co
6 A circle is drawn so that it just fits in the corner of a Point A is 1 cm from one edge of the page and 2 cm another edge. Find the radius of the circle.
7 Two circles of th have a common touches the large tangent. Prove th is $\frac{1}{4}$ of the radius

CHALLENGE SET 10

1 The diameter of a circle is doubled. What effect does this have on its area?
2 Wheel A has diameter 10 cm and wheel B has diameter 6 cm. If A rotates at 120 revolutions each second, at what speed will B rotate?
3 Find all possible triangles with perimeter 13 cm if the side lengths must be whole numbers.
4 Show that the sum of the areas of the largest semi-circle is equal to the sum of the areas of the smaller two semi-circles.
5 John can paint a room by himself in 3 hours. If Jack helps John paint the room they would take 2 hours. How long would Jack take to paint the room working by himself?
6 Two circles of radii 4 cm and 2 cm touch on a straight line at A and B as shown. Find the distance between A and B.
7 A rope is stretched tightly over an aeroplane hangar from A to B. The shape of the hangar is semi-cylindrical. Find the length of the rope.

Chapter 1

Algebra (notation and equations)

Contents:

Algebra is a very powerful tool which is used to make problem solving easier. Algebra involves using letters or **pronumerals** to represent unknown values or **variables** which can change depending on the situation.

Many worded problems can be converted to algebraic symbols to make algebraic **equations**. We will learn how to **solve** equations to find **solutions** to the problems.

Algebra can also be used to construct **formulae**, which are equations that connect two or more variables. Many people use formulae as part of their jobs, so an understanding of how to **substitute** into formulae and **rearrange** them is essential. Builders, nurses, pharmacists, engineers, financial planners, and computer programmers all use formulae which rely on algebra.

OPENING PROBLEM

Holly bought XBC shares for $2.50 each and NLG shares for $4.00 each.

Things to think about:

- How much would Holly pay in total for 500 XBC shares and 600 NGL shares?
- What would Holly pay in total for x XBC shares and $(x + 100)$ NGL shares?
- Bob knows that Holly paid a total of $5925 for her XBC and NGL shares. He also knows that she bought 100 more NGL shares than XBC shares. How can Bob use algebra to find how many of each share type Holly bought?

A   ALGEBRAIC NOTATION

The ability to convert worded sentences and problems into algebraic symbols and to understand algebraic notation is essential in the problem solving process.

Notice that:
- $x^2 + 3x$ is an algebraic **expression**, whereas
- $x^2 + 3x = 8$ is an **equation**, and
- $x^2 + 3x > 28$ is an **inequality** or **inequation**.

When we simplify **repeated sums**, we use **product** notation:

For example:
$$x + x$$
$$= 2 \text{ 'lots' of } x$$
$$= 2 \times x$$
$$= 2x$$

and
$$x + x + x$$
$$= 3 \text{ 'lots' of } x$$
$$= 3 \times x$$
$$= 3x$$

When we simplify **repeated products**, we use **index** notation:

For example: $x \times x = x^2$ and $x \times x \times x = x^3$

Example 1 ◄))) Self Tutor

Write, in words, the meaning of:

   a $x - 5$ **b** $a + b$ **c** $3x^2 + 7$

   a $x - 5$ is "5 less than x".

   b $a + b$ is "the sum of a and b" or "b more than a"

   c $3x^2 + 7$ is "7 more than three times the square of x".

Example 2 ◄))) Self Tutor

Write the following as algebraic expressions:

   a the sum of p and the square of q **b** the square of the sum of p and q

   c b less than double a

   a $p + q^2$ **b** $(p + q)^2$ **c** $2a - b$

Example 3 ◄))) Self Tutor

Write, in sentence form, the meaning of:

   a $D = ct$ **b** $A = \dfrac{b + c}{2}$

   a D is equal to the product of c and t.

   b A is equal to a half of the sum of b and c,

      or, A is the average of b and c.

Example 4 ◄))) Self Tutor

Write 'S is the sum of a and the product of g and t' as an equation.

The product of g and t is gt.

The sum of a and gt is $a + gt$.

So, the equation is $S = a + gt$.

EXERCISE 1A

1 Write, in words, the meaning of:

   a $2a$ **b** pq **c** $\sqrt{m}$ **d** a^2

   e $a - 3$ **f** $b + c$ **g** $2x + c$ **h** $(2a)^2$

   i $2a^2$ **j** $a - c^2$ **k** $a + b^2$ **l** $(a + b)^2$

2 Write the following as algebraic expressions:
- **a** the sum of a and c
- **b** the sum of p, q and r
- **c** the product of a and b
- **d** the sum of r and the square of s
- **e** the square of the sum of r and s
- **f** the sum of the squares of r and s
- **g** the sum of twice a and b
- **h** the difference between p and q if $p > q$
- **i** a less than the square of b
- **j** half the sum of a and b
- **k** the sum of a and a quarter of b
- **l** the square root of the sum of m and n
- **m** the sum of x and its reciprocal
- **n** a quarter of the sum of a and b
- **o** the square root of the sum of the squares of x and y

3 Write, in sentence form, the meaning of:

- **a** $L = a + b$
- **b** $K = \dfrac{a + b}{2}$
- **c** $M = 3d$
- **d** $N = bc$
- **e** $T = bc^2$
- **f** $F = ma$
- **g** $K = \sqrt{\dfrac{n}{t}}$
- **h** $c = \sqrt{a^2 + b^2}$
- **i** $A = \dfrac{a + b + c}{3}$

4 Write the following as algebraic equations:
- **a** S is the sum of p and r
- **b** D is the difference between a and b where $b > a$
- **c** A is the average of k and l
- **d** M is the sum of a and its reciprocal
- **e** K is the sum of t and the square of s
- **f** N is the product of g and h
- **g** y is the sum of x and the square of x
- **h** P is the square root of the sum of d and e

> The **difference** between two numbers is the larger one minus the smaller one.

B   ALGEBRAIC SUBSTITUTION

To **evaluate** an algebraic expression, we **substitute** numerical values for the unknown, then calculate the result.

Consider the number crunching machine alongside:

If we place any number x into the machine, it calculates $5x - 7$. So, x is multiplied by 5, and then 7 is subtracted.

input, x

$5x - 7$ calculator

output
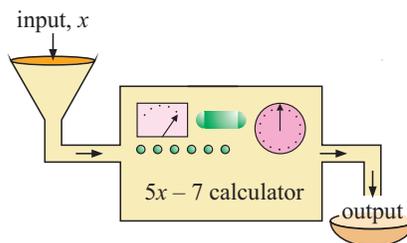
For example: if $x = 2$,

$$5x - 7$$
$$= 5 \times 2 - 7$$
$$= 10 - 7$$
$$= 3$$

and if $x = -2$,

$$5x - 7$$
$$= 5 \times (-2) - 7$$
$$= -10 - 7$$
$$= -17$$

Notice that when we substitute a negative number such as -2, we place it in brackets. This helps us to get the sign of each term correct.

Example 5

🔊 **Self Tutor**

If $p = 4$, $q = -2$ and $r = 3$, find the value of:

a $3q - 2r$ **b** $2pq - r$ **c** $\dfrac{p - 2q + 2r}{p + r}$

a $3q - 2r$
$= 3 \times (-2) - 2 \times 3$
$= -6 - 6$
$= -12$

b $2pq - r$
$= 2 \times 4 \times (-2) - 3$
$= -16 - 3$
$= -19$

c $\dfrac{p - 2q + 2r}{p + r}$
$= \dfrac{4 - 2 \times (-2) + 2 \times 3}{4 + 3}$
$= \dfrac{4 + 4 + 6}{4 + 3}$
$= \dfrac{14}{7}$
$= 2$

Example 6

🔊 **Self Tutor**

If $a = 3$, $b = -2$ and $c = -1$, evaluate:

a b^2 **b** $ab - c^3$

a b^2
$= (-2)^2$
$= (-2) \times (-2)$
$= 4$

b $ab - c^3$
$= 3 \times (-2) - (-1)^3$
$= -6 - (-1)$
$= -6 + 1$
$= -5$

> Notice the use of brackets.

Example 7

🔊 **Self Tutor**

If $p = 4$, $q = -3$ and $r = 2$, evaluate:

a $\sqrt{p - q + r}$ **b** $\sqrt{p + q^2}$

a $\sqrt{p - q + r}$
$= \sqrt{4 - (-3) + 2}$
$= \sqrt{4 + 3 + 2}$
$= \sqrt{9}$
$= 3$

b $\sqrt{p + q^2}$
$= \sqrt{4 + (-3)^2}$
$= \sqrt{4 + 9}$
$= \sqrt{13}$
≈ 3.61

EXERCISE 1B

1 If $p = 5$, $q = 3$ and $r = -4$ find the value of:

a $5p$ **b** $4q$ **c** $3pq$ **d** pqr

e $3p - 2q$ **f** $5r - 4q$ **g** $4q - 2r$ **h** $2pr + 5q$

2 If $w = 3$, $x = 1$ and $y = -2$, evaluate:

a $\dfrac{y}{w}$

b $\dfrac{y+w}{x}$

c $\dfrac{3x-y}{w}$

d $\dfrac{5w-2x}{y-x}$

e $\dfrac{y-x+w}{2(y-w)}$

f $\dfrac{xy+w}{y-x}$

g $\dfrac{x-wy}{y+w-2x}$

h $\dfrac{y}{x} - w$

3 If $a = -3$, $b = -4$ and $c = -1$, evaluate:

a c^2

b b^3

c $a^2 + b^2$

d $(a+b)^2$

e $b^3 + c^3$

f $(b+c)^3$

g $(2a)^2$

h $2a^2$

4 If $p = 4$, $q = -1$ and $r = 2$, evaluate:

a $\sqrt{p} + q$

b $\sqrt{p+q}$

c $\sqrt{r-q}$

d $\sqrt{p-pq}$

e $\sqrt{pr-q}$

f $\sqrt{p^2+q^2}$

g $\sqrt{p+r+2q}$

h $\sqrt{2q-5r}$

C   LINEAR EQUATIONS

Linear equations are equations which can be written in the form $ax + b = 0$, where x is the **unknown variable** and a, b are **constants**.

INVESTIGATION   SOLVING EQUATIONS

Linear equations like $5x - 3 = 12$ can be solved using a table of values on a **graphics calculator**.
We try to find the value of x which makes the expression $5x - 3$ equal to 12. This is the **solution** to the equation.

To do this investigation you may need the calculator instructions on pages **21** and **22**.

What to do:

1 Enter the **function** $Y_1 = 5X - 3$ into your calculator.

2 Set up a **table** that calculates the value of $y = 5x - 3$ for x values from -5 to 5.

3 Scroll down the table until you find the value of x that makes y equal to 12.
As we can see, the solution is $x = 3$.

X	Y1	
-1	-8	
0	-3	
1	2	
2	7	
3	12	
4	17	
5	22	
X=3		

4 Use your calculator and the method given above to solve the following equations:

a $7x + 1 = -20$
b $8 - 3x = -4$
c $\dfrac{x}{4} + 2 = 1$
d $\tfrac{1}{3}(2x - 1) = 3$

5 The solutions to the following equations are *not integers*. Change your table to investigate x values from -5 to 5 in intervals of 0.5.

a $2x - 3 = -6$
b $6 - 4x = 8$
c $x - 5 = -3.5$

6 Use a calculator to solve the following equations:

a $3x + 2 = 41$
b $5 - 4x = 70$
c $\dfrac{2x}{3} + 5 = 2\tfrac{2}{3}$

ALGEBRAIC SOLUTION TO LINEAR EQUATIONS

The following steps should be followed to solve linear equations:

Step 1:   Decide how the expression containing the unknown has been '**built up**'.

Step 2:   Perform **inverse** operations on **both sides** of the equation to 'undo' how the equation is 'built up'. In this way we **isolate** the unknown.

Step 3:   Check your solution by substitution.

Example 8   ◀) **Self Tutor**

Solve for x: **a** $4x - 1 = 7$ **b** $5 - 3x = 6$

> The inverse of
> $+$ is $-$
> and
> $\times$ is $\div$.

a $4x - 1 = 7$

$\therefore$ $4x - 1 + 1 = 7 + 1$ {adding 1 to both sides}

$\therefore$ $4x = 8$

$\therefore$ $\dfrac{4x}{4} = \dfrac{8}{4}$ {divide both sides by 4}

$\therefore$ $x = 2$ *Check:* $4 \times 2 - 1 = 8 - 1 = 7$ ✓

b $5 - 3x = 6$

$\therefore$ $5 - 3x - 5 = 6 - 5$ {subtracting 5 from both sides}

$\therefore$ $-3x = 1$

$\therefore$ $\dfrac{-3x}{-3} = \dfrac{1}{-3}$ {dividing both sides by -3}

$\therefore$ $x = -\frac{1}{3}$ *Check:* $5 - 3 \times (-\frac{1}{3}) = 5 + 1 = 6$ ✓

Example 9   ◀) **Self Tutor**

Solve for x: **a** $\dfrac{x}{5} - 3 = -1$ **b** $\frac{1}{5}(x - 3) = -1$

a $\dfrac{x}{5} - 3 = -1$

$\therefore$ $\dfrac{x}{5} - 3 + 3 = -1 + 3$ {adding 3 to both sides}

$\therefore$ $\dfrac{x}{5} = 2$

$\therefore$ $\dfrac{x}{5} \times 5 = 2 \times 5$ {multiplying both sides by 5}

$\therefore$ $x = 10$ *Check:* $\frac{10}{5} - 3 = 2 - 3 = -1$ ✓

b $\quad\quad\quad \frac{1}{5}(x - 3) = -1$

$\therefore \quad \frac{1}{5}(x - 3) \times 5 = -1 \times 5 \quad\quad$ {multiplying both sides by 5}

$\therefore \quad\quad x - 3 = -5$

$\therefore \quad x - 3 + 3 = -5 + 3 \quad\quad$ {adding 3 to both sides}

$\therefore \quad\quad x = -2 \quad\quad\quad$ *Check:* $\frac{1}{5}((-2) - 3) = \frac{1}{5} \times -5 = -1$ ✓

EXERCISE 1C.1

1 Solve for x:

    a $\;x + 9 = 4$ **b** $\;5x = 45$ **c** $\;-24 = -6x$ **d** $\;3 - x = 12$

    e $\;2x + 5 = 17$ **f** $\;3x - 2 = -14$ **g** $\;3 - 4x = -17$ **h** $\;8 = 9 - 2x$

2 Solve for x:

    a $\;\dfrac{x}{4} = 12$ **b** $\;\frac{1}{2}x = 6$ **c** $\;5 = \dfrac{x}{-2}$ **d** $\;\dfrac{x}{3} + 4 = -2$

    e $\;\dfrac{x + 3}{5} = -2$ **f** $\;\frac{1}{3}(x + 2) = 3$ **g** $\;\dfrac{2x - 1}{3} = 7$ **h** $\;\frac{1}{2}(5 - x) = -2$

THE DISTRIBUTIVE LAW

In situations where the unknown appears more than once, we need to **expand** any brackets, **collect like terms**, and then **solve** the equation.

To expand the brackets we use the **distributive law**: $\quad\quad a(b + c) = ab + ac$

Example 10   ◀ᴺ) **Self Tutor**

Solve for x: $\quad\quad 3(2x - 5) - 2(x - 1) = 3$

$$3(2x - 5) - 2(x - 1) = 3$$

$\therefore \quad 3 \times 2x + 3 \times (-5) - 2 \times x - 2 \times (-1) = 3 \quad\quad$ {expanding brackets}

$\therefore \quad\quad 6x - 15 - 2x + 2 = 3$

$\therefore \quad\quad 4x - 13 = 3 \quad\quad$ {collecting like terms}

$\therefore \quad 4x - 13 + 13 = 3 + 13 \quad\quad$ {adding 13 to both sides}

$\therefore \quad\quad 4x = 16$

$\therefore \quad\quad x = 4 \quad\quad$ {dividing both sides by 4}

Check: $\;3(2 \times 4 - 5) - 2(4 - 1) = 3 \times 3 - 2 \times 3 = 3$ ✓

If the unknown appears on **both sides** of the equation, we

- **expand** any brackets and **collect like terms**
- move the **unknown to one side** of the equation and the remaining terms to the other side
- **simplify** and then **solve** the equation.

Example 11

Solve for x:

a $3x - 4 = 2x + 6$ **b** $4 - 3(2 + x) = x$

a
$$3x - 4 = 2x + 6$$
$$\therefore \quad 3x - 4 - 2x = 2x + 6 - 2x \qquad \{\text{subtracting } 2x \text{ from both sides}\}$$
$$\therefore \quad x - 4 = 6$$
$$\therefore \quad x - 4 + 4 = 6 + 4 \qquad \{\text{adding 4 to both sides}\}$$
$$\therefore \quad x = 10$$

Check: LHS $= 3 \times 10 - 4 = 26$,   RHS $= 2 \times 10 + 6 = 26$. ✓

b
$$4 - 3(2 + x) = x$$
$$\therefore \quad 4 - 6 - 3x = x \qquad \{\text{expanding}\}$$
$$\therefore \quad -2 - 3x = x$$
$$\therefore \quad -2 - 3x + 3x = x + 3x \qquad \{\text{adding } 3x \text{ to both sides}\}$$
$$\therefore \quad -2 = 4x$$
$$\therefore \quad \frac{-2}{4} = \frac{4x}{4} \qquad \{\text{dividing both sides by 4}\}$$
$$\therefore \quad -\tfrac{1}{2} = x$$
$$\text{i.e.,} \quad x = -\tfrac{1}{2}$$

Check: LHS $= 4 - 3(2 + (-\tfrac{1}{2})) = 4 - 3 \times \tfrac{3}{2} = 4 - 4\tfrac{1}{2} = -\tfrac{1}{2} =$ RHS ✓

EXERCISE 1C.2

1 Solve for x:

a $2(x + 8) + 5(x - 1) = 60$ **b** $2(x - 3) + 3(x + 2) = -5$

c $3(x + 3) - 2(x + 1) = 0$ **d** $4(2x - 3) + 2(x + 2) = 32$

e $3(4x + 1) - 2(3x - 4) = -7$ **f** $5(x + 2) - 2(3 - 2x) = -14$

2 Solve for x:

a $2x - 3 = 3x + 6$ **b** $3x - 4 = 5 - x$

c $4 - 5x = 3x - 8$ **d** $-x = 2x + 4$

e $12 - 7x = 3x + 7$ **f** $5x - 9 = 1 - 3x$

g $4 - x - 2(2 - x) = 6 + x$ **h** $5 - 3(1 - x) = 2 - 3x$

i $5 - 2x - (2x + 1) = -6$ **j** $3(4x + 2) - x = -7 + x$

3 Solve for x:

a $2(3x + 1) - 3 = 6x - 1$ **b** $3(4x + 1) = 6(2x + 1)$

c Comment on your solutions to **a** and **b**.

D | RATIONAL EQUATIONS

Rational equations are equations involving fractions. We write all fractions in the equation with the same **lowest common denominator (LCD)**, and then equate the numerators.

Consider the following rational equations:

$$\frac{x}{2} = \frac{x}{3} \qquad \text{LCD is } 2 \times 3 = 6$$

$$\frac{5}{2x} = \frac{3x}{5} \qquad \text{LCD is } 2x \times 5 = 10x$$

$$\frac{x-7}{3} = \frac{x}{2x-1} \qquad \text{LCD is } 3 \times (2x-1) = 3(2x-1)$$

Example 12   ◀》 **Self Tutor**

Solve for x: $\dfrac{x}{2} = \dfrac{3+x}{5}$

> Notice the insertion of brackets here.

$\dfrac{x}{2} = \dfrac{3+x}{5}$ has LCD $= 10$

$\therefore \quad \dfrac{x}{2} \times \dfrac{5}{5} = \dfrac{2}{2} \times \left(\dfrac{3+x}{5}\right)$ {to create a common denominator}

$\therefore \quad 5x = 2(3+x)$ {equating numerators}

$\therefore \quad 5x = 6 + 2x$ {expanding brackets}

$\therefore \quad 5x - 2x = 6 + 2x - 2x$ {subtracting $2x$ from both sides}

$\therefore \quad 3x = 6$

$\therefore \quad x = 2$ {dividing both sides by 3}

Example 13   ◀》 **Self Tutor**

Solve for x: $\dfrac{4}{x} = \dfrac{3}{4}$

$\dfrac{4}{x} = \dfrac{3}{4}$ has LCD $= 4x$

$\therefore \quad \dfrac{4}{x} \times \dfrac{4}{4} = \dfrac{3}{4} \times \dfrac{x}{x}$ {to create a common denominator}

$\therefore \quad 16 = 3x$ {equating numerators}

$\therefore \quad x = \dfrac{16}{3}$ {dividing both sides by 3}

Example 14   ◀)) **Self Tutor**

Solve for x: $\dfrac{2x+1}{3-x} = \dfrac{3}{4}$

$\dfrac{2x+1}{3-x} = \dfrac{3}{4}$ has LCD $= 4(3-x)$

> Notice the use of brackets here.

$\therefore \quad \dfrac{4}{4} \times \left(\dfrac{2x+1}{3-x}\right) = \dfrac{3}{4} \times \left(\dfrac{3-x}{3-x}\right)$ {to create a common denominator}

$\therefore \quad 4(2x+1) = 3(3-x)$ {equating numerators}

$\therefore \quad 8x+4 = 9 - 3x$ {expanding the brackets}

$\therefore \quad 8x+4+3x = 9 - 3x + 3x$ {adding $3x$ to both sides}

$\therefore \quad 11x+4 = 9$

$\therefore \quad 11x+4-4 = 9-4$ {subtracting 4 from both sides}

$\therefore \quad 11x = 5$

$\therefore \quad x = \frac{5}{11}$ {dividing both sides by 11}

Example 15   ◀)) **Self Tutor**

Solve for x: $\dfrac{x}{3} - \dfrac{1-2x}{6} = -4$

$\dfrac{x}{3} - \dfrac{1-2x}{6} = -4$ has LCD $= 6$

$\therefore \quad \dfrac{x}{3} \times \dfrac{2}{2} - \left(\dfrac{1-2x}{6}\right) = -4 \times \dfrac{6}{6}$ {to create a common denominator}

$\therefore \quad 2x - (1-2x) = -24$ {equating numerators}

$\therefore \quad 2x - 1 + 2x = -24$ {expanding}

$\therefore \quad 4x - 1 = -24$

$\therefore \quad 4x - 1 + 1 = -24 + 1$ {adding 1 to both sides}

$\therefore \quad 4x = -23$

$\therefore \quad x = -\frac{23}{4}$ {dividing both sides by 4}

EXERCISE 1D

1 Solve for x:

a $\dfrac{x}{2} = \dfrac{4}{7}$

b $\dfrac{5}{8} = \dfrac{x}{6}$

c $\dfrac{x}{2} = \dfrac{x-2}{3}$

d $\dfrac{x+1}{3} = \dfrac{2x-1}{4}$

e $\dfrac{2x}{3} = \dfrac{5-x}{2}$

f $\dfrac{3x+2}{5} = \dfrac{2x-1}{2}$

g $\dfrac{2x-1}{3} = \dfrac{4-x}{6}$

h $\dfrac{4x+7}{7} = \dfrac{5-x}{2}$

i $\dfrac{3x+1}{6} = \dfrac{4x-1}{-2}$

2 Solve for x:

a $\dfrac{5}{x} = \dfrac{2}{3}$ **b** $\dfrac{6}{x} = \dfrac{3}{5}$ **c** $\dfrac{4}{3} = \dfrac{5}{x}$ **d** $\dfrac{3}{2x} = \dfrac{7}{6}$

e $\dfrac{3}{2x} = \dfrac{7}{3}$ **f** $\dfrac{7}{3x} = -\dfrac{1}{6}$ **g** $\dfrac{5}{4x} = -\dfrac{1}{12}$ **h** $\dfrac{4}{7x} = \dfrac{3}{2x}$

3 Solve for x:

a $\dfrac{2x+3}{x+1} = \dfrac{5}{3}$ **b** $\dfrac{x+1}{1-2x} = \dfrac{2}{5}$ **c** $\dfrac{2x-1}{4-3x} = -\dfrac{3}{4}$

d $\dfrac{x+3}{2x-1} = \dfrac{1}{3}$ **e** $\dfrac{4x+3}{2x-1} = 3$ **f** $\dfrac{3x-2}{x+4} = -3$

g $\dfrac{6x-1}{3-2x} = 5$ **h** $\dfrac{5x+1}{x+4} = 4$ **i** $2 + \dfrac{2x+5}{x-1} = -3$

4 Solve for x:

a $\dfrac{x}{2} - \dfrac{x}{6} = 4$ **b** $\dfrac{x}{4} - 3 = \dfrac{2x}{3}$

c $\dfrac{x}{8} + \dfrac{x+2}{2} = -1$ **d** $\dfrac{x+2}{3} + \dfrac{x-3}{4} = 1$

e $\dfrac{2x-1}{3} - \dfrac{5x-6}{6} = -2$ **f** $\dfrac{x}{4} = 4 - \dfrac{x+2}{3}$

g $\dfrac{2x-7}{3} - 1 = \dfrac{x-4}{6}$ **h** $\dfrac{x+1}{3} - \dfrac{x}{6} = \dfrac{2x-3}{2}$

i $\dfrac{x}{5} - \dfrac{2x-5}{3} = \dfrac{3}{4}$ **j** $\dfrac{x+1}{3} + \dfrac{x-2}{6} = \dfrac{x+4}{12}$

k $\dfrac{x-6}{5} - \dfrac{2x-1}{10} = \dfrac{x-1}{2}$ **l** $\dfrac{2x+1}{4} - \dfrac{1-4x}{2} = \dfrac{3x+7}{6}$

E   LINEAR INEQUATIONS

The speed limit when passing roadworks is often 25 kilometres per hour.

This can be written as a **linear inequation** using the variable s to represent the speed of a car in km per h. $s \leqslant 25$ reads 's is less than or equal to 25'.

We can also represent the allowable speeds on a **number line**:

The number line shows that any speed of 25 km per h or less is an acceptable speed. We say that these are **solutions** of the inequation.

RULES FOR HANDLING INEQUATIONS

Notice that $5 > 3$ and $3 < 5$,
and that $-3 < 2$ and $2 > -3$.

This suggests that if we **interchange** the LHS and RHS of an inequation, then we must **reverse** the inequation sign. $>$ is the reverse of $<$, $\geqslant$ is the reverse of $\leqslant$, and so on.

You may also remember from previous years that:

- If we **add** or **subtract** the same number to both sides, the inequation sign is *maintained*. For example, if $5 > 3$, then $5 + 2 > 3 + 2$.
- If we **multiply** or **divide** both sides by a **positive** number, the inequation sign is *maintained*. For example, if $5 > 3$, then $5 \times 2 > 3 \times 2$.
- If we **multiply** or **divide** both sides by a **negative** number, the inequation sign is *reversed*. For example, if $5 > 3$, then $5 \times -1 < 3 \times -1$.

The method of solution of linear inequalities is thus identical to that of linear equations with the exceptions that:

- **interchanging** the sides **reverses** the inequation sign
- **multiplying** or **dividing** both sides by a **negative** number **reverses** the inequation sign.

GRAPHING SOLUTIONS

Suppose our solution to an inequation is $x \geqslant 4$, so *every* number which is 4 or greater than 4 is a possible value for x. We could represent this on a number line by:

The filled-in circle indicates that 4 **is** included.

The arrowhead indicates that all numbers on the number line in this direction are included.
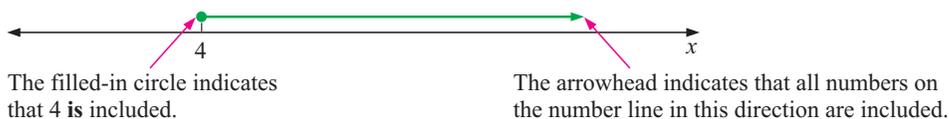
Likewise if our solution is $x < 5$ our representation would be

The hollow circle indicates that 5 **is not** included.
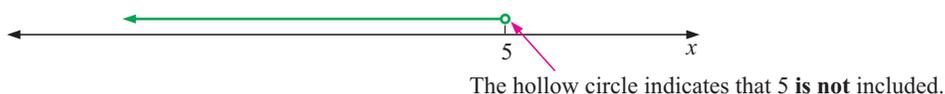
Example 16   ◀) Self Tutor

Solve for x and graph the solutions: **a** $3x - 4 \leqslant 2$ **b** $3 - 2x < 7$

a $3x - 4 \leqslant 2$

$\therefore \quad 3x - 4 + 4 \leqslant 2 + 4$ {adding 4 to both sides}

$\therefore \quad 3x \leqslant 6$

$\therefore \quad \dfrac{3x}{3} \leqslant \dfrac{6}{3}$ {dividing both sides by 3}

$\therefore \quad x \leqslant 2$
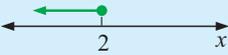
Check: If $x = 1$ then $3x - 4 = 3 \times 1 - 4 = -1$ and $-1 < 2$ is true.

b

$$3 - 2x < 7$$

$$\therefore \quad 3 - 2x - 3 < 7 - 3 \qquad \{\text{subtracting 3 from both sides}\}$$

$$\therefore \quad -2x < 4$$

$$\therefore \quad \frac{-2x}{-2} > \frac{4}{-2} \qquad \{\text{dividing both sides by } -2, \text{ so reverse the sign}\}$$

$$\therefore \quad x > -2$$

Notice the reversal of the inequation sign in **b** line 4 as we are dividing by -2.

Check: If $x = 3$ then $3 - 2x = 3 - 2 \times 3 = -3$

and $-3 < 7$ is true.

Example 17
◀⬤ **Self Tutor**

Solve for x and graph the solutions: $-5 < 9 - 2x$

$$-5 < 9 - 2x$$

$$\therefore \quad -5 + 2x < 9 - 2x + 2x \qquad \{\text{adding } 2x \text{ to both sides}\}$$

$$\therefore \quad 2x - 5 < 9$$

$$\therefore \quad 2x - 5 + 5 < 9 + 5 \qquad \{\text{adding 5 to both sides}\}$$

$$\therefore \quad 2x < 14$$

$$\therefore \quad \frac{2x}{2} < \frac{14}{2} \qquad \{\text{dividing both sides by 2}\}$$

$$\therefore \quad x < 7$$

Check: If $x = 5$ then $-5 < 9 - 2 \times 5$,   i.e., $-5 < -1$ which is true.

Example 18
◀⬤ **Self Tutor**

Solve for x and graph the solutions: $3 - 5x \geqslant 2x + 7$

$$3 - 5x \geqslant 2x + 7$$

$$\therefore \quad 3 - 5x - 2x \geqslant 2x + 7 - 2x \qquad \{\text{subtracting } 2x \text{ from both sides}\}$$

$$\therefore \quad 3 - 7x \geqslant 7$$

$$\therefore \quad 3 - 7x - 3 \geqslant 7 - 3 \qquad \{\text{subtracting 3 from both sides}\}$$

$$\therefore \quad -7x \geqslant 4$$

$$\therefore \quad \frac{-7x}{-7} \leqslant \frac{4}{-7} \qquad \{\text{dividing both sides by } -7, \text{ so reverse the sign}\}$$

$$\therefore \quad x \leqslant -\tfrac{4}{7}$$

Check: If $x = -1$ then $3 - 5 \times (-1) \geqslant 2 \times (-1) + 7$,   i.e., $8 \geqslant 5$ which is true.

EXERCISE 1E

1 Solve for x and graph the solutions:

   a $3x + 2 < 0$ **b** $5x - 7 > 2$ **c** $2 - 3x \geqslant 1$

   d $5 - 2x \leqslant 11$ **e** $2(3x - 1) < 4$ **f** $5(1 - 3x) \geqslant 8$

2 Solve for x and graph the solutions:

   a $7 \geqslant 2x - 1$ **b** $-13 < 3x + 2$ **c** $20 > -5x$

   d $-3 \geqslant 4 - 3x$ **e** $3 < 5 - 2x$ **f** $2 \leqslant 5(1 - x)$

3 Solve for x and graph the solutions:

   a $3x + 2 > x - 5$ **b** $2x - 3 < 5x - 7$

   c $5 - 2x \geqslant x + 4$ **d** $7 - 3x \leqslant 5 - x$

   e $3x - 2 > 2(x - 1) + 5x$ **f** $1 - (x - 3) \geqslant 2(x + 5) - 1$

4 Solve for x:

   a $3x + 1 > 3(x + 2)$ **b** $5x + 2 < 5(x + 1)$ **c** $2x - 4 \geqslant 2(x - 2)$

   d Comment on your solutions to **a**, **b** and **c**.

F   PROBLEM SOLVING

Many problems can be translated into **algebraic equations**. When problems are solved using algebra, we follow these steps:

Step 1:   Decide on the unknown quantity and allocate a variable.

Step 2:   Decide which operations are involved.

Step 3:   Translate the problem into an equation.

Step 4:   Solve the equation by isolating the variable.

Step 5:   Check that your solution does satisfy the original problem.

Step 6:   Write your answer in sentence form. Remember, there is usually no variable in the original problem.

Example 19   ◀) Self Tutor

When a number is trebled and subtracted from 7, the result is -11. Find the number.

Let x be the number, so $3x$ is the number trebled.

$\therefore$ $7 - 3x$ is this number subtracted from 7.

So, $7 - 3x = -11$

$\therefore$ $7 - 3x - 7 = -11 - 7$ {subtracting 7 from both sides}

$\therefore$ $-3x = -18$

$\therefore$ $x = 6$ {dividing both sides by -3}

So, the number is 6.   *Check:* $7 - 3 \times 6 = 7 - 18 = -11$ ✓

Example 20   ◀ɔ **Self Tutor**

What number must be added to both the numerator and the denominator of the fraction $\frac{1}{3}$ to get the fraction $\frac{7}{8}$?

Let x be the number.

$$\therefore \quad \frac{1+x}{3+x} = \frac{7}{8} \qquad \text{where the LCD is } \; 8(3+x)$$

$$\therefore \quad \frac{8}{8} \times \left(\frac{1+x}{3+x}\right) = \frac{7}{8} \times \left(\frac{3+x}{3+x}\right) \qquad \{\text{to get a common denominator}\}$$

$$\therefore \quad 8(1+x) = 7(3+x) \qquad \{\text{equating numerators}\}$$

$$\therefore \quad 8 + 8x = 21 + 7x \qquad \{\text{expanding brackets}\}$$

$$\therefore \quad 8 + 8x - 7x = 21 + 7x - 7x \qquad \{\text{subtracting } 7x \text{ from both sides}\}$$

$$\therefore \quad 8 + x = 21$$

$$\therefore \quad x = 13 \qquad \text{So, the number is 13.}$$

Example 21   ◀ɔ **Self Tutor**

Sarah's age is one third her father's age. In 13 years' time her age will be a half of her father's age. How old is Sarah now?

Let Sarah's present age be x years, so her father's present age is $3x$ years.

Table of ages:

	Now	13 years time
Sarah	x	$x + 13$
Father	$3x$	$3x + 13$

So, $3x + 13 = 2(x + 13)$

$\therefore \quad 3x + 13 = 2x + 26$

$\therefore \quad 3x - 2x = 26 - 13$

$\therefore \quad x = 13$

$\therefore$ Sarah's present age is 13 years.

EXERCISE 1F

1 When three times a certain number is subtracted from 15, the result is -6. Find the number.

2 Five times a certain number, minus 5, is equal to 7 more than three times the number. What is the number?

3 Three times the result of subtracting a certain number from 7 gives the same answer as adding eleven to the number. Find the number.

4 I think of a number. If I divide the sum of 6 and the number by 3, the result is 4 more than one quarter of the number. Find the number.

5 The sum of two numbers is 15. When one of these numbers is added to three times the other, the result is 27. What are the numbers?

6 What number must be added to both the numerator and the denominator of the fraction $\frac{2}{5}$ to get the fraction $\frac{7}{8}$?

7 What number must be subtracted from both the numerator and the denominator of the fraction $\frac{3}{4}$ to get the fraction $\frac{1}{3}$?

8 Eli is now one quarter of his father's age. In 5 years' time his age will be one third of his father's age. How old is Eli now?

9 When Maria was born, her mother was 24 years old. At present, Maria's age is 20% of her mother's age. How old is Maria now?

10 Five years ago, Jacob was one sixth of the age of his brother. In three years' time his age doubled will match his brother's age. How old is Jacob now?

G MONEY AND INVESTMENT PROBLEMS

Problems involving money can be made easier to understand by constructing a table and placing the given information into it.

Example 22　　　　　　　　　　　　　　　　　　　　🔊 **Self Tutor**

Britney has only 2-cent and 5-cent stamps. Their total value is $1.78, and there are two more 5-cent stamps than there are 2-cent stamps. How many 2-cent stamps are there?

If there are x 2-cent stamps then there are $(x+2)$ 5-cent stamps.

Type	Number	Value
2-cent	x	$2x$ cents
5-cent	$x+2$	$5(x+2)$ cents

$$\therefore \quad 2x + 5(x+2) = 178 \qquad \text{\{equating values in cents\}}$$
$$\therefore \quad 2x + 5x + 10 = 178$$
$$\therefore \quad 7x + 10 = 178$$
$$\therefore \quad 7x = 168$$
$$\therefore \quad x = 24 \qquad \text{So, there are 24, 2-cent stamps.}$$

EXERCISE 1G

1 Michaela has 5-cent and 10-cent stamps with a total value of €5.75. If she has 5 more 10-cent stamps than 5-cent stamps, how many of each stamp does she have?

2 The school tuck-shop has milk in 600 mL and 1 litre cartons. If there are 54 cartons and 40 L of milk in total, how many 600 mL cartons are there?

3 Aaron has a collection of American coins. He has three times as many 10 cent coins as 25 cent coins, and he has some 5 cent coins as well. If he has 88 coins with total value $11.40, how many of each type does he have?

4 Tickets to a football match cost €8, €15 or €20 each. The number of €15 tickets sold was double the number of €8 tickets sold. 6000 more €20 tickets were sold than €15 tickets. If the gate receipts totalled €783 000, how many of each type of ticket were sold?

5 Kelly blends coffee. She mixes brand A costing $6 per kilogram with brand B costing $8 per kilogram. How many kilograms of each brand does she have to mix to make 50 kg of coffee costing her $7.20 per kg?

6 Su Li has 13 kg of almonds costing $5 per kilogram. How many kilograms of cashews costing $12 per kg should be added to get a mixture of the two nut types costing $7.45 per kg?

7 Answer the questions posed in the **Opening Problem** on page **30**.

Example 23 ◀) Self Tutor

I invest in oil shares which earn me 12% yearly, and in coal mining shares which earn me 10% yearly. If I invest $3000 more in oil shares than in coal mining shares and my total yearly earnings amount to $910, how much do I invest in each type of share?

Let the amount I invest in coal mining shares be x.

Type of Shares	Amount invested ($)	Interest	Earnings ($)
Coal	x	10%	10% of x
Oil	$(x + 3000)$	12%	12% of $(x + 3000)$
		Total	910

From the earnings column of the table,

10% of x + 12% of $(x + 3000) = 910$

$\therefore \quad 0.1x \ + \ 0.12(x + 3000) = 910$

$\therefore \quad 0.1x + 0.12x + 360 = 910$

$\therefore \quad 0.22x + 360 = 910$

$\therefore \quad 0.22x = 550$

$\therefore \quad x = \dfrac{550}{0.22}$

$\therefore \quad x = 2500$

$\therefore$ I invest $2500 in coal shares and $5500 in oil shares.

---

8 Qantas shares pay a yearly return of 9% while Telstra shares pay 11%. John invests $1500 more on Telstra shares than on Qantas shares, and his total yearly earnings from the two investments is $1475. How much did he invest in Qantas shares?

9 I invested twice as much money in technology shares as I invested in mining shares. Technology shares earn me 10% yearly and mining shares earn me 9% yearly. My yearly income from these shares is $1450. Find how much I invested in each type of share.

10 Wei has three types of shares: A, B and C. A shares pay 8%, B shares pay 6%, and C shares pay 11% dividends. Wei has twice as much invested in B shares as A shares, and has $\$50\,000$ invested altogether. The yearly return from the share dividends is $\$4850$. How much is invested in each type of share?

11 Mrs Jones invests £4000 at 10% annual return and £6000 at 12% annual return. How much should she invest at 15% return so that her total annual return is 13% of the total amount she has invested?

H   MOTION PROBLEMS

Motion problems are problems concerned with speed, distance travelled, and time taken. These variables are related by the formulae:

$$\text{speed} = \frac{\text{distance}}{\text{time}} \qquad \text{distance} = \text{speed} \times \text{time} \qquad \text{time} = \frac{\text{distance}}{\text{speed}}$$

Since $\text{speed} = \dfrac{\text{distance}}{\text{time}}$,   it is usually measured in either:

- kilometres per hour, denoted km/h or $\text{km}\,\text{h}^{-1}$,   or
- metres per second, denoted m/s or $\text{m}\,\text{s}^{-1}$.

Example 24   ◀) Self Tutor

A car travels for 2 hours at a certain speed and then 3 hours more at a speed $10\ \text{km}\,\text{h}^{-1}$ faster than this. If the entire distance travelled is 455 km, find the car's speed in the first two hours of travel.

> Distance = speed × time

Let the speed in the first 2 hours be $s\ \text{km}\,\text{h}^{-1}$.

	Speed $(km\,h^{-1})$	Time (h)	Distance (km)
First section	s	2	$2s$
Second section	$(s+10)$	3	$3(s+10)$
		Total	455

So, $2s + 3(s+10) = 455$

$\therefore \quad 2s + 3s + 30 = 455$

$\therefore \qquad 5s = 425$

$\therefore \qquad s = 85$

$\therefore$ the car's speed in the first two hours was $85\ \text{km}\,\text{h}^{-1}$.

EXERCISE 1H

1 Joe can run twice as fast as Pete. They start at the same point and run in opposite directions for 40 minutes. The distance between them is now 16 km. How fast does Joe run?

2 A car leaves a country town at 60 km per hour. Two hours later, a second car leaves the town; it catches the first car after 5 more hours. Find the speed of the second car.

3 A boy cycles from his house to a friend's house at 20 $km\,h^{-1}$ and home again at 25 $km\,h^{-1}$. If his round trip takes $\frac{9}{10}$ of an hour, how far is it to his friend's house?

4 A motor cyclist makes a trip of 500 km. If he had increased his speed by 10 $km\,h^{-1}$, he could have covered 600 km in the same time. What was his original speed?

5 Normally I drive to work at 60 $km\,h^{-1}$. If I drive at 72 $km\,h^{-1}$ I cut 8 minutes off my time for the trip. What distance do I travel?

6 My motor boat normally travels at 24 $km\,h^{-1}$ in still water. One day I travelled 36 km against a constant current in a river. If I had travelled with the current, I would have travelled 48 km in the same time. How fast was the current?

I   MIXTURE PROBLEMS

The following problems are concerned with the concentration of a mixture when one liquid is added to another.

For example, a 6% cordial mixture contains 6% cordial and 94% water. If we add more cordial to the mixture then it will become more **concentrated**. Alternatively, if we add more water then the mixture will become more **diluted**.

Example 25   ◀) Self Tutor

How much water should be added to 2 litres of 5% cordial mixture to produce a 3% cordial mixture?

The unknown is the amount of water to add. Call it x litres.

x litres of water

2 litres of 5% cordial mixture

$(2 + x)$ litres of 3% cordial mixture
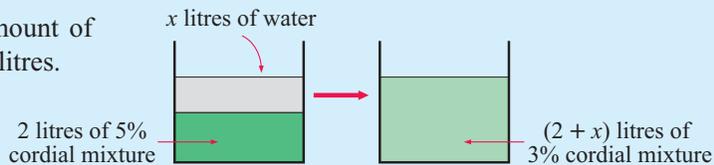
From the diagrams we can write an equation for the total amount of water in the mixture:

x L + 95% of 2 L = 97% of $(x + 2)$ L

$\therefore \quad x + \frac{95}{100} \times 2 = \frac{97}{100} \times (x + 2)$

$$\therefore \quad \frac{x}{1} \times \frac{100}{100} + \frac{190}{100} = \frac{97(x+2)}{100} \quad \text{\{as LCD} = 100\}$$

$$\therefore \quad 100x + 190 = 97(x+2) \quad \text{\{equating numerators\}}$$

$$\therefore \quad 100x + 190 = 97x + 194 \quad \text{\{expanding brackets\}}$$

$$\therefore \quad 100x - 97x = 194 - 190$$

$$\therefore \quad 3x = 4$$

$$\therefore \quad x = 1\tfrac{1}{3}$$

$\therefore$ $1\tfrac{1}{3}$ litres of water must be added.

EXERCISE 1I

1 How much water must be added to 1 litre of 5% cordial mixture to produce a 4% cordial mixture?

2 How much water must be added to 5 L of 8% weedkiller mixture to make a 5% weedkiller mixture?

3 How many litres of 3% cordial mixture must be added to 24 litres of 6% cordial mixture to make a 4% cordial mixture?

4 How many litres of 15% weedkiller mixture must be added to 5 litres of 10% weedkiller mixture to make a 12% weedkiller mixture?

REVIEW SET 1A

1 Write in algebraic form:
    a "3 more than the square of x"   **b** "the square of the sum of 3 and x"

2 Write, in words, the meaning of:
    a $\sqrt{a} + 3$ **b** $\sqrt{a+3}$

3 If $p = 1$, $q = -2$ and $r = -3$,   find the value of $\dfrac{4q - p}{r}$.

4 Solve for x:
    a $5 - 2x = 3x + 4$ **b** $\dfrac{x-1}{2} - \dfrac{2-3x}{7} = \dfrac{1}{3}$

5 Solve the following and graph the solutions: $5 - 2x \geqslant 3(x + 6)$

6 If a number is increased by 5 and then trebled, the result is six more than two thirds of the number. Find the number.

7 A drinks stall sells small, medium and large cups of fruit drink for €1.50, €2 and €2.50 respectively. In one morning, three times as many medium cups were sold as small cups, and the number of large cups sold was 140 less than the number of medium cups. If the total of the drink sales was €1360, how many of each size cup were sold?

8 Ray drives between towns A and B at an average speed of 75 km h^{-1}. Mahmoud drives the same distance at 80 km h^{-1} and takes 10 minutes less. What is the distance between A and B?

REVIEW SET 1B

1 Write, in words, the meaning of:

   a $\dfrac{a+b}{2}$ **b** $a + \dfrac{b}{2}$

2 Write in algebraic form:

   a "the sum of a and the square root of b"

   b "the square root of the sum of a and b"

3 If $a = -1$, $b = 4$ and $c = -6$, find the value of $\dfrac{2b+3c}{2a}$.

4 Solve the following inequation and graph the solution set: $5x + 2(3-x) < 8 - x$

5 Solve for x:

   a $2(x-3) + 4 = 5$ **b** $\dfrac{2x+3}{3} - \dfrac{3x+1}{4} = 2$

6 What number must be added to both the numerator and denominator of $\frac{3}{4}$ in order to finish with $\frac{1}{3}$?

7 X shares pay 8% dividend and Y shares pay 9%. Reiko has invested ¥200 000 more on X shares than she has on Y shares. Her total earnings for the year for the two investments is ¥271 000. How much did she invest in X shares?

8 Carlos cycles for 2 hours at a fast speed, then cycles for 3 more hours at a speed 10 km h^{-1} slower. If the entire distance travelled is 90 km, find Carlos' speed for the first two hours of travel.
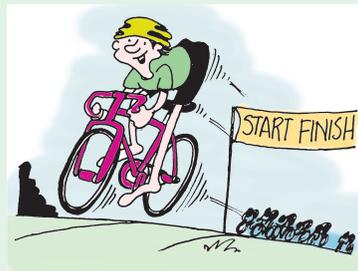
Chapter 2

Indices

Contents:

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices** or **exponents** to represent such expressions easily.

For example, the expression $3 \times 3 \times 3 \times 3$ can be represented as 3^4.

Indices have many applications in areas such as finance, engineering, physics, biology, electronics and computer science.

Problems encountered in these areas may involve situations where quantities increase or decrease over time. Such problems are often examples of **exponential growth** or **decay**.

OPENING PROBLEM

1 $3^3 = 27$, so the last digit of 3^3 is 7.

- What is the last digit of 3^{100}?
  Is there an easy way to find it?
  Why can't the answer be found using a calculator?

- What is the last digit of $2^{50} + 3^{50}$?

2 Which is larger: $(3^3)^3$ or $3^{(3^3)}$?
What is the largest number you can write using three 10s?

3 We know that $2^3 = 2 \times 2 \times 2 = 8$, but what do these numbers mean:

   a 2^0 **b** 2^{-1} **c** $2^{3.5}$?

After studying the concepts of this chapter, you should be able to answer the questions above.

A   INDEX NOTATION

To simplify the product $3 \times 3 \times 3 \times 3$, we can write 3^4.

The 3 is called the **base** of the expression. The 4 is called the **power** or **index** or **exponent**.

3^4 reads "three to the power of four" or "three with index four".

3^4 — power, index or exponent (4); base (3)

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \ldots \times a}_{n \text{ factors}}$$

Example 1   ◀) Self Tutor

Find the integer equal to: **a** 2^5 **b** $2^3 \times 5^2 \times 7$

a 2^5
     $= 2 \times 2 \times 2 \times 2 \times 2$
     $= 32$

b $2^3 \times 5^2 \times 7$
     $= 2 \times 2 \times 2 \times 5 \times 5 \times 7$
     $= 8 \times 25 \times 7$
     $= 1400$

EXERCISE 2A.1

1 Find the integer equal to:

    a 2^4 **b** 5^3 **c** 2^6 **d** 7^3

    e $2 \times 3^3 \times 5^2$ **f** $2^4 \times 3^2 \times 7$ **g** $3^3 \times 5^3 \times 11$ **h** $2^4 \times 5^2 \times 13$

2 By dividing continuously by the primes 2, 3, 5, 7, ..., write as a product of prime factors in index form:

    a 54   **b** 72   **c** 100   **d** 240

    e 1890   **f** 882   **g** 1089   **h** 3375

3 Copy and complete the values of these common powers. Try to remember them.

    a $2^1 = ...,$ $2^2 = ...,$ $2^3 = ...,$ $2^4 = ...,$ $2^5 = ...,$ $2^6 = ...$

    b $3^1 = ...,$ $3^2 = ...,$ $3^3 = ...,$ $3^4 = ...$

    c $5^1 = ...,$ $5^2 = ...,$ $5^3 = ...,$ $5^4 = ...$

    d $7^1 = ...,$ $7^2 = ...,$ $7^3 = ...$

4 The following numbers can be written as 2^n. Find n.

    a 16   **b** 128   **c** 512

5 The following numbers can be written as 5^n. Find n.

    a 125   **b** 625   **c** 15 625

6 By considering $2^1, 2^2, 2^3, 2^4, 2^5, ...$ and looking for a pattern, find the last digit of 2^{22}.

7 Answer the **Opening Problem** question **1** on page **52**.

NEGATIVE BASES

So far we have only considered **positive** bases raised to a power.

However, the base can also be negative. To indicate this we need to use brackets.

Notice that $(-2)^2 = -2 \times -2$ whereas $-2^2 = -(2^2)$

                         $= 4$ $= -1 \times 2 \times 2$

                                                          $= -4$

Consider the statements below:

    $(-1)^1 = -1$ $(-2)^1 = -2$

    $(-1)^2 = -1 \times -1 = 1$ $(-2)^2 = -2 \times -2 = 4$

    $(-1)^3 = -1 \times -1 \times -1 = -1$ $(-2)^3 = -2 \times -2 \times -2 = -8$

    $(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$ $(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the pattern above it can be seen that:

> - a **negative** base raised to an **odd** power is **negative**
> - a **negative** base raised to an **even** power is **positive**.

Example 2   ◀ᴺ **Self Tutor**

Evaluate:

a $(-3)^4$ b -3^4 c $(-3)^5$ d $-(-3)^5$

a $(-3)^4$ b -3^4 c $(-3)^5$ d $-(-3)^5$

= 81 $= -1 \times 3^4$ $= -243$ $= -1 \times (-3)^5$

                         $= -81$ $= -1 \times -243$

                                                                            $= 243$

> Notice the effect of the brackets.

EXERCISE 2A.2

1 Simplify:

   a $(-1)^3$ b $(-1)^4$ c $(-1)^{12}$ d $(-1)^{17}$
   e $(-1)^6$ f -1^6 g $-(-1)^6$ h $(-2)^3$
   i -2^3 j $-(-2)^3$ k $-(-5)^2$ l $-(-5)^3$

CALCULATOR USE

Different calculators have different keys for entering powers. However, they all perform the operation in a similar manner.

The power keys are: $\boxed{x^2}$ squares the number in the display.

                         $\boxed{\wedge}$ raises the number in the display to whatever power is required.
                                                On some calculators this key is $\boxed{y^x}$, $\boxed{a^x}$ or $\boxed{x^y}$.

Example 3   ◀ᴺ **Self Tutor**

Find, using your calculator:

a 4^7 b $(-3)^6$ c -11^4 d 2^{-2}

> Not all calculators will use these key sequences. If you have problems, refer to the calculator instructions on page **12**.

                                                                                   Answer

a Press: 4 $\boxed{\wedge}$ 7 $\boxed{\text{ENTER}}$ 16 384

b Press: $\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{)}$ $\boxed{\wedge}$ 6 $\boxed{\text{ENTER}}$ 729

c Press: $\boxed{(-)}$ 11 $\boxed{\wedge}$ 4 $\boxed{\text{ENTER}}$ −14 641

d Press: 2 $\boxed{\wedge}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$ 0.25

EXERCISE 2A.3

1 Use your calculator to find the value of the following, recording the entire display:

    a 2^9 **b** $(-5)^5$ **c** -3^5 **d** 7^5 **e** 8^3

    f $(-9)^4$ **g** -9^4 **h** 1.16^{11} **i** -0.981^{14} **j** $(-1.14)^{23}$

2 Use your calculator to find the values of the following:

    a 7^{-1} **b** $\dfrac{1}{7^1}$ **c** 3^{-2} **d** $\dfrac{1}{3^2}$

    e 4^{-3} **f** $\dfrac{1}{4^3}$ **g** 13^0 **h** 172^0

3 From your answers to question **2**, discuss what happens when a number is raised:

    a to a negative power   **b** to the power zero.

B   INDEX LAWS

In the previous exercise you should have developed **index laws** for zero and negative powers. The following is a more complete list of index laws:

If the bases a and b are both positive and the indices m and n are integers then:

$a^m \times a^n = a^{m+n}$ To **multiply** numbers with the **same base**, keep the base and **add** the indices.

$\dfrac{a^m}{a^n} = a^{m-n}$ To **divide** numbers with the same base, keep the base and **subtract** the indices.

$(a^m)^n = a^{mn}$ When **raising** a **power** to a **power**, keep the base and **multiply** the indices.

$(ab)^n = a^n b^n$ The power of a product is the product of the powers.

$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ The power of a quotient is the quotient of the powers.

$a^0 = 1, \quad a \neq 0$ Any non-zero number raised to the power of zero is 1.

$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \dfrac{1}{a}$.

Most of these laws can be demonstrated by simple examples:

- $3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) = 3^5$
- $\dfrac{3^5}{3^3} = \dfrac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3} = 3^2$
- $(3^2)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6$

Example 4   ◀》 **Self Tutor**

Simplify using $a^m \times a^n = a^{m+n}$:

  a $11^5 \times 11^3$ **b** $a^4 \times a^5$ **c** $x^4 \times x^a$

a $11^5 \times 11^3$	**b** $a^4 \times a^5$	**c** $x^4 \times x^a$
$= 11^{5+3}$	$= a^{4+5}$	$= x^{4+a}$
$= 11^8$	$= a^9$	$= x^{a+4}$

Example 5   ◀》 **Self Tutor**

Simplify using $\dfrac{a^m}{a^n} = a^{m-n}$:   **a** $\dfrac{7^8}{7^5}$ **b** $\dfrac{b^6}{b^m}$

a $\dfrac{7^8}{7^5}$	**b** $\dfrac{b^6}{b^m}$
$= 7^{8-5}$	$= b^{6-m}$
$= 7^3$	

Example 6   ◀》 **Self Tutor**

Simplify using $(a^m)^n = a^{mn}$:

  a $(2^4)^3$ **b** $(x^3)^5$ **c** $(b^7)^m$

a $(2^4)^3$	**b** $(x^3)^5$	**c** $(b^7)^m$
$= 2^{4 \times 3}$	$= x^{3 \times 5}$	$= b^{7 \times m}$
$= 2^{12}$	$= x^{15}$	$= b^{7m}$

EXERCISE 2B

1 Simplify using $a^m \times a^n = a^{m+n}$:

  a $7^3 \times 7^2$ **b** $5^4 \times 5^3$ **c** $a^7 \times a^2$ **d** $a^4 \times a$

  e $b^8 \times b^5$ **f** $a^3 \times a^n$ **g** $b^7 \times b^m$ **h** $m^4 \times m^2 \times m^3$

2 Simplify using $\dfrac{a^m}{a^n} = a^{m-n}$:

  a $\dfrac{5^9}{5^2}$ **b** $\dfrac{11^{13}}{11^9}$ **c** $7^7 \div 7^4$ **d** $\dfrac{a^6}{a^2}$

  e $\dfrac{b^{10}}{b^7}$ **f** $\dfrac{p^5}{p^m}$ **g** $\dfrac{y^a}{y^5}$ **h** $b^{2x} \div b$

3 Simplify using $(a^m)^n = a^{m \times n}$:

  a $(3^2)^4$ **b** $(5^3)^5$ **c** $(2^4)^7$ **d** $(a^5)^2$

  e $(p^4)^5$ **f** $(b^5)^n$ **g** $(x^y)^3$ **h** $(a^{2x})^5$

Example 7

Express in simplest form with a prime number base:

a 4^4 **b** 9×3^a **c** 49^{x+2}

a 4^4

$= (2^2)^4$

$= 2^{2 \times 4}$

$= 2^8$

b 9×3^a

$= 3^2 \times 3^a$

$= 3^{2+a}$

$= 3^{a+2}$

c 49^{x+2}

$= (7^2)^{x+2}$

$= 7^{2(x+2)}$

$= 7^{2x+4}$

4 Express in simplest form with a prime number base:

a 8	**b** 25	**c** 81	**d** 4^3
e 9^2	**f** $3^a \times 9$	**g** $5^t \div 5$	**h** $3^k \times 9^k$
i $\dfrac{16}{2^x}$	**j** $\dfrac{3^{x+1}}{3^{x-1}}$	**k** $(5^4)^{x-1}$	**l** $2^x \times 2^{2-x}$
m $\dfrac{2^y}{4^x}$	**n** $\dfrac{4^y}{8^x}$	**o** $\dfrac{3^{x+1}}{3^{1-x}}$	**p** $\dfrac{2^t \times 4^t}{8^{t-1}}$

Example 8

Remove the brackets of:

a $(3a)^2$ **b** $\left(\dfrac{2x}{y}\right)^3$

a $(3a)^2$

$= 3^2 \times a^2$

$= 9a^2$

b $\left(\dfrac{2x}{y}\right)^3$

$= \dfrac{2^3 \times x^3}{y^3}$

$= \dfrac{8x^3}{y^3}$

> Each factor within the brackets has to be raised to the power outside them.

5 Remove the brackets of:

a $(ab)^3$	**b** $(ac)^4$	**c** $(bc)^5$	**d** $(abc)^3$
e $(2a)^4$	**f** $(5b)^2$	**g** $(3n)^4$	**h** $(2bc)^3$
i $(4ab)^3$	**j** $\left(\dfrac{a}{b}\right)^3$	**k** $\left(\dfrac{m}{n}\right)^4$	**l** $\left(\dfrac{2c}{d}\right)^5$

Example 9

Express the following in simplest form, without brackets:

a $(3a^3b)^4$ **b** $\left(\dfrac{x^2}{2y}\right)^3$

$$a \quad (3a^3b)^4$$
$$= 3^4 \times (a^3)^4 \times b^4$$
$$= 81 \times a^{3 \times 4} \times b^4$$
$$= 81a^{12}b^4$$

$$b \quad \left(\frac{x^2}{2y}\right)^3$$
$$= \frac{(x^2)^3}{2^3 \times y^3}$$
$$= \frac{x^{2 \times 3}}{8 \times y^3}$$
$$= \frac{x^6}{8y^3}$$

6 Express the following in simplest form, without brackets:

a $(2b^4)^3$

b $\left(\dfrac{3}{x^2y}\right)^2$

c $(5a^4b)^2$

d $\left(\dfrac{m^3}{2n^2}\right)^4$

e $\left(\dfrac{3a^3}{b^5}\right)^3$

f $(2m^3n^2)^5$

g $\left(\dfrac{4a^4}{b^2}\right)^2$

h $(5x^2y^3)^3$

Example 10   ◀) Self Tutor

Simplify using the index laws:

a $3x^2 \times 5x^5$

b $\dfrac{20a^9}{4a^6}$

c $\dfrac{b^3 \times b^7}{(b^2)^4}$

a $3x^2 \times 5x^5$
$$= 3 \times 5 \times x^2 \times x^5$$
$$= 15 \times x^{2+5}$$
$$= 15x^7$$

b $\dfrac{20a^9}{4a^6}$
$$= \tfrac{20}{4} \times a^{9-6}$$
$$= 5a^3$$

c $\dfrac{b^3 \times b^7}{(b^2)^4}$
$$= \frac{b^{3+7}}{b^{2 \times 4}}$$
$$= \frac{b^{10}}{b^8}$$
$$= b^{10-8}$$
$$= b^2$$

---

7 Simplify the following expressions using one or more of the index laws:

a $\dfrac{a^3}{a}$

b $4b^2 \times 2b^3$

c $\dfrac{m^5n^4}{m^2n^3}$

d $\dfrac{14a^7}{2a^2}$

e $\dfrac{12a^2b^3}{3ab}$

f $\dfrac{18m^7a^3}{4m^4a^3}$

g $10hk^3 \times 4h^4$

h $\dfrac{m^{11}}{(m^2)^8}$

i $\dfrac{p^2 \times p^7}{(p^3)^2}$

Example 11   ◀) **Self Tutor**

Simplify, giving answers in simplest rational form:

  a 3^0 **b** 2^{-2} **c** $5^0 - 5^{-1}$ **d** $\left(\frac{2}{5}\right)^{-2}$

> Notice that
> $$\left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2$$

a $3^0 = 1$ **b** $\quad 2^{-2}$ **c** $\quad 5^0 - 5^{-1}$ **d** $\quad \left(\frac{2}{5}\right)^{-2}$

$$= \frac{1}{2^2} \qquad\qquad = 1 - \tfrac{1}{5} \qquad\qquad = \left(\tfrac{5}{2}\right)^2$$
$$= \tfrac{1}{4} \qquad\qquad\quad = \tfrac{4}{5} \qquad\qquad\quad = \tfrac{25}{4}$$
$$= 6\tfrac{1}{4}$$

8   Simplify, giving answers in simplest rational form:

  a 5^0 **b** 3^{-1} **c** 6^{-1} **d** 8^0

  e 3^2 **f** 3^{-2} **g** 2^3 **h** 2^{-3}

  i 5^2 **j** 5^{-2} **k** 10^2 **l** 10^{-2}

9   Simplify, giving answers in simplest rational form:

  a $\left(\frac{2}{3}\right)^0$ **b** $\dfrac{4^3}{4^3}$ **c** $3y^0$ **d** $(3y)^0$

  e 2×3^0 **f** 6^0 **g** $\dfrac{5^2}{5^4}$ **h** $\dfrac{2^{10}}{2^{15}}$

  i $\left(\frac{1}{3}\right)^{-1}$ **j** $\left(\frac{2}{5}\right)^{-1}$ **k** $\left(\frac{4}{3}\right)^{-1}$ **l** $\left(\frac{1}{12}\right)^{-1}$

  m $\left(\frac{2}{3}\right)^{-1}$ **n** $5^0 - 5^{-1}$ **o** $7^{-1} + 7^0$ **p** $2^0 + 2^1 + 2^{-1}$

  q $\left(\frac{3}{4}\right)^{-2}$ **r** $\left(1\frac{1}{2}\right)^{-3}$ **s** $\left(\frac{5}{2}\right)^{-2}$ **t** $\left(2\frac{1}{3}\right)^{-2}$

Example 12   ◀) **Self Tutor**

Write the following without brackets or negative indices:

  a $(5x)^{-1}$ **b** $5x^{-1}$ **c** $(3b^2)^{-2}$

a $\quad (5x)^{-1}$ **b** $\quad 5x^{-1}$ **c** $\quad (3b^2)^{-2}$

$$= \frac{1}{5x} \qquad\qquad = \frac{5}{x} \qquad\qquad = \frac{1}{(3b^2)^2}$$
$$= \frac{1}{3^2 b^4}$$
$$= \frac{1}{9b^4}$$

> In $5x^{-1}$ the index -1 refers to the x only.

10   Write the following without brackets or negative indices:

  a $(2a)^{-1}$ **b** $2a^{-1}$ **c** $3b^{-1}$ **d** $(3b)^{-1}$

  e $\left(\frac{2}{b}\right)^{-2}$ **f** $(2b)^{-2}$ **g** $(3n)^{-2}$ **h** $(3n^{-2})^{-1}$

    i ab^{-1} **j** $(ab)^{-1}$ **k** ab^{-2} **l** $(ab)^{-2}$

    m $(2ab)^{-1}$ **n** $2(ab)^{-1}$ **o** $2ab^{-1}$ **p** $\dfrac{(ab)^2}{b^{-1}}$

11 In **Chapter 1** we saw that kilometres per hour could be written as km/h or $\text{km}\,\text{h}^{-1}$. Write these units in index form:

    a m/s   **b** cubic metres/hour   **c** square centimetres per second

    d cubic centimetres per minute   **e** grams per second

    f metres per second per second

Example 13　　　　　　　　　　　　　　　　　◀) **Self Tutor**

Write the following as powers of 2, 3 or 5:

    a $\dfrac{1}{8}$ **b** $\dfrac{1}{9^n}$ **c** $\dfrac{25}{5^4}$

  a $\dfrac{1}{8}$

    $= \dfrac{1}{2^3}$

    $= 2^{-3}$

  b $\dfrac{1}{9^n}$

    $= \dfrac{1}{(3^2)^n}$

    $= \dfrac{1}{3^{2n}}$

    $= 3^{-2n}$

  c $\dfrac{25}{5^4}$

    $= \dfrac{5^2}{5^4}$

    $= 5^{2-4}$

    $= 5^{-2}$

---

12 Write the following as powers of 2, 3 or 5:

    a $\dfrac{1}{3}$ **b** $\dfrac{1}{2}$ **c** $\dfrac{1}{5}$ **d** $\dfrac{1}{4}$

    e $\dfrac{1}{27}$ **f** $\dfrac{1}{25}$ **g** $\dfrac{1}{8^x}$ **h** $\dfrac{1}{16^y}$

    i $\dfrac{1}{81^a}$ **j** $\dfrac{9}{3^4}$ **k** 25×5^{-4} **l** $\dfrac{5^{-1}}{5^2}$

    m $2 \div 2^{-3}$ **n** 1 **o** 6^{-3} **p** 4×10^2

13 The water lily *Nymphaea Mathematicus* doubles its size every day. From the time it was planted until it completely covered a pond, it took 26 days.

How many days did it take to cover half the pond?

14 Suppose you have the following six coins in your pocket: 5 cents, 10 cents, 20 cents, 50 cents, $1, $2. How many different sums of money can you make?

Hint: Simplify the problem to a smaller number of coins and look for a pattern.

C EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $3^x = 9$ and $3^2 \times 4^x = 8$ are both exponential equations.

Notice that if $3^x = 9$ then $3^x = 3^2$. Thus $x = 2$ is a solution to the exponential equation $3^x = 9$, and it is in fact the only solution to the equation.

In general:

> If $a^x = a^k$ then $x = k$.
>
> If the base numbers are the same, we can **equate indices**.

Example 14 ◀) **Self Tutor**

Solve for x: **a** $3^x = 27$ **b** $2^{x+1} = \frac{1}{8}$

a $\qquad 3^x = 27$

$\therefore \quad 3^x = 3^3$

$\therefore \quad x = 3$

b $\qquad 2^{x+1} = \frac{1}{8}$

$\therefore \quad 2^{x+1} = \frac{1}{2^3}$

$\therefore \quad 2^{x+1} = 2^{-3}$

$\therefore \quad x + 1 = -3$

$\therefore \quad x = -4$

Once we have the same base we then equate the indices.

Example 15 ◀) **Self Tutor**

Solve for x: **a** $4^x = \frac{1}{2}$ **b** $25^{x+1} = 5$

a $\qquad 4^x = \frac{1}{2}$

$\therefore \quad (2^2)^x = \frac{1}{2^1}$

$\therefore \quad 2^{2x} = 2^{-1}$

$\therefore \quad 2x = -1$

$\therefore \quad x = -\frac{1}{2}$

b $\qquad 25^{x+1} = 5$

$\therefore \quad (5^2)^{x+1} = 5^1$

$\therefore \quad 5^{2(x+1)} = 5^1$

$\therefore \quad 2x + 2 = 1$

$\therefore \quad 2x = -1$

$\therefore \quad x = -\frac{1}{2}$

Remember to use the index laws correctly!

EXERCISE 2C

1 Solve for x:

a $2^x = 2$	**b** $2^x = 4$	**c** $3^x = 27$	**d** $2^x = 1$
e $2^x = \frac{1}{2}$	**f** $3^x = \frac{1}{3}$	**g** $2^x = \frac{1}{8}$	**h** $2^{x+1} = 8$
i $2^{x-2} = \frac{1}{4}$	**j** $3^{x+1} = \frac{1}{27}$	**k** $2^{x+1} = 64$	**l** $2^{1-2x} = \frac{1}{2}$

2 Solve for x:

a $4^x = 32$ **b** $8^x = \frac{1}{4}$ **c** $27^x = \frac{1}{9}$ **d** $49^x = \frac{1}{7}$

e $4^x = \frac{1}{8}$ **f** $25^x = \frac{1}{5}$ **g** $8^{x+2} = 32$ **h** $8^{1-x} = \frac{1}{4}$

i $4^{x-1} = \frac{1}{16}$ **j** $9^{x-2} = 3$ **k** $(\frac{1}{2})^{x+1} = 2$ **l** $(\frac{1}{3})^{x+2} = 9$

m $4^{2x} = 8^{-x}$ **n** $(\frac{1}{4})^{1+x} = 8$ **o** $(\frac{1}{49})^x = 7$ **p** $(\frac{1}{8})^{x+1} = 32$

HISTORICAL NOTE

350 years ago the French mathematician **Pierre de Fermat** wrote the following in the margin of his copy of Diophantus' *Arithmetica*:

"*To resolve the cube into the sum of two cubes, a fourth power into two fourth powers or in general any power higher than the second into two of the same kind, is impossible; of which fact I have found a remarkable proof. The margin is too small to contain it.*"

This text is known as **Fermat's Last Theorem**. Using algebra, it can be written as:

"*For all whole numbers $n > 2$ there are no positive integers x, y, z such that $x^n + y^n = z^n$.*"

If Fermat did in fact have such a remarkable proof, it has never been found. Many mathematical minds have worked on the problem since. In fact, the Academy of Science in Gottingen offered a 100 000 DM prize in 1908 for the solution to the theorem and it has never been claimed.

In 1880, **Sophie Germain** provided a partial proof of Fermat's Last Theorem. She proved that if x, y and z are integers and $x^5 + y^5 = z^5$, then either x, y or z must be divisible by 5.

In 1993, Professor **Andrew Wiles** of Princeton University presented a proof of Fermat's Last Theorem at Cambridge, but it was found to be incomplete. He amended his proof in 1994 and after much scrutiny by mathematicians, it is now generally accepted as the "theorem of the twentieth century".

Millions of hours have been spent during the last 350 years by mathematicians, both amateur and professional, trying to prove or disprove Fermat's Last Theorem. Even though all but one were unsuccessful, thousands of new ideas and discoveries were made in the process. Some would say that Fermat's Last Theorem is really quite useless, but the struggle for a proof has been extremely worthwhile.
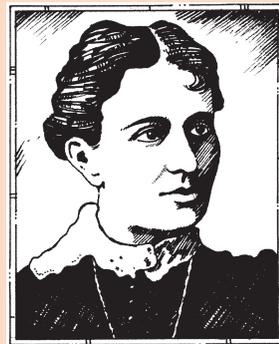
D   SCIENTIFIC NOTATION (STANDARD FORM)

Observe the pattern:

$$10\,000 = 10^4$$
$$1000 = 10^3$$
$$100 = 10^2$$
$$10 = 10^1$$
$$1 = 10^0$$
$$\tfrac{1}{10} = 10^{-1}$$
$$\tfrac{1}{100} = 10^{-2}$$
$$\tfrac{1}{1000} = 10^{-3}$$

As we divide by 10, the **exponent** or **power** of 10 decreases by one.

We can use this pattern to simplify the writing of very large and very small numbers.

For example,

$$5\,000\,000$$
$$= 5 \times 1\,000\,000$$
$$= 5 \times 10^6$$

and

$$0.000\,003$$
$$= \frac{3}{1\,000\,000}$$
$$= \frac{3}{1} \times \frac{1}{1\,000\,000}$$
$$= 3 \times 10^{-6}$$

SCIENTIFIC NOTATION

Scientific notation or **standard form** involves writing any given number as *a number between* 1 *and* 10, multiplied by a *power of* 10,

i.e., $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer.

Example 16   ◀)) Self Tutor

Write in scientific notation:

a 23 600 000

b 0.000 023 6

a   23 600 000
$$= 2.36 \times 10^7$$

b   0.000 023 6
$$= 2.36 \times 10^{-5}$$

Example 17 ◀))) **Self Tutor**

Write as an ordinary decimal number:

a 2.57×10^4 **b** 7.853×10^{-3}

a 2.57×10^4
$= 2.5700 \times 10\,000$
$= 25\,700$

b 7.853×10^{-3}
$= 0\,007.853 \div 10^3$
$= 0.007\,853$

Remember that
$10^{-3} = \frac{1}{10^3}$

EXERCISE 2D

1 Write using scientific notation:

a 230   **b** 53 900   **c** 0.0361   **d** 0.006 80

e 3.26   **f** 0.5821   **g** 361 000 000   **h** 0.000 001 674

2 Write as an ordinary decimal number:

a 2.3×10^3 **b** 2.3×10^{-2} **c** 5.64×10^5 **d** 7.931×10^{-4}

e 9.97×10^0 **f** 6.04×10^7 **g** 4.215×10^{-1} **h** 3.621×10^{-8}

3 Express the following quantities using scientific notation:

a There are approximately 4 million red blood cells in a drop of blood.

b The thickness of a coin is about 0.0008 m.

c The earth's radius is about 6.38 million metres.

d A typical human cell has a diameter of about 0.000 02 m.

4 Express the following quantities as ordinary decimal numbers:

a The sun has diameter 1.392×10^6 km.

b A piece of paper is about 1.8×10^{-2} cm thick.

c A test tube holds 3.2×10^7 bacteria.

d A mushroom weighs 8.2×10^{-6} tonnes.

e The number of minutes in an average person's life is around 3.7×10^7.

f A blood capillary is about 2.1×10^{-5} m in diameter.

Example 18 ◀))) **Self Tutor**

Simplify, writing your answer in scientific notation:

a $(3 \times 10^4) \times (8 \times 10^3)$ **b** $(5 \times 10^{-4})^3$

a $(3 \times 10^4) \times (8 \times 10^3)$
$= 24 \times 10^{4+3}$
$= 2.4 \times 10^1 \times 10^7$
$= 2.4 \times 10^8$

b $(5 \times 10^{-4})^3$
$= 5^3 \times (10^{-4})^3$
$= 125 \times 10^{-4 \times 3}$
$= 1.25 \times 10^2 \times 10^{-12}$
$= 1.25 \times 10^{-10}$

5 Simplify the following products, writing your answers in scientific notation:

   a $(3 \times 10^3) \times (4 \times 10^7)$

   b $(4 \times 10^3) \times (7 \times 10^5)$

   c $(8 \times 10^{-4}) \times (7 \times 10^{-5})$

   d $(9 \times 10^{-5}) \times (6 \times 10^{-2})$

   e $(3 \times 10^5)^2$

   f $(4 \times 10^7)^2$

   g $(2 \times 10^{-3})^4$

   h $(8 \times 10^{-3})^3$

   i $(6 \times 10^{-1}) \times (4 \times 10^3) \times (5 \times 10^{-4})$

   j $(5 \times 10^{-3})^2 \times (8 \times 10^{11})$

Example 19 🔊 **Self Tutor**

Simplify, writing your answer in scientific notation:

   a $\dfrac{8 \times 10^4}{2 \times 10^{-3}}$

   b $\dfrac{2 \times 10^{-3}}{5 \times 10^{-8}}$

   a $\dfrac{8 \times 10^4}{2 \times 10^{-3}}$

   $= \frac{8}{2} \times 10^{4-(-3)}$

   $= 4 \times 10^7$

   b $\dfrac{2 \times 10^{-3}}{5 \times 10^{-8}}$

   $= \frac{2}{5} \times 10^{-3-(-8)}$

   $= 0.4 \times 10^5$

   $= 4 \times 10^{-1} \times 10^5$

   $= 4 \times 10^4$

---

6 Simplify the following divisions, writing your answers in scientific notation:

   a $\dfrac{8 \times 10^6}{4 \times 10^3}$

   b $\dfrac{9 \times 10^{-3}}{3 \times 10^{-1}}$

   c $\dfrac{4 \times 10^6}{2 \times 10^{-2}}$

   d $\dfrac{2.5 \times 10^{-4}}{(5 \times 10^7)^2}$

   e $\dfrac{(8 \times 10^{-2})^2}{2 \times 10^{-6}}$

   f $\dfrac{(5 \times 10^{-3})^{-2}}{(2 \times 10^4)^{-1}}$

7 Use a calculator to find, correct to 3 significant figures:

   a $(4.7 \times 10^5) \times (8.5 \times 10^7)$

   b $(2.7 \times 10^{-3}) \times (9.6 \times 10^9)$

   c $(3.4 \times 10^7) \div (4.8 \times 10^{15})$

   d $(7.3 \times 10^{-7}) \div (1.5 \times 10^4)$

   e $(2.83 \times 10^3)^2$

   f $(5.96 \times 10^{-5})^2$

8 If $M = \dfrac{ab}{c}$, find M when $a = 3.12 \times 10^4$, $b = 5.69 \times 10^{11}$ and $c = 8.29 \times 10^{-7}$.

9 If $G = \dfrac{(p+q)^2}{r^3}$, find G when $p = 5.17 \times 10^{-3}$, $q = 6.89 \times 10^{-4}$ and $r = 4.73 \times 10^{-5}$.

10 **a** How many times larger is 3×10^{11} than 3×10^8?

   b Which is the smaller of 5×10^{-16} and 5×10^{-21}, and by how many times?

11 Use your calculator to answer the following:

a A rocket travels in space at 4×10^5 km h^{-1}. How far will it travel in:

i 30 days   **ii** 20 years?   (Assume 1 year $= 365.25$ days)

b A bullet travelling at an average speed of 2×10^3 km h^{-1} hits a target 500 m away. Find the time of the bullet's flight in seconds.

c Mars has volume 1.31×10^{21} m^3 whereas Pluto has volume 4.93×10^{19} m^3.
How many times bigger is Mars than Pluto?

d Microbe C has mass 2.63×10^{-5} grams whereas microbe D has mass 8×10^{-7} grams. Which microbe is heavier? How many times is it heavier than the other one?

E   RATIONAL (FRACTIONAL) INDICES

The index laws we saw earlier in the chapter can also be applied to rational indices, or indices which are written as a fraction.

INVESTIGATION   RATIONAL (FRACTIONAL) INDICES

This investigation will help you discover the meaning of numbers raised to rational indices.

Remember that
$a^m \times a^n = a^{m+n}$
and $(a^m)^n = a^{m \times n}$.

What to do:

1 Notice that $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$ and $\sqrt{5} \times \sqrt{5} = 5$.
Copy and complete the following:

a $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = ... = ... = ...$ **b** $\sqrt{3} \times \sqrt{3} = ...$

c $13^{\frac{1}{2}} \times 13^{\frac{1}{2}} = ... = ... = ...$ **d** $\sqrt{13} \times \sqrt{13} = ...$

2 Notice that $(7^{\frac{1}{3}})^3 = 7^{\frac{1}{3} \times 3} = 7^1 = 7$ and $(\sqrt[3]{7})^3 = 7$.
Copy and complete the following:

a $(8^{\frac{1}{3}})^3 = ... = ... = ...$ **b** $(\sqrt[3]{8})^3 = ...$

c $(27^{\frac{1}{3}})^3 = ... = ... = ...$ **d** $(\sqrt[3]{27})^3 = ...$

$\sqrt[3]{7}$ is read as "the cube root of 7".

3 Suggest a rule for:

a $a^{\frac{1}{2}} = ...$ **b** $a^{\frac{1}{3}} = ...$ **c** the general case: $a^{\frac{1}{n}} = ...$

From the investigation, we can conclude that $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$.

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$ where $\sqrt[n]{a}$ is called the 'nth root of a'.

Example 20   ◄� **Self Tutor**

Simplify: **a** $16^{\frac{1}{2}}$ **b** $8^{\frac{1}{3}}$ **c** $16^{-\frac{1}{2}}$ **d** $8^{-\frac{1}{3}}$

a $16^{\frac{1}{2}}$
$= \sqrt{16}$
$= 4$

b $8^{\frac{1}{3}}$
$= \sqrt[3]{8}$
$= 2$

c $16^{-\frac{1}{2}}$
$= \dfrac{1}{16^{\frac{1}{2}}}$
$= \dfrac{1}{\sqrt{16}}$
$= \frac{1}{4}$

d $8^{-\frac{1}{3}}$
$= \dfrac{1}{8^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{8}}$
$= \frac{1}{2}$

Example 21   ◄ঠ **Self Tutor**

Write each of the following in index form:

a $\sqrt{3}$ **b** $\sqrt[3]{7}$ **c** $\dfrac{1}{\sqrt[4]{7}}$

a $\sqrt{3}$
$= 3^{\frac{1}{2}}$

b $\sqrt[3]{7}$
$= 7^{\frac{1}{3}}$

c $\dfrac{1}{\sqrt[4]{7}}$
$= \dfrac{1}{7^{\frac{1}{4}}}$
$= 7^{-\frac{1}{4}}$

Example 22   ◄ঠ **Self Tutor**

Write the following as powers of 2:   **a** $\sqrt[3]{4}$ **b** $\dfrac{1}{\sqrt{8}}$

a $\sqrt[3]{4}$
$= (2^2)^{\frac{1}{3}}$
$= 2^{2 \times \frac{1}{3}}$
$= 2^{\frac{2}{3}}$

b $\dfrac{1}{\sqrt{8}}$
$= 8^{-\frac{1}{2}}$ $\{$as $\dfrac{1}{\sqrt{a}} = a^{-\frac{1}{2}}\}$
$= (2^3)^{-\frac{1}{2}}$
$= 2^{-\frac{3}{2}}$

Remember that
$(a^m)^n = a^{m \times n}$

EXERCISE 2E

1 Evaluate the following without using a calculator:

a $4^{\frac{1}{2}}$ **b** $4^{-\frac{1}{2}}$ **c** $9^{\frac{1}{2}}$ **d** $9^{-\frac{1}{2}}$

e $36^{\frac{1}{2}}$ **f** $36^{-\frac{1}{2}}$ **g** $8^{\frac{1}{3}}$ **h** $8^{-\frac{1}{3}}$

i $1000^{\frac{1}{3}}$ **j** $1000^{-\frac{1}{3}}$ **k** $27^{\frac{1}{3}}$ **l** $27^{-\frac{1}{3}}$

2 Write each of the following in index form:

a $\sqrt{11}$ **b** $\dfrac{1}{\sqrt{11}}$ **c** $\sqrt{12}$ **d** $\dfrac{1}{\sqrt{12}}$

e $\sqrt[3]{26}$ **f** $\dfrac{1}{\sqrt[3]{26}}$ **g** $\sqrt[4]{7}$ **h** $\dfrac{1}{\sqrt[5]{7}}$

3 Write the following as powers of 2:

a $\sqrt[4]{2}$ **b** $\sqrt[5]{2}$ **c** $\sqrt[4]{4}$ **d** $\sqrt[3]{16}$

e $\dfrac{1}{\sqrt[4]{8}}$ **f** $\dfrac{1}{\sqrt[3]{16}}$ **g** $\dfrac{1}{\sqrt[7]{8}}$ **h** $\dfrac{1}{\sqrt[5]{64}}$

4 Write the following as powers of 3:

a $\sqrt[3]{3}$ **b** $\sqrt[4]{27}$ **c** $\dfrac{1}{\sqrt[4]{3}}$ **d** $\dfrac{1}{\sqrt[3]{9}}$

Example 23 ◀)) **Self Tutor**

Evaluate without using a calculator: **a** $8^{\frac{4}{3}}$ **b** $32^{-\frac{2}{5}}$

a $8^{\frac{4}{3}}$

$= (2^3)^{\frac{4}{3}}$

$= 2^{3 \times \frac{4}{3}}$

$= 2^4$

$= 16$

b $32^{-\frac{2}{5}}$

$= (2^5)^{-\frac{2}{5}}$

$= 2^{5 \times -\frac{2}{5}}$

$= 2^{-2}$

$= \frac{1}{4}$

> The first step is to write the base number in power form.

---

5 Evaluate without using a calculator:

a $4^{\frac{3}{2}}$ **b** $4^{\frac{5}{2}}$ **c** $8^{\frac{2}{3}}$ **d** $8^{\frac{5}{3}}$

e $16^{\frac{3}{4}}$ **f** $9^{\frac{3}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $4^{-\frac{1}{2}}$

i $16^{-\frac{3}{4}}$ **j** $8^{-\frac{2}{3}}$ **k** $27^{-\frac{4}{3}}$ **l** $32^{-0.6}$

Example 24

◀) **Self Tutor**

Evaluate to 3 significant figures where appropriate: **a** $8^{\frac{2}{3}}$ **b** $10^{-\frac{2}{3}}$

Answer

a Press: 8 $\boxed{\wedge}$ $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\text{ENTER}}$ $\qquad$ 4

b Press: 10 $\boxed{\wedge}$ $\boxed{(}$ $\boxed{(-)}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\text{ENTER}}$ ≈ 0.215

6 Use your calculator to evaluate, correct to 3 significant figures, where necessary:

a $4^{\frac{3}{2}}$ **b** $27^{\frac{2}{3}}$ **c** $8^{\frac{4}{3}}$ **d** $9^{\frac{2}{5}}$ **e** $10^{\frac{3}{7}}$

f $15^{\frac{5}{3}}$ **g** $10^{\frac{2}{7}}$ **h** $18^{\frac{7}{3}}$ **i** $16^{\frac{3}{11}}$ **j** $146^{\frac{4}{9}}$

k $4^{-\frac{5}{2}}$ **l** $27^{-\frac{5}{3}}$ **m** $15^{-\frac{2}{5}}$ **n** $53^{-\frac{3}{7}}$ **o** $3^{-\frac{7}{5}}$

CHESS BOARD CALCULATIONS

LINKS
click here

Areas of interaction:
Approaches to learning/Human ingenuity

REVIEW SET 2A

1 Simplify:

a $-(-1)^{10}$ **b** $-(-3)^3$ **c** $3^0 - 3^{-1}$

2 Simplify using the index laws:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\dfrac{5(x^2y)^2}{(5x^2)^2}$

3 Write the following as powers of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

4 Write without brackets or negative indices:

a b^{-3} **b** $(ab)^{-1}$ **c** ab^{-1}

5 Evaluate, giving answers in scientific notation:

a $(2 \times 10^5)^3$ **b** $(3 \times 10^8) \div (4 \times 10^{-5})$

6 Find the value of x, without using your calculator:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$

7 Evaluate without using a calculator:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

8   Evaluate, correct to 3 significant figures, using your calculator:

   a $3^{\frac{3}{4}}$
   b $27^{-\frac{1}{5}}$
   c $\sqrt[4]{100}$

9   How many times smaller is 8×10^7 than 8×10^9?

10 If $N = \dfrac{ab^3}{c^2}$,   find the value of N when $a = 2.39 \times 10^{-11}$, $b = 8.97 \times 10^5$

   and $c = 1.09 \times 10^{-3}$.   Give your answer correct to 3 significant figures.

REVIEW SET 2B

1   Simplify:

   a $-(-2)^3$
   b $5^{-1} - 5^0$

2   Simplify using the index laws:

   a $(a^7)^3$
   b $pq^2 \times p^3 q^4$
   c $\dfrac{8ab^5}{2a^4b^4}$

3   Write as powers of 2:

   a $\frac{1}{16}$
   b $2^x \times 4$
   c $4^x \div 8$

4   Write without brackets or negative indices:

   a $x^{-2} \times x^{-3}$
   b $2(ab)^{-2}$
   c $2ab^{-2}$

5   Solve for x without using a calculator:

   a $2^{x+1} = 32$
   b $4^{x+1} = \left(\frac{1}{8}\right)^x$

6   Evaluate, giving answers in scientific notation:

   a $(3 \times 10^9) \div (5 \times 10^{-2})$
   b $(9 \times 10^{-3})^2$

7   Evaluate without using a calculator:

   a $16^{\frac{3}{4}}$
   b $25^{-\frac{3}{2}}$

8   Use your calculator to evaluate, correct to 3 significant figures:

   a $4^{\frac{2}{3}}$
   b $20^{-\frac{1}{2}}$
   c $\sqrt[3]{30}$

9   How many times larger is 3×10^{-14} than 3×10^{-20}?

10 If $K = \dfrac{m^2 n}{p^3}$,   find K when $m = 5.62 \times 10^{11}$, $n = 7.97 \times 10^{-9}$

   and $p = 8.44 \times 10^{-4}$.   Give your answer correct to 3 significant figures.

Chapter 3

Algebraic expansion and simplification

Contents:

The study of **algebra** is an important part of the problem solving process. When we convert real life problems into algebraic equations, we often obtain expressions that need to be **expanded** and **simplified**.

OPENING PROBLEM

Ethel is planning a rectangular flower bed with a lawn of constant width around it. The lawn's outer boundary is also rectangular. The shorter side of the flower bed is x m long.

x m
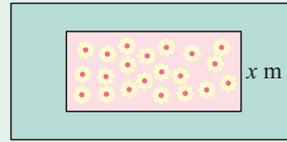
- If the flower bed's length is 4 m longer than its width, what is its width?
- If the width of the lawn's outer boundary is double the width of the flower bed, what are the dimensions of the flower bed?
- Wooden strips form the boundaries of the flower bed and lawn. Find, in terms of x, the total length L of wood required.

A   COLLECTING LIKE TERMS

In algebra,

like terms are terms which contain the same variables (or letters) to the same indices.

For example:

- xy and $-2xy$ are **like terms**.
- x^2 and $3x$ are **unlike terms** because the powers of x are not the same.

Algebraic expressions can often be simplified by adding or subtracting like terms. We call this **collecting like terms**.

Consider $2a + 3a = \underbrace{a + a}_{\text{2 lots of } a} + \underbrace{a + a + a}_{\text{3 lots of } a} = \underbrace{5a}_{\text{5 lots of } a}$

Example 1   ◄)) Self Tutor

Where possible, simplify by collecting the terms:

a $4x + 3x$ **b** $5y - 2y$ **c** $2a - 1 + a$ **d** $mn - 2mn$ **e** $a^2 - 4a$

a $4x + 3x = 7x$

b $5y - 2y = 3y$

c $2a - 1 + a = 3a - 1$ {since $2a$ and a are like terms}

d $mn - 2mn = -mn$ {since mn and $-2mn$ are like terms}

e $a^2 - 4a$ cannot be simplified since a^2 and $-4a$ are unlike terms.

Example 2 ◀)) **Self Tutor**

Simplify by collecting like terms:

a $-a - 1 + 3a + 4$

b $5a - b^2 + 2a - 3b^2$

a $-a - 1 + 3a + 4$

$= -a + 3a - 1 + 4$

$= 2a + 3$

$\{-a$ and $3a$ are like terms

-1 and 4 are like terms$\}$

b $5a - b^2 + 2a - 3b^2$

$= 5a + 2a - b^2 - 3b^2$

$= 7a - 4b^2$

$\{5a$ and $2a$ are like terms

$-b^2$ and $-3b^2$ are like terms$\}$

EXERCISE 3A

1 Simplify, where possible, by collecting like terms:

a $5 + a + 4$ **b** $6 + 3 + a$ **c** $m - 2 + 5$ **d** $x + 1 + x$

e $f + f - 3$ **f** $5a + a$ **g** $5a - a$ **h** $a - 5a$

i $x^2 + 2x$ **j** $d^2 + d^2 + d$ **k** $5g + 5$ **l** $x^2 - 5x^2 + 5$

m $2a + 3a - 5$ **n** $2a + 3a - a$ **o** $4xy + xy$ **p** $3x^2z - x^2z$

2 Simplify, where possible:

a $7a - 7a$ **b** $7a - a$ **c** $7a - 7$

d $xy + 2yx$ **e** $cd - 2cd$ **f** $4p^2 - p^2$

g $x + 3 + 2x + 4$ **h** $2 + a + 3a - 4$ **i** $2y - x + 3y + 3x$

j $3m^2 + 2m - m^2 - m$ **k** $ab + 4 - 3 + 2ab$ **l** $x^2 + 2x - x^2 - 5$

m $x^2 + 5x + 2x^2 - 3x$ **n** $ab + b + a + 4$ **o** $2x^2 - 3x - x^2 - 7x$

3 Simplify, where possible:

a $4x + 6 - x - 2$ **b** $2c + d - 2cd$ **c** $3ab - 2ab + ba$

d $x^2 + 2x^2 + 2x^2 - 5$ **e** $p^2 - 6 + 2p^2 - 1$ **f** $3a + 7 - 2a - 10$

g $-3a + 2b - a - b$ **h** $a^2 + 2a - a^3$ **i** $2a^2 - a^3 - a^2 + 2a^3$

j $4xy - x - y$ **k** $xy^2 + x^2y + x^2y$ **l** $4x^3 - 2x^2 - x^3 - x^2$

B   PRODUCT NOTATION

In algebra we agree:

- to **leave out** the "$\times$" **signs** between any multiplied quantities provided that at least one of them is an unknown (letter)
- to write **numerals (numbers) first** in any product
- where products contain two or more letters, we write them in **alphabetical order**.

For example:
- $2a$ is used rather than $2 \times a$ or $a2$
- $2ab$ is used rather than $2ba$.

ALGEBRAIC PRODUCTS

The **product** of two or more factors is the result obtained by multiplying them together.

Consider the factors $-3x$ and $2x^2$. Their product $-3x \times 2x^2$ can be simplified by following the steps below:

Step 1: Find the product of the **signs**.

Step 2: Find the product of the **numerals** or numbers.

Step 3: Find the product of the **variables** or letters.

For $-3x$, the sign is $-$, the numeral is 3, and the variable is x.

So, $-3x \times 2x^2 = -6x^3$

$$- \times + = -$$
$$3 \times 2 = 6$$
$$x \times x^2 = x^3$$

Example 3   ◀) **Self Tutor**

Simplify the following products:

a $-3 \times 4x$

b $2x \times -x^2$

c $-4x \times -2x^2$

a	$-3 \times 4x$	b	$2x \times -x^2$	c	$-4x \times -2x^2$
	$= -12x$		$= -2x^3$		$= 8x^3$

EXERCISE 3B

1 Write the following algebraic products in simplest form:

a $c \times b$

b $a \times 2 \times b$

c $y \times xy$

d $pq \times 2q$

2 Simplify the following:

a $2 \times 3x$

b $4x \times 5$

c $-2 \times 7x$

d $3 \times -2x$

e $2x \times x$

f $3x \times 2x$

g $-2x \times x$

h $-3x \times 4$

i $-2x \times -x$

j $-3x \times x^2$

k $-x^2 \times -2x$

l $3d \times -2d$

m $(-a)^2$

n $(-2a)^2$

o $2a^2 \times a^2$

p $a^2 \times -3a$

3 Simplify the following:

a $2 \times 5x + 3x \times 4$

b $5 \times 3x - 2y \times y$

c $3 \times x^2 + 2x \times 4x$

d $a \times 2b + b \times 3a$

e $4 \times x^2 - 3x \times x$

f $3x \times y - 2x \times 2y$

g $3a \times b + 2a \times 2b$

h $4c \times d - 3c \times 2d$

i $3a \times b - 2c \times a$

C | THE DISTRIBUTIVE LAW

Consider the expression $2(x+3)$.   We say that 2 is the **coefficient** of the expression in the brackets. We can **expand** the brackets using the **distributive law**:

$$a\,(b+c) = ab + ac$$

The distributive law says that we must multiply the coefficient by each term within the brackets, and add the results.

Geometric Demonstration:
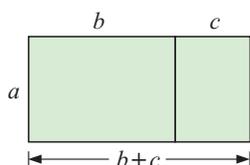
The overall area is $a(b+c)$.

However, this could also be found by adding the areas of the two small rectangles,   i.e., $ab + ac$.

So, $a(b+c) = ab + ac$.   {equating areas}

Example 4   ◀) **Self Tutor**

Expand the following:

  a $3(4x+1)$ **b** $2x(5-2x)$ **c** $-2x(x-3)$

a $3(4x+1)$
$= 3 \times 4x + 3 \times 1$
$= 12x + 3$

b $2x(5-2x)$
$= 2x(5 + -2x)$
$= 2x \times 5 + 2x \times -2x$
$= 10x - 4x^2$

c $-2x(x-3)$
$= -2x(x + -3)$
$= -2x \times x + -2x \times -3$
$= -2x^2 + 6x$

With practice, we do not need to write all of these steps.

Example 5   ◀) **Self Tutor**

Expand and simplify:

  a $2(3x-1) + 3(5-x)$ **b** $x(2x-1) - 2x(5-x)$

a $2(3x-1) + 3(5-x)$
$= 6x - 2 + 15 - 3x$
$= 3x + 13$

b $x(2x-1) - 2x(5-x)$
$= 2x^2 - x - 10x + 2x^2$
$= 4x^2 - 11x$

Notice in **b** that the minus sign in front of $2x$ affects *both* terms inside the following bracket.

EXERCISE 3C

1 Expand and simplify:

    a $3(x+1)$ **b** $2(5-x)$ **c** $-(x+2)$ **d** $-(3-x)$

    e $4(a+2b)$ **f** $3(2x+y)$ **g** $5(x-y)$ **h** $6(-x^2+y^2)$

    i $-2(x+4)$ **j** $-3(2x-1)$ **k** $x(x+3)$ **l** $2x(x-5)$

    m $-3(x+2)$ **n** $-4(x-3)$ **o** $-(3-x)$ **p** $-2(x-y)$

    q $a(a+b)$ **r** $-a(a-b)$ **s** $x(2x-1)$ **t** $2x(x^2-x-2)$

2 Expand and simplify:

    a $1+2(x+2)$ **b** $13-4(x+3)$ **c** $3(x-2)+5$

    d $4(3-x)-10$ **e** $x(x-1)+x$ **f** $2x(3-x)+x^2$

    g $2a(b-a)+3a^2$ **h** $4x-3x(x-1)$ **i** $7x^2-5x(x+2)$

3 Expand and simplify:

    a $3(x-4)+2(5+x)$ **b** $2a+(a-2b)$ **c** $2a-(a-2b)$

    d $3(y+1)+6(2-y)$ **e** $2(y-3)-4(2y+1)$ **f** $3x-4(2-3x)$

    g $2(b-a)+3(a+b)$ **h** $x(x+4)+2(x-3)$ **i** $x(x+4)-2(x-3)$

    j $x^2+x(x-1)$ **k** $-x^2-x(x-2)$ **l** $x(x+y)-y(x+y)$

    m $-4(x-2)-(3-x)$ **n** $5(2x-1)-(2x+3)$ **o** $4x(x-3)-2x(5-x)$

D   THE PRODUCT $(a+b)(c+d)$

Consider the product $(a+b)(c+d)$.

It has two **factors**, $(a+b)$ and $(c+d)$.

We can evaluate this product by using the distributive law several times.

$$(a+b)(c+d) = a(c+d) + b(c+d)$$
$$= ac + ad + bc + bd$$

So, $(a+b)(c+d) = ac+ad+bc+bd$

This is sometimes called the FOIL rule.

The final result contains four terms:

ac is the product of the   **F**irst terms of each bracket.
ad is the product of the   **O**uter terms of each bracket.
bc is the product of the   **I**nner terms of each bracket.
bd is the product of the   **L**ast terms of each bracket.

Example 6 ◀) **Self Tutor**

Expand and simplify: $(x+3)(x+2)$.

$(x+3)(x+2)$

$= x \times x + x \times 2 + 3 \times x + 3 \times 2$
$= x^2 + 2x + 3x + 6$
$= x^2 + 5x + 6$

> In practice we do not include the second line of these examples.

Example 7 ◀) **Self Tutor**

Expand and simplify: $(2x+1)(3x-2)$

$(2x+1)(3x-2)$

$= 2x \times 3x + 2x \times -2 + 1 \times 3x + 1 \times -2$
$= 6x^2 - 4x + 3x - 2$
$= 6x^2 - x - 2$

EXERCISE 3D

1 Consider the figure alongside:
Give an expression for the area of:

   a rectangle 1   **b** rectangle 2
   c rectangle 3   **d** rectangle 4
   e the overall rectangle.

What can you conclude?
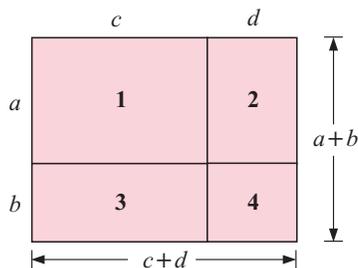
2 Use the rule $(a+b)(c+d) = ac + ad + bc + bd$ to expand and simplify:

   a $(x+3)(x+7)$ **b** $(x+5)(x-4)$ **c** $(x-3)(x+6)$
   d $(x+2)(x-2)$ **e** $(x-8)(x+3)$ **f** $(2x+1)(3x+4)$
   g $(1-2x)(4x+1)$ **h** $(4-x)(2x+3)$ **i** $(3x-2)(1+2x)$
   j $(5-3x)(5+x)$ **k** $(7-x)(4x+1)$ **l** $(5x+2)(5x+2)$

Example 8 ◀) **Self Tutor**

Expand and simplify:

   a $(x+3)(x-3)$ **b** $(3x-5)(3x+5)$

> What do you notice about the two middle terms?

   a $(x+3)(x-3)$ **b** $(3x-5)(3x+5)$
           $= x^2 - 3x + 3x - 9$ $= 9x^2 + 15x - 15x - 25$
           $= x^2 - 9$ $= 9x^2 - 25$

3 Expand and simplify:

   a $(x+2)(x-2)$ **b** $(a-5)(a+5)$ **c** $(4+x)(4-x)$

   d $(2x+1)(2x-1)$ **e** $(5a+3)(5a-3)$ **f** $(4+3a)(4-3a)$

Example 9   ◀) **Self Tutor**

Expand and simplify:

   a $(3x+1)^2$ **b** $(2x-3)^2$

> *What do you notice about the two middle terms?*

   a $(3x+1)^2$

      $= (3x+1)(3x+1)$

      $= 9x^2 + 3x + 3x + 1$

      $= 9x^2 + 6x + 1$

   b $(2x-3)^2$

      $= (2x-3)(2x-3)$

      $= 4x^2 - 6x - 6x + 9$

      $= 4x^2 - 12x + 9$

---

4 Expand and simplify:

   a $(x+3)^2$ **b** $(x-2)^2$ **c** $(3x-2)^2$

   d $(1-3x)^2$ **e** $(3-4x)^2$ **f** $(5x-y)^2$

E   DIFFERENCE OF TWO SQUARES

a^2 and b^2 are perfect squares and so $a^2 - b^2$ is called the **difference of two squares**.

Notice that $(a+b)(a-b) = a^2 \underbrace{- ab + ab} - b^2 = a^2 - b^2$

                       the middle two terms add to zero

Thus, $\boxed{(a+b)(a-b) = a^2 - b^2}$

Geometric Demonstration:

Consider the figure alongside:

The shaded area

$=$ area of large square $-$ area of small square

$= a^2 - b^2$
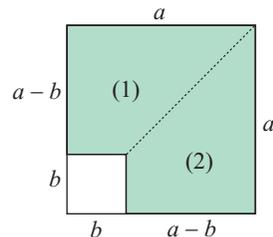
Cutting along the dotted line and flipping (2) over, we can form a rectangle.

The rectangle's area is $(a+b)(a-b)$.

$\therefore$ $(a+b)(a-b) = a^2 - b^2$

COMPUTER DEMO
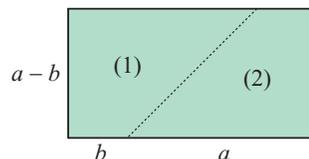
Example 10 ◄⑴ **Self Tutor**

Expand and simplify:

  a $(x+5)(x-5)$ **b** $(3-y)(3+y)$

a $(x+5)(x-5)$
$= x^2 - 5^2$
$= x^2 - 25$

b $(3-y)(3+y)$
$= 3^2 - y^2$
$= 9 - y^2$

Example 11 ◄⑴ **Self Tutor**

Expand and simplify:

  a $(2x-3)(2x+3)$ **b** $(5-3y)(5+3y)$

a $(2x-3)(2x+3)$
$= (2x)^2 - 3^2$
$= 4x^2 - 9$

b $(5-3y)(5+3y)$
$= 5^2 - (3y)^2$
$= 25 - 9y^2$

Example 12 ◄⑴ **Self Tutor**

Expand and simplify: $(3x+4y)(3x-4y)$

$(3x+4y)(3x-4y)$
$= (3x)^2 - (4y)^2$
$= 9x^2 - 16y^2$

EXERCISE 3E

1 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

  a $(x+2)(x-2)$ **b** $(x-2)(x+2)$ **c** $(2+x)(2-x)$
  d $(2-x)(2+x)$ **e** $(x+1)(x-1)$ **f** $(1-x)(1+x)$
  g $(x+7)(x-7)$ **h** $(c+8)(c-8)$ **i** $(d-5)(d+5)$
  j $(x+y)(x-y)$ **k** $(4+d)(4-d)$ **l** $(5+e)(5-e)$

2 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

  a $(2x-1)(2x+1)$ **b** $(3x+2)(3x-2)$ **c** $(4y-5)(4y+5)$
  d $(2y+5)(2y-5)$ **e** $(3x+1)(3x-1)$ **f** $(1-3x)(1+3x)$
  g $(2-5y)(2+5y)$ **h** $(3+4a)(3-4a)$ **i** $(4+3a)(4-3a)$

3 Expand and simplify using the rule $(a+b)(a-b) = a^2 - b^2$:

  a $(2a+b)(2a-b)$ **b** $(a-2b)(a+2b)$ **c** $(4x+y)(4x-y)$
  d $(4x+5y)(4x-5y)$ **e** $(2x+3y)(2x-3y)$ **f** $(7x-2y)(7x+2y)$

INVESTIGATION THE PRODUCT OF THREE CONSECUTIVE INTEGERS

Con was trying to multiply $19 \times 20 \times 21$ without a calculator. Aimee told him to 'cube the middle integer and then subtract the middle integer' to get the answer.

What to do:

1 Find $19 \times 20 \times 21$ using a calculator.

2 Find $20^3 - 20$ using a calculator. Does Aimee's rule seem to work?

3 Check that Aimee's rule works for the following products:

    a $4 \times 5 \times 6$ **b** $9 \times 10 \times 11$ **c** $49 \times 50 \times 51$

4 Let the middle integer be x, so the other integers must be $(x-1)$ and $(x+1)$. Find the product $(x-1) \times x \times (x+1)$ by expanding and simplifying. Have you proved Aimee's rule?

    Hint: Use the difference between two squares expansion.

F   PERFECT SQUARES EXPANSION

$(a+b)^2$ and $(a-b)^2$ are called **perfect squares**.

Notice that
$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + ab + ab + b^2 \quad \{\text{using 'FOIL'}\}$$
$$= a^2 + 2ab + b^2$$

> Notice that the middle two terms are identical.

Thus, we can state the perfect square expansion rule:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

We can remember the rule as follows:

Step 1: Square the *first term*.

Step 2: Add twice the product of the *first* and *last terms*.

Step 3: Add on the square of the *last term*.

Notice that
$$(a-b)^2 = (a+(-b))^2$$
$$= a^2 + 2a(-b) + (-b)^2$$
$$= a^2 - 2ab + b^2$$

Once again, we have the square of the first term, twice the product of the first and last terms, and the square of the last term.

Example 13 ◀⑨ **Self Tutor**

Expand and simplify:

a $(x+3)^2$ **b** $(x-5)^2$

a $(x+3)^2$
$= x^2 + 2 \times x \times 3 + 3^2$
$= x^2 + 6x + 9$

b $(x-5)^2$
$= (x+-5)^2$
$= x^2 + 2 \times x \times (-5) + (-5)^2$
$= x^2 - 10x + 25$

Example 14 ◀⑨ **Self Tutor**

Expand and simplify using the perfect square expansion rule:

a $(5x+1)^2$ **b** $(4-3x)^2$

a $(5x+1)^2$
$= (5x)^2 + 2 \times 5x \times 1 + 1^2$
$= 25x^2 + 10x + 1$

b $(4-3x)^2$
$= (4+-3x)^2$
$= 4^2 + 2 \times 4 \times (-3x) + (-3x)^2$
$= 16 - 24x + 9x^2$

EXERCISE 3F

1 Consider the figure alongside:
Give an expression for the area of:

a square 1 **b** rectangle 2 **c** rectangle 3
d square 4 **e** the overall square.

What can you conclude?

2 Use the rule $(a+b)^2 = a^2 + 2ab + b^2$ to expand and simplify:

a $(x+5)^2$ **b** $(x+4)^2$ **c** $(x+7)^2$
d $(a+2)^2$ **e** $(3+c)^2$ **f** $(5+x)^2$

3 Expand and simplify using the perfect square expansion rule:

a $(x-3)^2$ **b** $(x-2)^2$ **c** $(y-8)^2$
d $(a-7)^2$ **e** $(5-x)^2$ **f** $(4-y)^2$

4 Expand and simplify using the perfect square expansion rule:

a $(3x+4)^2$ **b** $(2a-3)^2$ **c** $(3y+1)^2$
d $(2x-5)^2$ **e** $(3y-5)^2$ **f** $(7+2a)^2$
g $(1+5x)^2$ **h** $(7-3y)^2$ **i** $(3+4a)^2$
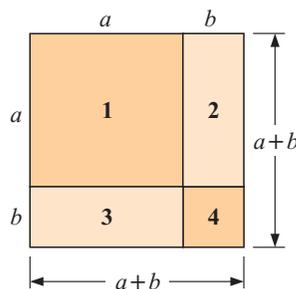
Example 15   ◀️ **Self Tutor**

Expand and simplify: **a** $(2x^2 + 3)^2$ **b** $5 - (x + 2)^2$

a $(2x^2 + 3)^2$
$= (2x^2)^2 + 2 \times 2x^2 \times 3 + 3^2$
$= 4x^4 + 12x^2 + 9$

b $5 - (x + 2)^2$
$= 5 - [x^2 + 4x + 4]$
$= 5 - x^2 - 4x - 4$
$= 1 - x^2 - 4x$

> Notice the use of square brackets in the second line. These remind us to change the signs inside them when they are removed.

5 Expand and simplify:

   a $(x^2 + 2)^2$　　　**b** $(y^2 - 3)^2$　　　**c** $(3a^2 + 4)^2$

   d $(1 - 2x^2)^2$　　　**e** $(x^2 + y^2)^2$　　　**f** $(x^2 - a^2)^2$

6 Expand and simplify:

   a $3x + 1 - (x + 3)^2$　　　　　　**b** $5x - 2 + (x - 2)^2$

   c $(x + 2)(x - 2) + (x + 3)^2$　　　**d** $(x + 2)(x - 2) - (x + 3)^2$

   e $(3 - 2x)^2 - (x - 1)(x + 2)$　　　**f** $(1 - 3x)^2 + (x + 2)(x - 3)$

   g $(2x + 3)(2x - 3) - (x + 1)^2$　　　**h** $(4x + 3)(x - 2) - (2 - x)^2$

   i $(1 - x)^2 + (x + 2)^2$　　　　　　**j** $(1 - x)^2 - (x + 2)^2$

G　　FURTHER EXPANSION

In this section we expand more complicated expressions by repeated use of the expansion laws.

Consider the expansion of $(a + b)(c + d + e)$.

Now　　$(a + b)(c + d + e)$　　　　　　Compare:　　$\square(c + d + e)$
$\quad = (a + b)c + (a + b)d + (a + b)e$　　　　　$= \square c + \square d + \square e$
$\quad = ac + bc + ad + bd + ae + be$

Notice that there are 6 terms in this expansion and that each term within the first bracket is multiplied by each term in the second.

2 terms in the first bracket $\times$ 3 terms in the second bracket $\longrightarrow$ 6 terms in the expansion.

Example 16   ◀️ **Self Tutor**

Expand and simplify: $(2x + 3)(x^2 + 4x + 5)$

$(2x + 3)(x^2 + 4x + 5)$

$= 2x^3 + 8x^2 + 10x$　　　　{all terms of 2nd bracket $\times\ 2x$}
$\quad + 3x^2 + 12x + 15$　　　{all terms of 2nd bracket $\times\ 3$}
$= 2x^3 + 11x^2 + 22x + 15$　　{collecting like terms}

Example 17

🔊 **Self Tutor**

Expand and simplify: $(x+2)^3$

$$(x+2)^3 = (x+2) \times (x+2)^2$$
$$= (x+2)(x^2 + 4x + 4)$$
$$= x^3 + 4x^2 + 4x \qquad \{\text{all terms in 2nd bracket} \times x\}$$
$$\quad + 2x^2 + 8x + 8 \qquad \{\text{all terms in 2nd bracket} \times 2\}$$
$$= x^3 + 6x^2 + 12x + 8 \quad \{\text{collecting like terms}\}$$

Example 18

🔊 **Self Tutor**

Expand and simplify:

a $x(x+1)(x+2)$ **b** $(x+1)(x-2)(x+2)$

> Always look for ways to make your expansions simpler. In **b** we can use the difference of two squares.

a $x(x+1)(x+2)$
$$= (x^2 + x)(x+2) \qquad \{\text{all terms in first bracket} \times x\}$$
$$= x^3 + 2x^2 + x^2 + 2x \qquad \{\text{expanding remaining factors}\}$$
$$= x^3 + 3x^2 + 2x \qquad \{\text{collecting like terms}\}$$

b $(x+1)(x-2)(x+2)$
$$= (x+1)(x^2 - 4) \qquad \{\text{difference of two squares}\}$$
$$= x^3 - 4x + x^2 - 4 \qquad \{\text{expanding factors}\}$$
$$= x^3 + x^2 - 4x - 4$$

EXERCISE 3G

> Each term of the first bracket is multiplied by each term of the second bracket.

1 Expand and simplify:

a $(x+3)(x^2 + x + 2)$ **b** $(x+4)(x^2 + x - 2)$

c $(x+2)(x^2 + x + 1)$ **d** $(x+5)(x^2 - x - 1)$

e $(2x+1)(x^2 + x + 4)$ **f** $(3x-2)(x^2 - x - 3)$

g $(x+2)(2x^2 - x + 2)$ **h** $(2x-1)(3x^2 - x + 2)$

2 Expand and simplify:

a $(x+1)^3$ **b** $(x+3)^3$ **c** $(x-1)^3$

d $(x-3)^3$ **e** $(2x+1)^3$ **f** $(3x-2)^3$

3 Expand and simplify:

a $x(x+2)(x+3)$ **b** $x(x-4)(x+1)$ **c** $x(x-3)(x-2)$

d $2x(x+3)(x+1)$ **e** $2x(x-4)(1-x)$ **f** $-x(3+x)(2-x)$

g $-3x(2x-1)(x+2)$ **h** $x(1-3x)(2x+1)$ **i** $2x^2(x-1)^2$

4 Expand and simplify:

    a $(x+3)(x+2)(x+1)$ **b** $(x-2)(x-1)(x+4)$

    c $(x-4)(x-1)(x-3)$ **d** $(2x-1)(x+2)(x-1)$

    e $(3x+2)(x+1)(x+3)$ **f** $(2x+1)(2x-1)(x+4)$

    g $(1-x)(3x+2)(x-2)$ **h** $(x-3)(1-x)(3x+2)$

H   THE BINOMIAL EXPANSION

Consider $(a+b)^n$. We note that:

- $a+b$ is called a **binomial** as it contains two terms
- any expression of the form $(a+b)^n$ is called a **power of a binomial**
- the **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

Now $(a+b)^3 = (a+b)^2(a+b)$
$$= (a^2 + 2ab + b^2)(a+b)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

So, the **binomial expansion** of $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

Example 19   ◀) **Self Tutor**

Expand and simplify using the rule
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

    a $(x+2)^3$ **b** $(2x-1)^3$

> We use brackets to assist our substitution.

    a We substitute $a = x$ and $b = 2$
      $\therefore$ $(x+2)^3 = x^3 + 3 \times x^2 \times 2 + 3 \times x \times 2^2 + 2^3$
                $= x^3 + 6x^2 + 12x + 8$

    b We substitute $a = (2x)$ and $b = (-1)$
      $\therefore$ $(2x-1)^3 = (2x)^3 + 3 \times (2x)^2 \times (-1) + 3 \times (2x) \times (-1)^2 + (-1)^3$
                  $= 8x^3 - 12x^2 + 6x - 1$

EXERCISE 3H

1 Use the binomial expansion for $(a+b)^3$ to expand and simplify:

    a $(x+1)^3$ **b** $(a+3)^3$ **c** $(x+5)^3$

    d $(x-1)^3$ **e** $(x-2)^3$ **f** $(x-3)^3$

    g $(3+a)^3$ **h** $(3x+2)^3$ **i** $(2x+3y)^3$

2 Copy and complete the argument $(a+b)^4 = (a+b)(a+b)^3$
$$= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$\vdots$$

3 Use the binomial expansion $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

    a $(x+1)^4$ **b** $(y+2)^4$ **c** $(3+a)^4$ **d** $(b+4)^4$

    e $(x-1)^4$ **f** $(y-2)^4$ **g** $(3-a)^4$ **h** $(b-4)^4$

4 Find the binomial expansion of $(a+b)^5$ by considering $(a+b)(a+b)^4$.
Hence, write down the binomial expansion for $(a-b)^5$.

REVIEW SET 3A

1 Expand and simplify:

    a $4x \times -8$ **b** $5x \times 2x^2$ **c** $-4x \times -6x$

    d $3x \times x - 2x^2$ **e** $4a \times c + 3c \times a$ **f** $2x^2 \times x - 3x \times x^2$

2 Expand and simplify:

    a $-3(x+6)$ **b** $2x(x^2 - 4)$

    c $2(x-5) + 3(2-x)$ **d** $3(1-2x) - (x-4)$

    e $2x - 3x(x-2)$ **f** $x(2x+1) - 2x(1-x)$

    g $x^2(x+1) - x(1-x^2)$ **h** $9(a+b) - a(4-b)$

3 Expand and simplify:

    a $(3x+2)(x-2)$ **b** $(2x-1)^2$ **c** $(4x+1)(4x-1)$

    d $(5-x)^2$ **e** $(3x-7)(2x-5)$ **f** $x(x+2)(x-2)$

    g $(3x+5)^2$ **h** $-(x-2)^2$ **i** $-2x(x-1)^2$

4 Expand and simplify:

    a $5 + 2x - (x+3)^2$ **b** $(x+2)^3$

    c $(3x-2)(x^2 + 2x + 7)$ **d** $(x-1)(x-2)(x-3)$

    e $x(x+1)^3$ **f** $(x^2+1)(x-1)(x+1)$

5
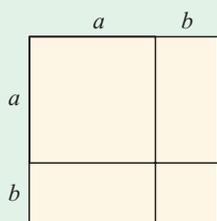
Explain how to use the given figure to show that $(a+b)^2 = a^2 + 2ab + b^2$.

REVIEW SET 3B

1 Expand and simplify:

    a $3x \times -2x^2$ **b** $2x^2 \times -3x$ **c** $-5x \times -8x$

    d $(2x)^2$ **e** $(-3x^2)^2$ **f** $4x \times -x^2$

2 Expand and simplify:

    a $-7(2x - 5)$ **b** $2(x - 3) + 3(2 - x)$

    c $-x(3 - 4x) - 2x(x + 1)$ **d** $2(3x + 1) - 5(1 - 2x)$

    e $3x(x^2 + 1) - 2x^2(3 - x)$ **f** $3(2a + b) - 5(b - 2a)$

3 Expand and simplify:

    a $(2x + 5)(x - 3)$ **b** $(3x - 2)^2$ **c** $(2x + 3)(2x - 3)$

    d $(5x - 1)(x - 2)$ **e** $(2x - 3)^2$ **f** $(1 - 5x)(1 + 5x)$

    g $(5 - 2x)^2$ **h** $-(x + 2)^2$ **i** $-3x(1 - x)^2$

4 Expand and simplify:

    a $(2x + 1)^2 - (x - 2)(3 - x)$ **b** $(x^2 - 4x + 3)(2x - 1)$

    c $(x + 3)^3$ **d** $(x + 1)(x - 2)(x + 5)$

    e $2x(x - 1)^3$ **f** $(4 - x^2)(x + 2)(x - 2)$

5 Use the binomial expansion $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

    a $(2x + 1)^4$ **b** $(x - 3)^4$
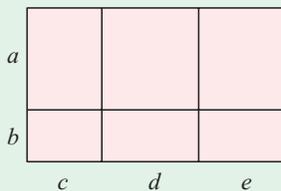
6 What algebraic fact can you derive by considering the area of the given figure in two different ways?

Chapter 4

Radicals (Surds)

Contents:

INTRODUCTION

In previous years we used the **Theorem of Pythagoras** to find the length of the third side of a triangle.

Our answers often involved radicals such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and so on.
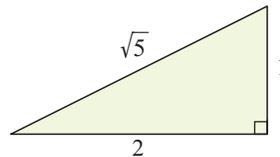
A **radical** is a number that is written using the radical sign $\sqrt{}$.

Radicals such as $\sqrt{4}$ and $\sqrt{9}$ are *rational* since $\sqrt{4} = 2$ and $\sqrt{9} = 3$.
Radicals such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are *irrational*. They have decimal expansions which neither terminate nor recur. Irrational radicals are also known as **surds**.

RESEARCH

- Where did the names radical and surd come from?

- Why do we use the word irrational to describe some numbers?

- Before we had calculators and computers, finding decimal representations for numbers like $\frac{1}{\sqrt{2}}$ to four or five decimal places was quite difficult and time consuming.
  Imagine having to find $\frac{1}{1.41421}$ correct to five decimal places using long division!
  A method was devised to do this calculation quickly. What was the process?
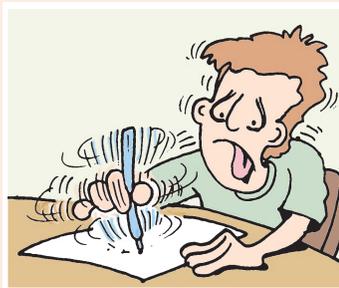
SQUARE ROOTS

The **square root of a** or $\sqrt{a}$ is the *positive* number which obeys the rule $\sqrt{a} \times \sqrt{a} = a$.
For $\sqrt{a}$ to have meaning we require a to be non-negative, i.e., $a \geqslant 0$.

For example, $\sqrt{5} \times \sqrt{5} = 5$ or $(\sqrt{5})^2 = 5$.

Note that $\sqrt{4} = 2$, not ± 2, since the square root of a number cannot be negative.

A   RADICALS ON A NUMBER LINE

If we convert a radical such as $\sqrt{5}$ to a decimal we can find its approximate position on a number line. $\sqrt{5} \approx 2.236\,067$, so $\sqrt{5}$ is close to $2\frac{1}{4}$.

We can also *construct* the position of $\sqrt{5}$ on a number line using a ruler and compass. Since $1^2 + 2^2 = (\sqrt{5})^2$,   we can use a right-angled triangle with sides of length 1, 2 and $\sqrt{5}$.

Step 1: Draw a number line and mark the numbers 0, 1, 2, and 3 on it, 1 cm apart.

Step 2: With compass point on 1, draw an arc above 2. Do the same with compass point on 3 using the same radius. Draw the perpendicular at 2 through the intersection of these arcs, and mark off 1 cm. Call this point A.
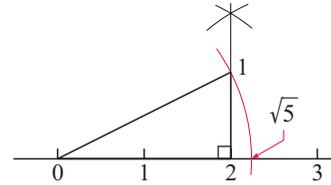
Step 3: Complete the right angled triangle. Its sides are 2, 1 and $\sqrt{5}$ cm.

Step 4: With centre O and radius OA, draw an arc through A to meet the number line. It meets the number line at $\sqrt{5}$.

EXERCISE 4A

1 Notice that $1^2 + 4^2 = 17 = (\sqrt{17})^2$.

Locate $\sqrt{17}$ on a number line using an accurate construction.

2   **a** The sum of the squares of which two positive integers is 13?

  b Accurately construct the position of $\sqrt{13}$ on a number line.

3 Can we construct the exact position of $\sqrt{6}$ on a number line using the method above?

4 7 cannot be written as the sum of two squares so the above method cannot be used for locating $\sqrt{7}$ on the number line.

However, $4^2 - 3^2 = 7$,   so $4^2 = 3^2 + (\sqrt{7})^2$.
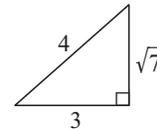
We can thus construct a right angled triangle with sides of length 4, 3 and $\sqrt{7}$.

Use such a triangle to accurately locate $\sqrt{7}$ on a number line.

B   OPERATIONS WITH RADICALS

ADDING AND SUBTRACTING RADICALS

We can add and subtract 'like radicals' in the same way as we do 'like terms' in algebra.

For example: • just as $3a + 2a = 5a$, $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$

  • just as $6b - 4b = 2b$, $6\sqrt{3} - 4\sqrt{3} = 2\sqrt{3}$.

Example 1   ◆ Self Tutor

Simplify: **a** $3\sqrt{2} - 4\sqrt{2}$ **b** $\sqrt{7} - 2(1 - \sqrt{7})$

a $3\sqrt{2} - 4\sqrt{2} = -1\sqrt{2}$ **b** $\sqrt{7} - 2(1 - \sqrt{7}) = \sqrt{7} - 2 + 2\sqrt{7}$
$\qquad\qquad\quad = -\sqrt{2}$ $\qquad\qquad\qquad\qquad\quad = 3\sqrt{7} - 2$

SIMPLIFYING PRODUCTS

We have established in previous years that:

$$\sqrt{a}\sqrt{a} = (\sqrt{a})^2 = a$$
$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$
$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Example 2   ◄╢ **Self Tutor**

Simplify: **a** $(\sqrt{2})^2$ **b** $(\sqrt{2})^3$ **c** $\left(\dfrac{4}{\sqrt{2}}\right)^2$

a $(\sqrt{2})^2$
$= \sqrt{2} \times \sqrt{2}$
$= 2$

b $(\sqrt{2})^3$
$= \sqrt{2} \times \sqrt{2} \times \sqrt{2}$
$= 2\sqrt{2}$

c $\left(\dfrac{4}{\sqrt{2}}\right)^2$
$= \dfrac{4^2}{(\sqrt{2})^2}$
$= \dfrac{16}{2}$
$= 8$

Example 3   ◄╢ **Self Tutor**

Simplifying: **a** $(3\sqrt{2})^2$ **b** $3\sqrt{3} \times (-2\sqrt{3})$

a $(3\sqrt{2})^2$
$= 3\sqrt{2} \times 3\sqrt{2}$
$= 9 \times 2$
$= 18$

b $3\sqrt{3} \times (-2\sqrt{3})$
$= 3 \times -2 \times \sqrt{3} \times \sqrt{3}$
$= -6 \times 3$
$= -18$

Example 4   ◄╢ **Self Tutor**

Write in simplest form:
a $\sqrt{2} \times \sqrt{5}$ **b** $3\sqrt{2} \times 4\sqrt{11}$

a $\sqrt{2} \times \sqrt{5}$
$= \sqrt{2 \times 5}$
$= \sqrt{10}$

b $3\sqrt{2} \times 4\sqrt{11}$
$= 3 \times 4 \times \sqrt{2} \times \sqrt{11}$
$= 12 \times \sqrt{2 \times 11}$
$= 12\sqrt{22}$

With practice you should not need the middle steps.

Example 5

◀) **Self Tutor**

Simplify: **a** $\dfrac{\sqrt{75}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{32}}{2\sqrt{2}}$

a $\dfrac{\sqrt{75}}{\sqrt{3}}$

$= \sqrt{\dfrac{75}{3}}$

$= \sqrt{25}$

$= 5$

b $\dfrac{\sqrt{32}}{2\sqrt{2}}$

$= \tfrac{1}{2}\sqrt{\dfrac{32}{2}}$

$= \tfrac{1}{2}\sqrt{16}$

$= \tfrac{1}{2} \times 4$

$= 2$

EXERCISE 4B.1

1 Simplify:

  a $3\sqrt{2} + 7\sqrt{2}$ **b** $11\sqrt{3} - 8\sqrt{3}$

  c $6\sqrt{5} - 7\sqrt{5}$ **d** $-\sqrt{2} + 2\sqrt{2}$

  e $\sqrt{3} - (2 - \sqrt{3})$ **f** $-\sqrt{2} - (3 + \sqrt{2})$

  g $5\sqrt{2} - \sqrt{3} + \sqrt{2} - \sqrt{3}$ **h** $\sqrt{7} - 2\sqrt{2} + \sqrt{7} - \sqrt{2}$

  i $3\sqrt{3} - \sqrt{2} - (1 - \sqrt{2})$ **j** $2(\sqrt{3} + 1) + 3(1 - \sqrt{3})$

  k $3(\sqrt{3} - \sqrt{2}) - (\sqrt{2} - \sqrt{3})$ **l** $3(\sqrt{3} - 1) - 2(2 - \sqrt{3})$

2 Simplify:

  a $(\sqrt{3})^2$ **b** $(\sqrt{3})^3$ **c** $(\sqrt{3})^5$ **d** $\left(\dfrac{1}{\sqrt{3}}\right)^2$

  e $(\sqrt{7})^2$ **f** $(\sqrt{7})^3$ **g** $\left(\dfrac{1}{\sqrt{7}}\right)^2$ **h** $\left(\dfrac{3}{\sqrt{7}}\right)^2$

  i $(\sqrt{5})^2$ **j** $(\sqrt{5})^4$ **k** $\left(\dfrac{5}{\sqrt{5}}\right)^2$ **l** $\left(\dfrac{10}{\sqrt{5}}\right)^2$

3 Simplify:

  a $(2\sqrt{2})^2$ **b** $(4\sqrt{2})^2$ **c** $(2\sqrt{3})^2$

  d $(3\sqrt{3})^2$ **e** $(2\sqrt{5})^2$ **f** $(3\sqrt{5})^2$

  g $(2\sqrt{7})^2$ **h** $(2\sqrt{10})^2$ **i** $(7\sqrt{10})^2$

  j $3\sqrt{2} \times 4\sqrt{2}$ **k** $5\sqrt{3} \times 2\sqrt{3}$ **l** $7\sqrt{2} \times 5\sqrt{2}$

  m $(-4\sqrt{2})^2$ **n** $(-7\sqrt{3})^2$ **o** $\sqrt{2} \times (-3\sqrt{2})$

  p $(-2\sqrt{3})(-5\sqrt{3})$ **q** $(-2\sqrt{7}) \times 3\sqrt{7}$ **r** $\sqrt{11} \times (-2\sqrt{11})$

4 Simplify:

  a $\sqrt{6\tfrac{1}{4}}$ **b** $\sqrt{1\tfrac{7}{9}}$ **c** $\sqrt{2\tfrac{14}{25}}$ **d** $\sqrt{7\tfrac{1}{9}}$

5 Simplify:

a $\sqrt{2} \times \sqrt{3}$

b $\sqrt{2} \times \sqrt{7}$

c $\sqrt{2} \times \sqrt{17}$

d $\sqrt{7} \times \sqrt{3}$

e $2\sqrt{2} \times 5\sqrt{3}$

f $(3\sqrt{2})^2$

g $5\sqrt{2} \times \sqrt{7}$

h $2\sqrt{6} \times 3\sqrt{5}$

i $-5\sqrt{2} \times 2\sqrt{7}$

j $(-\sqrt{7}) \times (-2\sqrt{3})$

k $(2\sqrt{3})^2 \times 2\sqrt{5}$

l $(2\sqrt{2})^3 \times 5\sqrt{3}$

6 Simplify:

a $\dfrac{\sqrt{8}}{\sqrt{2}}$

b $\dfrac{\sqrt{3}}{\sqrt{27}}$

c $\dfrac{\sqrt{18}}{\sqrt{3}}$

d $\dfrac{\sqrt{2}}{\sqrt{50}}$

e $\dfrac{\sqrt{75}}{\sqrt{5}}$

f $\dfrac{\sqrt{5}}{\sqrt{75}}$

g $\dfrac{\sqrt{18}}{\sqrt{2}}$

h $\dfrac{\sqrt{3}}{\sqrt{60}}$

i $\dfrac{3\sqrt{6}}{\sqrt{2}}$

j $\dfrac{4\sqrt{12}}{\sqrt{3}}$

k $\dfrac{4\sqrt{6}}{\sqrt{24}}$

l $\dfrac{3\sqrt{98}}{2\sqrt{2}}$

7 a Is $\sqrt{9} + \sqrt{16} = \sqrt{9 + 16}$?   Is $\sqrt{25} - \sqrt{16} = \sqrt{25 - 16}$?

b Are $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ and $\sqrt{a} - \sqrt{b} = \sqrt{a - b}$ possible laws for radical numbers?

8 a Prove that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for all positive numbers a and b.

   Hint: Consider $(\sqrt{a}\sqrt{b})^2$ and $(\sqrt{ab})^2$.

b Prove that $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0$ and $b > 0$.

SIMPLEST RADICAL FORM

A radical is in **simplest form** when the number under the radical sign is the smallest possible integer.

Example 6   ◀) **Self Tutor**

Write $\sqrt{8}$ in simplest form.

$\sqrt{8}$

$= \sqrt{4 \times 2}$

$= \sqrt{4} \times \sqrt{2}$

$= 2\sqrt{2}$

We look for the largest perfect square that can be taken out as a factor of this number.

$\sqrt{32} = \sqrt{4 \times 8} = 2\sqrt{8}$ is not in simplest form as $\sqrt{8}$ can be further simplified into $2\sqrt{2}$.

In simplest form, $\sqrt{32} = 4\sqrt{2}$.

Example 7

Self Tutor

Write $\sqrt{432}$ in simplest radical form.

$\sqrt{432}$
$= \sqrt{2^4 \times 3^3}$
$= \sqrt{2^4} \times \sqrt{3^3}$
$= 4 \times 3\sqrt{3}$
$= 12\sqrt{3}$

> It may be useful to do a prime factorisation of the number under the radical sign.

EXERCISE 4B.2

1 Write in the form $k\sqrt{2}$ where k is an integer:

    a $\sqrt{18}$ **b** $\sqrt{50}$ **c** $\sqrt{72}$ **d** $\sqrt{98}$

    e $\sqrt{162}$ **f** $\sqrt{200}$ **g** $\sqrt{20\,000}$ **h** $\sqrt{2\,000\,000}$

2 Write in the form $k\sqrt{3}$ where k is an integer:

    a $\sqrt{12}$ **b** $\sqrt{27}$ **c** $\sqrt{48}$ **d** $\sqrt{300}$

3 Write in the form $k\sqrt{5}$ where $k \in \mathbb{Z}$:

    a $\sqrt{20}$ **b** $\sqrt{80}$ **c** $\sqrt{320}$ **d** $\sqrt{500}$

4 Write in simplest radical form:

    a $\sqrt{99}$ **b** $\sqrt{52}$ **c** $\sqrt{40}$ **d** $\sqrt{63}$

    e $\sqrt{48}$ **f** $\sqrt{125}$ **g** $\sqrt{147}$ **h** $\sqrt{175}$

    i $\sqrt{176}$ **j** $\sqrt{150}$ **k** $\sqrt{275}$ **l** $\sqrt{2000}$

5 Write in simplest radical form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}, \ n \in \mathbb{Z}$:

    a $\dfrac{4 + \sqrt{8}}{2}$ **b** $\dfrac{6 - \sqrt{12}}{2}$ **c** $\dfrac{4 + \sqrt{18}}{4}$ **d** $\dfrac{8 - \sqrt{32}}{4}$

    e $\dfrac{12 + \sqrt{72}}{6}$ **f** $\dfrac{18 + \sqrt{27}}{6}$ **g** $\dfrac{14 - \sqrt{50}}{8}$ **h** $\dfrac{5 - \sqrt{200}}{10}$

C   EXPANSIONS WITH RADICALS

The rules for expanding radical expressions containing brackets are identical to those for ordinary algebra.

$$a(b + c) = ab + ac$$
$$(a + b)(c + d) = ac + ad + bc + bd$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Example 8 ◀) **Self Tutor**

Simplify: **a** $2(2+\sqrt{3})$ **b** $\sqrt{2}(5-2\sqrt{2})$

a $2(2+\sqrt{3})$
$= 2\times 2 + 2\times\sqrt{3}$
$= 4 + 2\sqrt{3}$

b $\sqrt{2}(5-2\sqrt{2})$
$= \sqrt{2}\times 5 + \sqrt{2}\times -2\sqrt{2}$
$= 5\sqrt{2} - 4$

> With practice you should not need the middle steps.

Example 9 ◀) **Self Tutor**

Expand and simplify:

a $-\sqrt{3}(2+\sqrt{3})$ **b** $-\sqrt{2}(\sqrt{2}-\sqrt{3})$

a $-\sqrt{3}(2+\sqrt{3})$
$= -\sqrt{3}\times 2 + -\sqrt{3}\times\sqrt{3}$
$= -2\sqrt{3} - 3$

b $-\sqrt{2}(\sqrt{2}-\sqrt{3})$
$= -\sqrt{2}\times\sqrt{2} + -\sqrt{2}\times -\sqrt{3}$
$= -2 + \sqrt{6}$

EXERCISE 4C

1 Expand and simplify:

a $4(3+\sqrt{2})$ **b** $3(\sqrt{2}+\sqrt{3})$ **c** $5(4-\sqrt{7})$

d $6(\sqrt{11}-4)$ **e** $\sqrt{2}(1+\sqrt{2})$ **f** $\sqrt{2}(\sqrt{2}-5)$

g $\sqrt{3}(2+2\sqrt{3})$ **h** $\sqrt{3}(\sqrt{3}-\sqrt{2})$ **i** $\sqrt{5}(6-\sqrt{5})$

j $\sqrt{5}(2\sqrt{5}-1)$ **k** $\sqrt{5}(2\sqrt{5}+\sqrt{3})$ **l** $\sqrt{7}(2+\sqrt{7}+\sqrt{2})$

2 Expand and simplify:

a $-\sqrt{2}(4+\sqrt{2})$ **b** $\sqrt{2}(3-\sqrt{2})$ **c** $-\sqrt{2}(\sqrt{2}-\sqrt{7})$

d $-\sqrt{3}(3+\sqrt{3})$ **e** $-\sqrt{3}(5-\sqrt{3})$ **f** $-\sqrt{3}(2\sqrt{3}+\sqrt{5})$

g $-\sqrt{5}(2\sqrt{2}-\sqrt{3})$ **h** $-2\sqrt{2}(\sqrt{2}+\sqrt{3})$ **i** $-2\sqrt{3}(1-2\sqrt{2})$

j $-\sqrt{7}(2\sqrt{7}+4)$ **k** $-\sqrt{11}(2-\sqrt{11})$ **l** $-(\sqrt{2})^3(4-2\sqrt{2})$

Example 10 ◀) **Self Tutor**

Expand and simplify:

a $(2+\sqrt{2})(3+\sqrt{2})$ **b** $(3+\sqrt{5})(1-\sqrt{5})$

a $(2+\sqrt{2})(3+\sqrt{2})$
$= (2+\sqrt{2})3 + (2+\sqrt{2})\sqrt{2}$
$= 6 + 3\sqrt{2} + 2\sqrt{2} + 2$
$= 8 + 5\sqrt{2}$

b $(3+\sqrt{5})(1-\sqrt{5})$
$= (3+\sqrt{5})(1+-\sqrt{5})$
$= (3+\sqrt{5})1 + (3+\sqrt{5})(-\sqrt{5})$
$= 3 + \sqrt{5} - 3\sqrt{5} - 5$
$= -2 - 2\sqrt{5}$

3 Expand and simplify:

   a $(2 + \sqrt{2})(3 + \sqrt{2})$ **b** $(3 + \sqrt{2})(3 + \sqrt{2})$

   c $(\sqrt{2} + 2)(\sqrt{2} - 1)$ **d** $(4 - \sqrt{3})(3 + \sqrt{3})$

   e $(2 + \sqrt{3})(2 - \sqrt{3})$ **f** $(2 - \sqrt{6})(5 + \sqrt{6})$

   g $(\sqrt{7} + 2)(\sqrt{7} - 3)$ **h** $(\sqrt{11} + \sqrt{2})(\sqrt{11} - \sqrt{2})$

   i $(3\sqrt{2} + 1)(3\sqrt{2} + 3)$ **j** $(6 - 2\sqrt{2})(2 + \sqrt{2})$

Example 11   ◄)) Self Tutor

Expand and simplify:

   a $(\sqrt{2} + 3)^2$ **b** $(\sqrt{5} - \sqrt{3})^2$

   a $(\sqrt{2} + 3)^2$
$$= (\sqrt{2})^2 + 2\sqrt{2}(3) + 3^2$$
$$= 2 + 6\sqrt{2} + 9$$
$$= 11 + 6\sqrt{2}$$

   b $(\sqrt{5} - \sqrt{3})^2$
$$= (\sqrt{5} + -\sqrt{3})^2$$
$$= (\sqrt{5})^2 + 2\sqrt{5}(-\sqrt{3}) + (-\sqrt{3})^2$$
$$= 5 - 2\sqrt{15} + 3$$
$$= 8 - 2\sqrt{15}$$

---

4 Expand and simplify:

   a $(1 + \sqrt{3})^2$ **b** $(\sqrt{2} + 5)^2$ **c** $(3 - \sqrt{2})^2$

   d $(1 + \sqrt{7})^2$ **e** $(\sqrt{3} - \sqrt{2})^2$ **f** $(4 - \sqrt{5})^2$

   g $(\sqrt{3} + \sqrt{5})^2$ **h** $(3 - \sqrt{6})^2$ **i** $(\sqrt{6} - \sqrt{3})^2$

   j $(2\sqrt{2} + 3)^2$ **k** $(3 - 2\sqrt{2})^2$ **l** $(3 - 5\sqrt{2})^2$

Example 12   ◄)) Self Tutor

Expand and simplify:

   a $(4 + \sqrt{2})(4 - \sqrt{2})$ **b** $(2\sqrt{2} + 3)(2\sqrt{2} - 3)$

   a $(4 + \sqrt{2})(4 - \sqrt{2})$
$$= 4^2 - (\sqrt{2})^2$$
$$= 16 - 2$$
$$= 14$$

   b $(2\sqrt{2} + 3)(2\sqrt{2} - 3)$
$$= (2\sqrt{2})^2 - 3^2$$
$$= 8 - 9$$
$$= -1$$

---

5 Expand and simplify:

   a $(3 + \sqrt{2})(3 - \sqrt{2})$ **b** $(\sqrt{3} - 1)(\sqrt{3} + 1)$

   c $(5 + \sqrt{3})(5 - \sqrt{3})$ **d** $(\sqrt{3} - 4)(\sqrt{3} + 4)$

   e $(\sqrt{7} - 3)(\sqrt{7} + 3)$ **f** $(2 + 5\sqrt{2})(2 - 5\sqrt{2})$

   g $(\sqrt{7} - \sqrt{11})(\sqrt{7} + \sqrt{11})$ **h** $(2\sqrt{5} + 6)(2\sqrt{5} - 6)$

i $(3\sqrt{2}+2)(3\sqrt{2}-2)$ **j** $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$

k $(\sqrt{3}-\sqrt{7})(\sqrt{3}+\sqrt{7})$ **l** $(2\sqrt{2}+1)(2\sqrt{2}-1)$

D   DIVISION BY RADICALS

In numbers like $\dfrac{6}{\sqrt{2}}$ and $\dfrac{9}{\sqrt{5}-\sqrt{2}}$ we have divided by a radical.

It is customary to 'simplify' these numbers by rewriting them without the radical in the denominator.

INVESTIGATION 1   DIVISION BY $\sqrt{a}$

In this investigation we consider fractions of the form $\dfrac{b}{\sqrt{a}}$ where a and b are real numbers. To remove the radical from the denominator, there are two methods we could use:

- 'splitting' the numerator
- rationalising the denominator

What to do:

1 Consider the fraction $\dfrac{6}{\sqrt{2}}$.

   a Since 2 is a factor of 6, 'split' the 6 into $3 \times \sqrt{2} \times \sqrt{2}$.

   b Simplify $\dfrac{6}{\sqrt{2}}$.

2 Can the method of 'splitting' the numerator be used to simplify $\dfrac{7}{\sqrt{2}}$?

3 Consider the fraction $\dfrac{7}{\sqrt{2}}$.

   a If we multiply this fraction by $\dfrac{\sqrt{2}}{\sqrt{2}}$, are we changing its value?

   b Simplify $\dfrac{7}{\sqrt{2}}$ by multiplying both its numerator and denominator by $\sqrt{2}$.

4 The method in **3** is called 'rationalising the denominator'. Will this method work for all fractions of the form $\dfrac{b}{\sqrt{a}}$ where a and b are real?

From the **Investigation** above, you should have found that for any fraction of the form $\dfrac{b}{\sqrt{a}}$,

we can remove the radical from the denominator by multiplying by $\dfrac{\sqrt{a}}{\sqrt{a}}$. Since $\dfrac{\sqrt{a}}{\sqrt{a}}=1$,

we do not change the value of the fraction.

Example 13   ◀) **Self Tutor**

Write with an integer denominator:

a $\dfrac{6}{\sqrt{3}}$ **b** $\dfrac{35}{\sqrt{7}}$

a $\dfrac{6}{\sqrt{3}}$

$= \dfrac{6}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{^2\cancel{6}\sqrt{3}}{\cancel{3}\ _1}$

$= 2\sqrt{3}$

b $\dfrac{35}{\sqrt{7}}$

$= \dfrac{35}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$

$= \dfrac{^5\cancel{35}\sqrt{7}}{\cancel{7}\ _1}$

$= 5\sqrt{7}$

> Multiplying the original number by $\dfrac{\sqrt{3}}{\sqrt{3}}$ or $\dfrac{\sqrt{7}}{\sqrt{7}}$ does not change its value.

EXERCISE 4D.1

1 Write with integer denominator:

a $\dfrac{1}{\sqrt{3}}$ **b** $\dfrac{3}{\sqrt{3}}$ **c** $\dfrac{9}{\sqrt{3}}$ **d** $\dfrac{11}{\sqrt{3}}$ **e** $\dfrac{\sqrt{2}}{3\sqrt{3}}$

f $\dfrac{2}{\sqrt{2}}$ **g** $\dfrac{6}{\sqrt{2}}$ **h** $\dfrac{12}{\sqrt{2}}$ **i** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **j** $\dfrac{1}{4\sqrt{2}}$

k $\dfrac{5}{\sqrt{5}}$ **l** $\dfrac{15}{\sqrt{5}}$ **m** $\dfrac{-3}{\sqrt{5}}$ **n** $\dfrac{200}{\sqrt{5}}$ **o** $\dfrac{1}{3\sqrt{5}}$

p $\dfrac{7}{\sqrt{7}}$ **q** $\dfrac{21}{\sqrt{7}}$ **r** $\dfrac{2}{\sqrt{11}}$ **s** $\dfrac{26}{\sqrt{13}}$ **t** $\dfrac{1}{(\sqrt{3})^3}$

RADICAL CONJUGATES

Radical expressions such as $3 + \sqrt{2}$ and $3 - \sqrt{2}$ which are identical except for opposing signs in the middle, are called **radical conjugates**.

> The radical conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$.

INVESTIGATION 2   **RADICAL CONJUGATES**

Fractions of the form $\dfrac{c}{a + \sqrt{b}}$ can also be simplified to remove the radical from the denominator. To do this we use radical conjugates.

What to do:

1 Expand and simplify:

a $(2 + \sqrt{3})(2 - \sqrt{3})$ **b** $(\sqrt{3} - 1)(\sqrt{3} + 1)$

2 What do you notice about your results in **1**?

3 Show that for any integers a and b, the following products are integers:

   a $(a + \sqrt{b})(a - \sqrt{b})$ **b** $(\sqrt{a} - b)(\sqrt{a} + b)$

4 **a** Copy and complete:

    To remove the radicals from the denominator of a fraction, we can multiply the denominator by its ......

  b What must we do to the numerator of the fraction to ensure we do not change its value?

From the **Investigation** above, we should have found that:

to remove the radicals from the denominator of a fraction, we multiply both the numerator and the denominator by the **radical conjugate** of the denominator.

Example 14   ◄) **Self Tutor**

Write $\dfrac{14}{3 - \sqrt{2}}$ with an integer denominator.

$$\dfrac{14}{3 - \sqrt{2}} = \left(\dfrac{14}{3 - \sqrt{2}}\right)\left(\dfrac{3 + \sqrt{2}}{3 + \sqrt{2}}\right)$$

$$= \dfrac{14}{9 - 2} \times (3 + \sqrt{2})$$

$$= 2(3 + \sqrt{2})$$

$$= 6 + 2\sqrt{2}$$

Example 15   ◄) **Self Tutor**

Write $\dfrac{1}{5 + \sqrt{2}}$:   **a** with integer denominator

                b in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

a $\dfrac{1}{5 + \sqrt{2}}$

$$= \left(\dfrac{1}{5 + \sqrt{2}}\right) \times \left(\dfrac{5 - \sqrt{2}}{5 - \sqrt{2}}\right)$$

$$= \dfrac{5 - \sqrt{2}}{25 - 2}$$

$$= \dfrac{5 - \sqrt{2}}{23}$$

b $\dfrac{5 - \sqrt{2}}{23} = \dfrac{5}{23} - \dfrac{1}{23}\sqrt{2}$

So, $a = \dfrac{5}{23}$ and $b = -\dfrac{1}{23}$.

EXERCISE 4D.2

1 Write with integer denominator:

a $\dfrac{1}{3+\sqrt{2}}$ **b** $\dfrac{2}{3-\sqrt{2}}$ **c** $\dfrac{1}{2+\sqrt{5}}$ **d** $\dfrac{\sqrt{2}}{2-\sqrt{2}}$

e $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$ **f** $\dfrac{\sqrt{3}}{4-\sqrt{3}}$ **g** $\dfrac{-2\sqrt{2}}{1-\sqrt{2}}$ **h** $\dfrac{1+\sqrt{5}}{2-\sqrt{5}}$

2 Write in the form $a+b\sqrt{2}$ where $a,b \in \mathbb{Q}$:

a $\dfrac{3}{\sqrt{2}-3}$ **b** $\dfrac{4}{2+\sqrt{2}}$ **c** $\dfrac{\sqrt{2}}{\sqrt{2}-5}$ **d** $\dfrac{-2\sqrt{2}}{\sqrt{2}+1}$

3 Write in the form $a+b\sqrt{3}$ where $a,b \in \mathbb{Q}$:

a $\dfrac{4}{1-\sqrt{3}}$ **b** $\dfrac{6}{\sqrt{3}+2}$ **c** $\dfrac{\sqrt{3}}{2-\sqrt{3}}$ **d** $\dfrac{1+2\sqrt{3}}{3+\sqrt{3}}$

4 **a** If a, b and c are integers, show that $(a+b\sqrt{c})(a-b\sqrt{c})$ is an integer.

b Write with an integer denominator: **i** $\dfrac{1}{1+2\sqrt{3}}$ **ii** $\dfrac{\sqrt{2}}{3\sqrt{2}-5}$.

5 **a** If a and b are integers, show that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is also an integer.

b Write with an integer denominator:

i $\dfrac{1}{\sqrt{2}+\sqrt{3}}$ **ii** $\dfrac{\sqrt{3}}{\sqrt{3}-\sqrt{5}}$

LINKS
click here

HOW A CALCULATOR CALCULATES RATIONAL NUMBERS

Areas of interaction:
Human ingenuity

REVIEW SET 4A

1 Simplify:

a $(2\sqrt{3})^2$ **b** $\left(\dfrac{4}{\sqrt{2}}\right)^2$ **c** $3\sqrt{2} \times 2\sqrt{5}$ **d** $\sqrt{12\tfrac{1}{4}}$

2 **a** Copy and complete: $1^2+3^2=(\ldots\ldots)^2$

b Use **a** to accurately construct the position of $\sqrt{10}$ on a number line using a ruler and compass.

3 Simplify:

a $\dfrac{\sqrt{15}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{35}}{\sqrt{7}}$ **c** $\dfrac{\sqrt{35}}{\sqrt{5}}$ **d** $\dfrac{\sqrt{2}}{\sqrt{20}}$

4 **a** Write $\sqrt{8}$ in simplest radical form.

    b Hence, simplify $5\sqrt{2} - \sqrt{8}$.

5 Write $\sqrt{98}$ in simplest radical form.

6 Expand and simplify:

    a $2(\sqrt{3}+1)$ **b** $\sqrt{2}(3-\sqrt{2})$ **c** $(1+\sqrt{7})^2$

    d $(2-\sqrt{5})^2$ **e** $(3+\sqrt{2})(3-\sqrt{2})$ **f** $(3+\sqrt{2})(1-\sqrt{2})$

7 Write with an integer denominator:

    a $\dfrac{10}{\sqrt{5}}$ **b** $\dfrac{\sqrt{3}+2}{\sqrt{3}+1}$ **c** $\dfrac{1+\sqrt{7}}{1-\sqrt{7}}$

8 Write in the form $a + b\sqrt{5}$ where $a, b \in \mathbb{Q}$:

    a $\dfrac{3}{2-\sqrt{5}}$ **b** $\dfrac{2\sqrt{5}}{\sqrt{5}+1}$

REVIEW SET 4B

1 Simplify:

    a $\sqrt{3}\sqrt{2}$ **b** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **c** $(3\sqrt{5})^2$ **d** $\sqrt{5\tfrac{4}{9}}$

2 Find the exact position of $\sqrt{12}$ on a number line using a ruler and compass construction. Explain your method.

    Hint: Look for two positive integers a and b such that $a^2 - b^2 = 12$.

3 Simplify: **a** $\dfrac{\sqrt{21}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{3}}{\sqrt{24}}$

4 Simplify: $\sqrt{3} - \sqrt{27}$

5 Write in simplest radical form: **a** $\sqrt{12}$ **b** $\sqrt{63}$

6 Expand and simplify:

    a $3(2-\sqrt{3})$ **b** $\sqrt{7}(\sqrt{2}-1)$ **c** $(3-\sqrt{2})^2$

    d $(\sqrt{3}+\sqrt{2})^2$ **e** $(2-\sqrt{5})(2+\sqrt{5})$ **f** $(2+\sqrt{3})(3-\sqrt{3})$

7 Write with integer denominator:

    a $\dfrac{24}{\sqrt{3}}$ **b** $\dfrac{1+\sqrt{2}}{2-\sqrt{2}}$ **c** $\dfrac{4-\sqrt{5}}{3+\sqrt{5}}$

8 Write in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

    a $\dfrac{18}{5-\sqrt{3}}$ **b** $\dfrac{-\sqrt{3}}{3+\sqrt{3}}$

Chapter 5

Sets and Venn diagrams

Contents:

OPENING PROBLEM

In a class of 30 students, 7 have black hair and 24 are right handed.   If 2 students neither have black hair nor are right handed, how many students:

a have both black hair and are right handed

b have black hair, but are not right handed?

A   SETS

A **set** is a collection of objects or things.

For example, the set of all factors of 12 is   {1, 2, 3, 4, 6, 12}.

Notice how we place the factors within curly brackets with commas between them.

We often use a capital letter to represent a set so that we can refer to it easily.

For example, we might let $F = \{1, 2, 3, 4, 6, 12\}$.   We can then say that 'F is the set of all factors of 12'.

Every object in a set is called an **element** or **member**.

SUBSETS

Suppose P and Q are two sets. P is a **subset** of Q if every element of P is also an element of Q.

Set notation:
- $\in$ reads *is an element of* or *is a member of* or *is in*
- $\notin$ reads *is not an element of* or *is not a member of* or *is not in*
- { } or $\varnothing$ is the symbol used to represent an **empty set** which has no elements or members. $\varnothing$ is called a **trivial subset**.
- $\subseteq$ reads *is a subset of*
- $n(S)$ reads *the number of elements in set S*

So, for the set $F = \{1, 2, 3, 4, 6, 12\}$ we can write:

$$4 \in F, \quad 7 \notin F, \quad \{2, 4, 6\} \subseteq F \quad \text{and} \quad n(F) = 6.$$

UNION AND INTERSECTION

If P and Q are two sets then:
- $P \cap Q$ is the **intersection** of P and Q and consists of all elements which are in **both** **P and Q**.
- $P \cup Q$ is the **union** of P and Q and consists of all elements which are in P **or** Q.

For example,   if $P = \{1, 2, 4, 5, 6, 8\}$ and $Q = \{0, 2, 3, 5, 6, 7\}$ then:

- $P \cap Q = \{2, 5, 6\}$ as 2, 5 and 6 are in **both** sets
- $P \cup Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ as these are the elements which are in P **or** Q.

DISJOINT SETS

Two sets are **disjoint** or **mutually exclusive** if they have no elements in common.
If P and Q are disjoint then $P \cap Q = \varnothing$.

Example 1   ◀) Self Tutor

For $A = \{2, 3, 5, 7, 11\}$ and $B = \{1, 3, 4, 6, 7, 8, 9\}$:
- **a** True or False: **i** $4 \in B$ **ii** $4 \notin A$?
- **b** List the sets: **i** $A \cap B$ **ii** $A \cup B$
- **c** Is **i** $A \cap B \subseteq A$ **ii** $\{3, 6, 10\} \subseteq B$?

- **a** **i** 4 is an element of set B, so $4 \in B$ is true.
  ii 4 is not an element of set A, so $4 \notin A$ is true.
- **b** **i** $A \cap B = \{3, 7\}$ since 3 and 7 are elements of both sets.
  ii Every element which is in either A or B is in the union of A and B.
  $\therefore$ $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$
- **c** **i** $A \cap B \subseteq A$ is true as every element of $A \cap B$ is also an element of A.
  ii $\{3, 6, 10\} \nsubseteq B$ as $10 \notin B$.

EXERCISE 5A

1 Write in set notation:
- **a** 7 is an element of set K
- **b** 6 is not an element of set M
- **c** $\{3, 4\}$ is a subset of $\{2, 3, 4\}$
- **d** $\{3, 4\}$ is not a subset of $\{1, 2, 4\}$.

2 Find **i** $A \cap B$ **ii** $A \cup B$ for:
- **a** $A = \{2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$
- **b** $A = \{2, 3, 4, 5\}$ and $B = \{6, 7, 8\}$
- **c** $A = \{2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$

3 Suppose $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$.
- **a** Find $A \cap B$.
- **b** Are A and B disjoint?

4 True or false:
- **a** $A \cap B \subseteq A$ and $A \cap B \subseteq B$ for any two sets A and B
- **b** $A \cap B \subseteq A \cup B$ for any two sets A and B?

5 For each of the following, is $R \subseteq S$?
- **a** $R = \varnothing$ and $S = \{1, 3, 5, 6\}$
- **b** $R = \{1, 2, 3, 4, 5\}$ and $S = \{1, 3, 5\}$

6   **a**   Explain why the empty set $\varnothing$ is always a subset of any other set.

  b   The subsets of $\{a,\ b\}$ are $\varnothing$, $\{a\}$, $\{b\}$ and $\{a,\ b\}$.   List the 8 subsets of $\{a,\ b,\ c\}$.

  c   How many subsets has a set containing n members?

B   SPECIAL NUMBER SETS

The following is a list of some special number sets you should be familiar with:

$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ ......\}$ is the set of all **natural** or **counting numbers**.

$\mathbb{Z} = \{0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ ......\}$ is the set of all **integers**.

$\mathbb{Z}^+ = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ ......\}$ is the set of all **positive integers**.

$\mathbb{Z}^- = \{-1,\ -2,\ -3,\ -4,\ -5,\ ......\}$ is the set of all **negative integers**.

$\mathbb{Q}$ is the set of all **rational numbers**, which are real numbers which can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

$\mathbb{Q}'$ is called the set of all **irrational numbers**. Numbers like $\sqrt{2}$, $\sqrt[3]{7}$ and π belong to $\mathbb{Q}'$.

$\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.

$$\xleftarrow{\hspace{4cm}} \underset{0}{|} \xrightarrow{\hspace{4cm}}$$

All of these sets have infinitely many elements and so are called **infinite sets**. If S is an infinite set we would write $n(S) = \infty$.

Example 2   ◀) **Self Tutor**

Show that $0.101\,010\,....$ is a rational number.

  Let $x = 0.101\,010\,....$
$\therefore \quad 100x = 10.101\,0.... = 10 + x$
$\therefore \quad 99x = 10$
$\therefore \quad x = \frac{10}{99}$

So, $0.101\,010....$ is actually the rational number $\frac{10}{99}$.

EXERCISE 5B

1   **a**   Explain why 4 and -7 are rational numbers.

  b   Explain why $\frac{3}{0}$ is not a rational number.

2   Show that these are rational numbers:

  a 0.6 **b** 0.13 **c** $1\frac{1}{3}$ **d** $-4\frac{1}{2}$ **e** $0.\overline{3}$

3   Explain why $\mathbb{Z}^- \cup \mathbb{Z}^+ \neq \mathbb{Z}$.

4 True or false:

  a $\mathbb{N} \subseteq \mathbb{Z}$ **b** $\mathbb{N} \cap \mathbb{Z}^- = \varnothing$ **c** $\mathbb{N} \cup \mathbb{Z}^- = \mathbb{Z}$

  d $\mathbb{Z} \subseteq \mathbb{Q}$ **e** $\mathbb{Z} \nsubseteq \mathbb{R}$ **f** $\mathbb{R} \subseteq \mathbb{Q}$

5 Find:

  a $\mathbb{Q} \cup \mathbb{Q}'$ **b** $\mathbb{Q} \cap \mathbb{Q}'$ **c** $\mathbb{Q}' \cap \mathbb{R}$ **d** $\mathbb{Q}' \cup \mathbb{R}$

C   SET BUILDER NOTATION

To describe the set of all integers between 3 and 8 we could list the set as $\{4, 5, 6, 7\}$ or we could use **set builder notation**.

This set could be written as: $\{x \mid 3 < x < 8, \ x \in \mathbb{Z}\}$

We read this as "the set of all integers x such that x lies between 3 and 8".

Set builder notation is very useful if the set contains a large number of elements and listing them would be time consuming and tedious.

The set of all real numbers between 3 and 8 would be written as $\{x \mid 3 < x < 8, \ x \in \mathbb{R}\}$

Open circles indicate that 3 and 8 are not included.
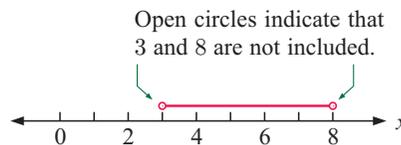
The set of all real numbers between 3 and 8 *inclusive* would be written as $\{x \mid 3 \leqslant x \leqslant 8, \ x \in \mathbb{R}\}$

Filled in circles indicate that 3 and 8 are included.
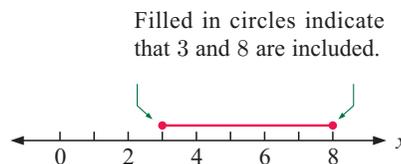
Example 3   ◀)) Self Tutor

Suppose $A = \{x \mid 0 < x \leqslant 7, \ x \in \mathbb{Z}\}$

  a Write down the meaning of the set builder notation.
  b List the elements of A.   **c** Find $n(A)$.
  d Illustrate A on a number line.

  a The set of all integers x such that x lies between 0 and 7, including 7.
  b $A = \{1, 2, 3, 4, 5, 6, 7\}$ **c** $n(A) = 7$

  d
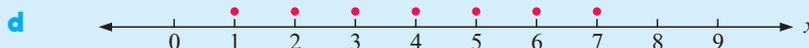
EXERCISE 5C

1 Are the following sets finite or infinite?

    a $\{x \mid -3 \leqslant x \leqslant 100, \; x \in \mathbb{Z}\}$ **b** $\{x \mid 1 \leqslant x \leqslant 2, \; x \in \mathbb{R}\}$

    c $\{x \mid x \geqslant 10, \; x \in \mathbb{Z}^+\}$ **d** $\{x \mid 0 \leqslant x \leqslant 1, \; x \in \mathbb{Q}\}$

2 For each of the following sets B:

    i write down the meaning of the set builder notation

    ii if possible list the elements of B

    iii find $n(B)$

    iv illustrate B on a number line.

    a $B = \{x \mid -3 \leqslant x \leqslant 4, \; x \in \mathbb{Z}\}$ **b** $B = \{x \mid -5 < x \leqslant -1, \; x \in \mathbb{N}\}$

    c $B = \{x \mid 2 < x < 3, \; x \in \mathbb{R}\}$ **d** $B = \{x \mid 1 \leqslant x \leqslant 2, \; x \in \mathbb{Q}\}$

3 Write in set builder notation:

    a the set of all integers between 100 and 300

    b the set of all real numbers greater than 50

    c the set of rational numbers between 7 and 8 inclusive.

4 For each of the following, is $C \subseteq D$?

    a $C = \{x \mid 0 \leqslant x \leqslant 10\,000, \; x \in \mathbb{R}\}$ and $D = \{x \mid x \in \mathbb{R}\}$

    b $C = \mathbb{Z}^+$ and $D = \mathbb{Z}$

    c $C = \{x \mid x \in \mathbb{Q}\}$ and $D = \{x \mid x \in \mathbb{Z}\}$

D   COMPLEMENT OF SETS

UNIVERSAL SETS

Suppose we are only interested in the single digit positive whole numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We would call this set our *universal set*.

> The **universal set** under consideration is represented by U.

If we are considering the possible results of rolling a die, the universal set would be $U = \{1, 2, 3, 4, 5, 6\}$.

COMPLEMENTARY SETS

If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$ then the *complementary set* of A is $A' = \{2, 4, 6, 8\}$.

> The **complement** of A, denoted A', is the set of all elements of U that are **not in** A.

Notice that:
- $A \cap A' = \varnothing$ {as A and A' have no members in common}
- $A \cup A' = U$ {all members of A and A' make up U}
- $n(A) + n(A') = n(U)$.

Example 4 ◀) **Self Tutor**

If $U = \{-3, -2, -1, 0, 1, 2, 3\}$, $A = \{-2, 0, 2, 3\}$ and $B = \{-3, -2, 1, 3\}$:
a list: **i** A' **ii** B' **iii** $A \cap B$ **iv** $A \cup B$ **v** $A \cap B'$ **vi** $A' \cup B'$
b find $n(A' \cup B')$.

---

a **i** $A' = \{-3, -1, 1\}$ {all elements of A not in U}
  ii $B' = \{-1, 0, 2\}$ {all elements of B not in U}
  iii $A \cap B = \{-2, 3\}$ {all elements common to A and B}
  iv $A \cup B = \{-3, -2, 0, 1, 2, 3\}$ {all elements in A or B or both}
  v $A \cap B' = \{-2, 0, 2, 3\} \cap \{-1, 0, 2\} = \{0, 2\}$
  vi $A' \cup B' = \{-3, -1, 1\} \cup \{-1, 0, 2\} = \{-3, -1, 0, 1, 2\}$
b $n(A' \cup B') = 5$ {as there are 5 members in this set}

EXERCISE 5D

1 For $U = \{2, 3, 4, 5, 6, 7\}$ and $B = \{2, 5, 7\}$:
  a list the set B' **b** check that $B \cap B' = \varnothing$ and that $B \cup B' = U$
  c check that $n(B) + n(B') = n(U)$.

2 Find D', the complement of D, given that:
  a $U = \{\text{integers}\}$ and $D = \{0\} \cup \mathbb{Z}$
  b $U = \mathbb{Z}^+$ and $D = \{\text{odd positive integers}\}$
  c $U = \mathbb{Z}$ and $D = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}\}$
  d $U = \mathbb{Q}$ and $D = \{x \mid x \leqslant 3 \ \text{or} \ x \geqslant 5, \ x \in \mathbb{Q}\}$

3 Suppose $U = \{0, 1, 2, 3, 4, 5, ..., 20\}$,
  $F = \{\text{factors of } 24\}$, and $M = \{\text{multiples of } 4\}$.
  a List the sets:
        i F **ii** M **iii** M'
        iv $F \cap M$ **v** $F \cup M$ **vi** $F \cap M'$
  b Find $n(F \cap M')$.

> The dots in U indicate that the integer list continues up to 20.

4 Suppose $U = \{x \mid 0 \leqslant x \leqslant 7, \ x \in \mathbb{Z}\}$, $A = \{0, 2, 4, 5\}$ and $B = \{2, 3, 5, 7\}$.
  a List the elements of:
        i U **ii** A' **iii** B' **iv** $A \cap B$ **v** $A \cup B$ **vi** $A' \cap B'$
  b Verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
  c Show that $(A \cup B)' = A' \cap B'$.

5 Suppose $U = \{x \mid 3 \leqslant x < 15,\ x \in \mathbb{Z}\}$, $R = \{x \mid 5 \leqslant x \leqslant 8,\ x \in \mathbb{Z}\}$, and
$T = \{4,\ 7,\ 10,\ 13\}$.

    a List the elements of:

        i U **ii** R **iii** R' **iv** T' **v** $R' \cup T$ **vi** $R \cap T'$

    b Find: **i** $n(R' \cup T)$ **ii** $n(R \cap T')$.

6 True or false?

    a If $n(U) = x$ and $n(B) = y$ then $n(B') = y - x$.

    b If $A \subseteq U$ then $A' = \{x \mid x \notin A,\ x \in U\}$

E   VENN DIAGRAMS

An alternative way of representing sets is to use a Venn diagram.

> A **Venn diagram** consists of a universal set U represented by a rectangle, and sets within it that are generally represented by circles.

For example:
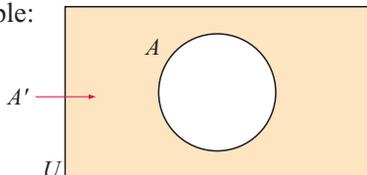
is a Venn diagram which shows set A within the universal set U.

A', the complement of A, is the shaded region outside the circle.

Suppose $U = \{1,\ 3,\ 4,\ 6,\ 9\}$, $A = \{1,\ 6,\ 9\}$ and $A' = (3,\ 4)$.

We can represent these sets by:
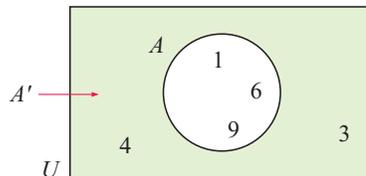
SUBSETS

If $B \subseteq A$ then every element of B is also in A.

The circle representing B is placed within the circle representing A.
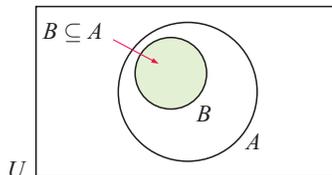
INTERSECTION

$A \cap B$ consists of all elements common to both A and B.

It is the shaded region where the circles representing A and B overlap.
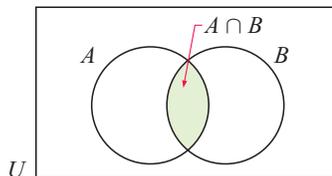
UNION

$A \cup B$ consists of all elements in A or B or both.

It is the shaded region which includes everywhere in either circle.
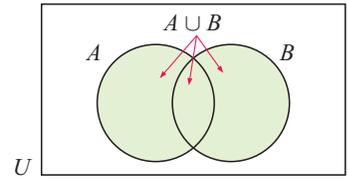
DISJOINT OR MUTUALLY EXCLUSIVE SETS

Disjoint sets do not have common elements.

They are represented by non-overlapping circles.
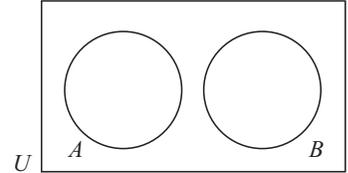
If the sets are disjoint and **exhaustive** then $B = A'$ and $A \cup B = U$.

We can represent this situation without using circles as shown.
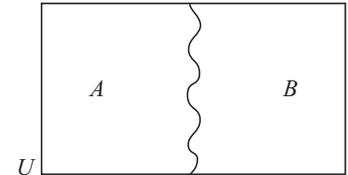
Example 5   ◀) **Self Tutor**

Suppose we are rolling a die, so the universal set $U = \{1, 2, 3, 4, 5, 6\}$.
Illustrate on a Venn diagram the sets:

  a $A = \{1, 2\}$ and $B = \{1, 3, 4\}$ **b** $A = \{1, 3, 5\}$ and $B = \{3, 5\}$
  c $A = \{2, 4, 6\}$ and $B = \{3, 5\}$ **d** $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

  a $A \cap B = \{1\}$
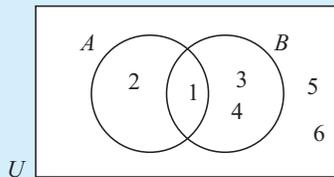
  b $A \cap B = \{3, 5\}$, $B \subseteq A$
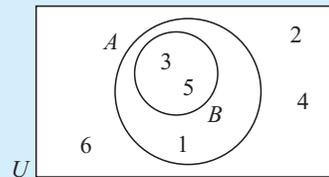
  c A and B are disjoint but
    $A \cup B \neq U$.
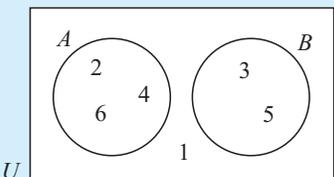
  d A and B are disjoint and exhaustive.
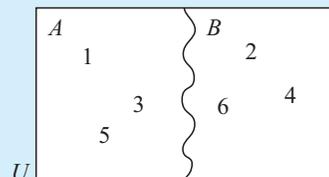
EXERCISE 5E.1

1 Consider the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.   Illustrate on a Venn diagram the sets:

    a $A = \{2, 3, 5, 7\}$ and $B = \{1, 2, 4, 6, 7, 8\}$

    b $A = \{2, 3, 5, 7\}$ and $B = \{4, 6, 8, 9\}$

    c $A = \{3, 4, 5, 6, 7, 8\}$ and $B = \{4, 6, 8\}$

    d $A = \{0, 1, 3, 7\}$ and $B = \{0, 1, 2, 3, 6, 7, 9\}$

2 Suppose $U = \{1, 2, 3, 4, 5, ..., 12\}$, $A = \{$factors of $8\}$ and $B = \{$primes $\leqslant 12\}$.

    a List the sets A and B.   **b** Find $A \cap B$ and $A \cup B$.

    c Represent A and B on a Venn diagram.

3 Suppose $U = \{x \mid x \leqslant 20, \ x \in \mathbb{Z}^+\}$, $R = \{$primes less than $20\}$ and $S = \{$composites less than $20\}$.

    a List the sets R and S.   **b** Find $R \cap S$ and $R \cup S$.

    c Represent R and S on a Venn diagram.

4

List the members of the set:

    a U **b** A **c** B **d** A'

    e B' **f** $A \cap B$ **g** $A \cup B$ **h** $(A \cup B)'$

VENN DIAGRAM REGIONS

We can use shading to show various sets. For example, for two intersecting sets, we have:

    A is shaded $A \cap B$ is shaded B' is shaded $A \cap B'$ is shaded

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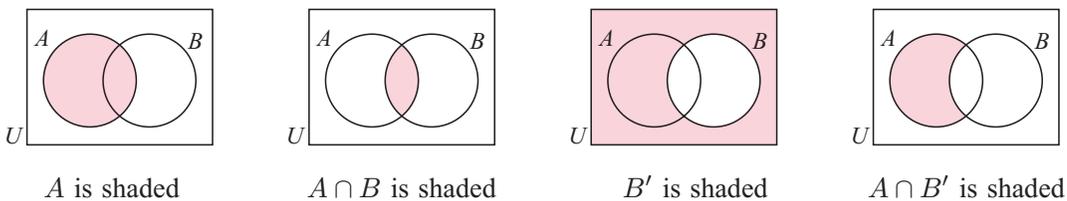
Example 6   ◀) **Self Tutor**

On separate Venn diagrams shade these regions for two overlapping sets A and B:

    a $A \cup B$ **b** $A' \cap B$ **c** $(A \cap B)'$

    $A \cup B$ means in A, $A' \cap B$ means outside $(A \cap B)'$ means outside

    B, or both. A intersected with B.   the intersection of A

                                                        and B.
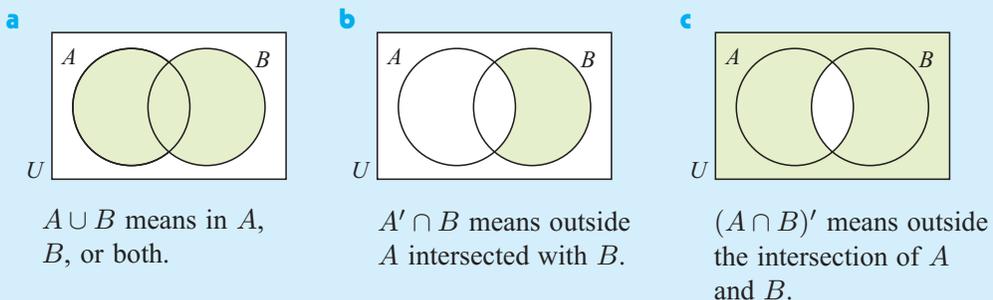
Click on the icon to **practise shading regions** representing various subsets. If you are correct you will be informed of this. The demonstration includes two and three intersecting sets.

DEMO

EXERCISE 5E.2

1
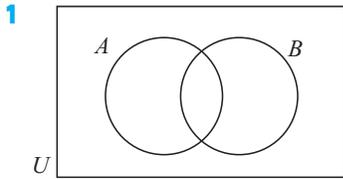
On separate Venn diagrams, shade:

a $A \cap B$ **b** $A \cap B'$

c $A' \cup B$ **d** $A \cup B'$

e $(A \cap B)'$ **f** $(A \cup B)'$

PRINTABLE VENN DIAGRAMS

2
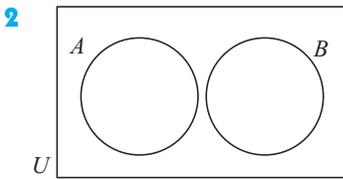
A and B are two disjoint sets. Shade on separate Venn diagrams:

a A **b** B **c** A'

d B' **e** $A \cap B$ **f** $A \cup B$

g $A' \cap B$ **h** $A \cup B'$ **i** $(A \cap B)'$

3
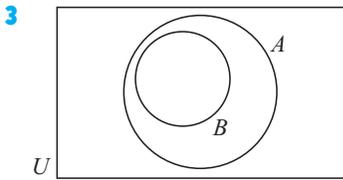
In the given Venn diagram, $B \subseteq A$.   Shade on separate Venn diagrams:

a A **b** B **c** A'

d B' **e** $A \cap B$ **f** $A \cup B$

g $A' \cap B$ **h** $A \cup B'$ **i** $(A \cap B)'$

NUMBERS IN REGIONS

Consider the Venn diagram for two intersecting sets A and B.

This Venn diagram has four regions:

is 'in A, but not in B'

is 'in B but not in A'

is 'in both A and B'

is 'neither in A nor in B'

Example 7

🔊 **Self Tutor**

If (5) means that there are 5 elements in the set $P \cap Q$, how many elements are there in:

a P
b Q'
c $P \cup Q$
d P, but not Q
e Q, but not P
f   neither P nor Q?

a $n(P) = 8 + 5 = 13$

b $n(Q') = 8 + 2 = 10$

c $n(P \cup Q) = 8 + 5 + 9$
$ = 22$

d $n(P, \text{ but not } Q) = 8$

e $n(Q, \text{ but not } P) = 9$

f $n(\text{neither } P \text{ nor } Q) = 2$

Example 8

🔊 **Self Tutor**

Given $n(U) = 40$, $n(A) = 24$, $n(B) = 27$ and $n(A \cap B) = 13$,   find:

a $n(A \cup B)$
b $n(A \cap B')$.

We see that $b = 13$ $\{$as $n(A \cap B) = 13\}$
$a + b = 24$ $\{$as $n(A) = 24\}$
$b + c = 27$ $\{$as $n(B) = 27\}$
$a + b + c + d = 40$ $\{$as $n(U) = 40\}$

Solving these, $b = 13$ $\therefore$ $a = 11$, $c = 14$, $d = 2$

a $n(A \cup B) = a + b + c$
$ = 38$

b $n(A \cap B') = a$
$ = 11$

EXERCISE 5E.3

1

If (3) means that there are 3 elements in the set $A \cap B$, give the number of elements in:

a B
b A'
c $A \cup B$
d A, but not B
e B, but not A
f   neither A nor B

2

Give the number of elements in:

a X'
b $X \cap Y$
c $X \cup Y$
d $X \cap Y'$
e $Y \cap X'$
f $X' \cap Y'$

3

(a) means that there are a elements in the shaded region.
Find:

a $n(Q)$ **b** $n(P')$ **c** $n(P \cap Q)$

d $n(P \cup Q)$ **e** $n((P \cap Q)')$ **f** $n((P \cup Q)')$

4
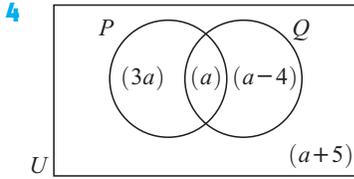
The Venn diagram shows us that $n(P \cap Q) = a$ and
$n(P) = a + 3a = 4a$.

a Find:

   i $n(Q)$ **ii** $n(P \cup Q)$

   iii $n(Q')$ **iv** $n(U)$

b Find a if $n(U) = 43$.

5
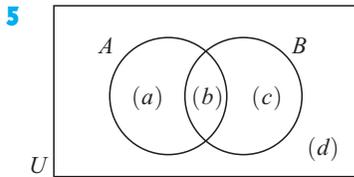
a For the given Venn diagram:

  i find $n(A \cup B)$

  ii find $n(A) + n(B) - n(A \cap B)$

b What have you proved in **a**?

6 Given $n(U) = 20$, $n(A) = 8$, $n(B) = 9$ and $n(A \cap B) = 2$, find:

  a $n(A \cup B)$ **b** $n(B \cap A')$

7 Given $n(U) = 35$, $n(N) = 16$, $n(N \cap R) = 4$, $n(N \cup R) = 31$, find:

  a $n(R)$ **b** $n((N \cup R)')$

8
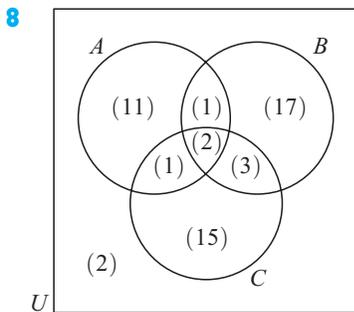
This Venn diagram contains the sets A, B and C. Find:

a $n(A)$ **b** $n(B)$

c $n(C)$ **d** $n(A \cap B)$

e $n(A \cup C)$ **f** $n(A \cap B \cap C)$

g $n(A \cup B \cup C)$ **h** $n((A \cup B) \cap C)$

PRINTABLE PAGE

9 On separate Venn diagrams, shade

  a A **b** B **c** C'

  d $A \cup B$ **e** $B \cap C$ **f** $A \cap B \cap C$

  g $(A \cup B)'$ **h** $A' \cup (B \cap C)$
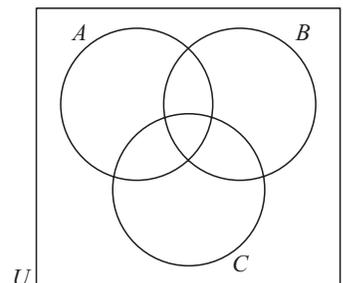
Example 9   ◀) **Self Tutor**

A tennis club has 42 members. 25 have fair hair, 19 have blue eyes and 10 have both fair hair and blue eyes.

   a Place this information on a Venn diagram.

   b Find the number of members with:

      i fair hair or blue eyes   **ii** blue eyes, but not fair hair.

   a Let F represent the fair hair set and B represent the blue eyes set.
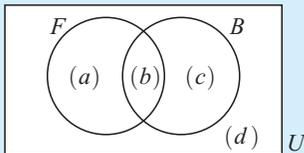
$$a + b + c + d = 42$$
$$a + b = 25$$
$$b + c = 19$$
$$b = 10$$
$$\therefore \quad a = 15, \quad c = 9, \quad d = 8$$
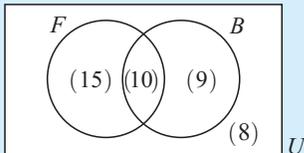
   b **i** $n(F \cup B) = 15 + 10 + 9$
$$= 34$$

      ii $n(B \cap F') = 9$

10 In James' apartment block there are 27 apartments with a dog, 33 with a cat, and 17 with a dog *and* a cat. 35 apartments have neither a cat nor a dog.

   a Place the information on a Venn diagram.

   b How many apartments are there in the block?

   c How many apartments contain:

      i a dog but not a cat   **ii** a cat but not a dog

      iii no dogs   **iv** no cats?

11 A riding club has 28 riders, 15 of whom ride dressage and 24 of whom showjump. All but 2 of the riders do at least one of these disciplines. How many members:

   a ride dressage only   **b** only showjump   **c** ride dressage *and* showjump?

12 46% of people in a town ride a bicycle and 45% ride a motor scooter. 16% ride neither a bicycle nor a scooter.

   a Illustrate this information on a Venn diagram.

   b How many people ride:

      i both a bicycle and a scooter   **ii** at least one of a bicycle or a scooter

      iii a bicycle only   **iv** exactly one of a bicycle or a scooter?

13 A bookstore sells books, magazines and newspapers. Their sales records indicate that 40% of customers buy books, 43% buy magazines, 40% buy newspapers, 11% buy books and magazines, 9% buy magazines and newspapers, 7% buy books and newspapers, and 4% buy all three.

a Illustrate this information on a Venn diagram like the one shown.

b What percentage of customers buy:
   i books only
   ii books or newspapers
   iii books but not magazines?
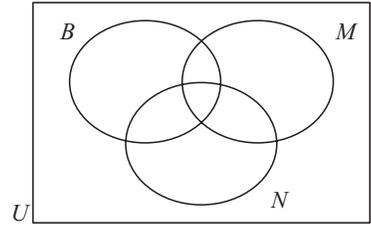
REVIEW SET 5A

1 Consider $A = \{x \mid 3 < x \leqslant 8, \ x \in \mathbb{Z}\}$.

    a List the elements of A.   **b** Is $3 \in A$?   **c** Find $n(A)$.

2 Write in set builder notation:

    a the set of all integers greater than 10

    b the set of all rationals between -2 and 3.

3 If $M = \{1 \leqslant x \leqslant 9, \ x \in \mathbb{Z}^+\}$, find all subsets of M whose elements when multiplied give a result of 35.

4 If $U = \{x \mid -5 \leqslant x \leqslant 5, \ x \in \mathbb{Z}\}$, $A = \{-2, 0, 1, 3, 5\}$ and $B = \{-4, -1, 0, 2, 3, 4\}$, list the elements of:

    a A' **b** $A \cap B$ **c** $A \cup B$ **d** $A' \cap B$ **e** $A \cup B'$

5
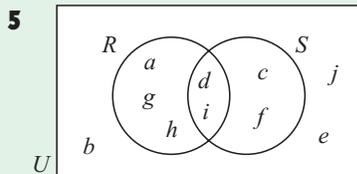
    a List the sets:
        i R **ii** S **iii** S'
        iv $R \cap S$ **v** $R \cup S$ **vi** $(R \cup S)'$

    b Find $n(R \cup S)$.

6 In the swimming pool, 4 people can swim butterfly and 11 can swim freestyle. The people who can swim butterfly are a subset of those who can swim freestyle. There are 15 people in the pool in total.

    a Display this information on a Venn diagram.

    b Hence, find the number of people who can swim:
        i freestyle but not butterfly   **ii** neither stroke.

7
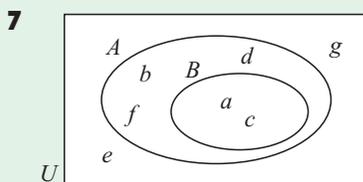
    a List the members of set:
        i U **ii** A **iii** B

    b True or false?
        i $A \subseteq B$ **ii** $A \cap B = B$
        iii $A \cup B = B$ **iv** $d \notin A$

8 The numbers in the brackets indicate the number of elements in that region of the Venn diagram.

What is the greatest possible value of $n(U)$?
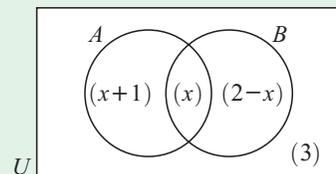
REVIEW SET 5B

1 Consider $B = \{y \mid 1 \leqslant y \leqslant 7, \ y \in \mathbb{Z}\}$.

    a Is B a finite or infinite set?   **b** Find $n(B)$.

2 List all subsets of the set $\{p, q, r\}$.

3 Suppose $U = \mathbb{Z}^+$, $F = \{\text{factors of } 36\}$ and $M = \{\text{multiples of 3 less than } 36\}$.

    a List the sets F and M.

    b Is $F \subseteq M$?

    c List the sets: **i** $F \cap M$ **ii** $F \cup M$

    d Verify that $n(F \cup M) = n(F) + n(M) - n(F \cap M)$.

4   **a** Write in set builder notation: "S is the set of all rational numbers between 0 and 3."

    b Which of the following numbers belong to S?

        i 3 **ii** $1\frac{1}{3}$ **iii** $\frac{2}{0}$ **iv** $1.\overline{3}$ **v** $\sqrt{2}$

5 Suppose $U = \{x \mid -8 < x < 0, \ x \in \mathbb{Z}^-\}$, $A = \{-7, -5, -3, -1\}$ and $B = \{-6, -4, -2\}$.

    a Write in simplest form: **i** $A \cap B$ **ii** $A \cup B$ **iii** A'

    b Illustrate A, B and U on a Venn diagram.

6
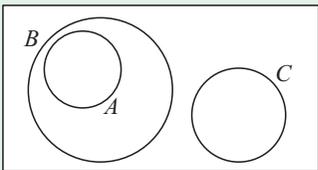
    a What is the relationship between sets:

        i A and B **ii** B and C **iii** A and C?

    b Copy and shade on separate Venn diagrams:

        i $A \cap B$ **ii** $A \cup C$

7 At a youth camp, 37 youths participated in at least one of canoeing and archery. If 24 went canoeing and 22 did archery, how many participated in both of these activities?

8 In the Venn diagram shown, $n(U) = 40$, $n(A) = 20$, $n(B) = 17$, and $n(A \cap B) = \frac{1}{2}n(A' \cap B')$. Find $n(A \cap B)$.
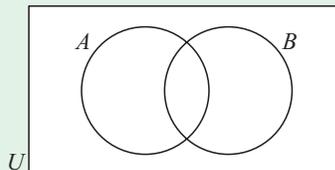
Chapter 6

Coordinate geometry

Contents:

THE 2-DIMENSIONAL COORDINATE SYSTEM

The position of any point in the **Cartesian** or **number plane** can be described by an **ordered pair** of numbers (x, y).

These numbers tell us the steps we need to take from the origin O to get to the required point.

x is the **horizontal step** from O, and is the **x-coordinate** of the point.

y is the **vertical step** from O, and is the **y-coordinate** of the point.

For example:
- to locate the point A(4, 2), we start at the origin O, move 4 units along the x-axis in the positive direction, and then 2 units in the positive y-direction
- to locate the point B(-2, 3), we start at O, move 2 units in the negative x-direction, and then 3 units in the positive y-direction.
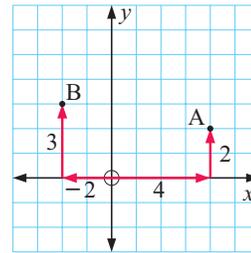
NOTATION

Given two points A and B in the number plane:
- (AB) is the **infinite line** passing through A and B
- [AB] is the **line segment** from A to B
- AB is the **distance** from A to B.

OPENING PROBLEM

Cyril and Hilda live in outback Australia. Both have cattle stations and each owns a small aeroplane. They both have a map of the region showing grid lines and an origin at Kimberley. Each unit on the grid corresponds to 1 km. Cyril's airstrip is at C(223, 178) and Hilda's is at H(-114, -281).
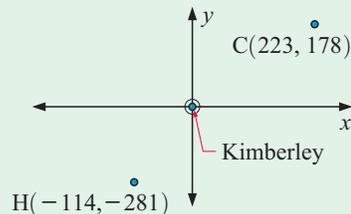
Can you find:
- **a** whose airstrip is closer to Kimberley
- **b** the distance between the airstrips at C and H
- **c** the position of the communications tower midway between C and H?

RESEARCH   RENÉ DESCARTES

Research the contributions to mathematics made by the French mathematician **René Descartes**. In particular, consider his work associated with the 2-dimensional number plane.

The library and internet may be appropriate sources of material.

Summarise your findings in no more than 300 words.

René Descartes

A   THE DISTANCE BETWEEN TWO POINTS

Consider finding the distance from A(2, 4) to B(5, 1). Let this distance be d units.

We plot the points on a set of axes, then join [AB] with a straight line. We then construct a right angled triangle, as shown.
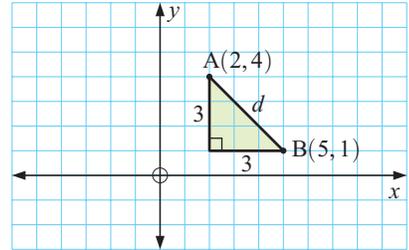
Notice that $d^2 = 3^2 + 3^2$ {Pythagoras}

$\quad \therefore \quad d^2 = 18$

$\quad \therefore \quad d = \sqrt{18} \qquad$ {as $\ d > 0$}

$\quad \therefore \quad d = 3\sqrt{2}$

So, the distance from A to B is $3\sqrt{2}$ units. We write this as $AB = 3\sqrt{2}$ units.

EXERCISE 6A.1

1 If necessary, use the theorem of Pythagoras to find the distance between:

    a A and B   **b** B and C

    c C and D   **d** A and C

    e B and D   **f** O and A.
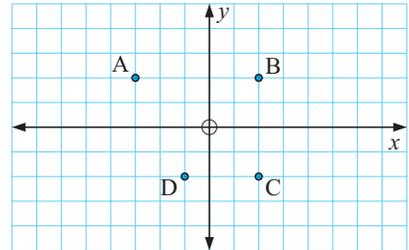
2 By plotting points and using the theorem of Pythagoras, find the distance between:

    a O(0, 0) and M(3, 5)   **b** C(1, 2) and D(3, 7)

    c A(-1, -2) and B(2, 3)   **d** A(1, 4) and B(2, -1)

    e P(3, 5) and Q(-1, 4)   **f** R(-2, 0) and S(0, 3).

THE DISTANCE FORMULA

Instead of graphing points and using the theorem of Pythagoras, we establish the **distance formula**.

Let A be at (x_1, y_1) and B be at (x_2, y_2).

In going from A to B,   the x-step $= x_2 - x_1$

$\qquad\qquad\qquad$ and   the y-step $= y_2 - y_1$.
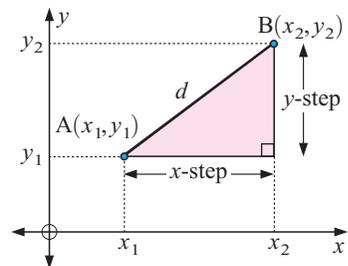
Using Pythagoras' theorem, $d^2 = (x\text{-step})^2 + (y\text{-step})^2$

$\qquad \therefore \quad d = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}$

$\qquad \therefore \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a number plane then the distance between them is given by:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{or} \quad AB = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}.$$

Example 1   ◄ᴏ Self Tutor

Find the distance from $A(-3, 2)$ to $B(2, 4)$.

$A(-3, 2) \quad B(2, 4)$
$\quad x_1\ y_1 \qquad x_2\ y_2$

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - -3)^2 + (4 - 2)^2} \\ &= \sqrt{5^2 + 2^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

We can use the distance formula to help us classify triangles as scalene, isosceles, equilateral or right angled.

To establish that a triangle is **right angled**, find the lengths of its sides and then use the **converse of the theorem of Pythagoras**.

If a triangle has sides a, b and c where $\quad a^2 + b^2 = c^2 \quad$ then the triangle is right angled. The right angle will be opposite the longest side c.

For example, if $PQ = \sqrt{13}$, $QR = \sqrt{14}$ and $PR = \sqrt{27}$
then $(PR)^2 = (PQ)^2 + (QR)^2$

So, triangle PQR is right angled with the right angle at Q.

Example 2   ◄ᴏ Self Tutor

Classify the triangle with vertices $A(1, 2)$, $B(3, 5)$, and $C(0, 3)$.

$$\begin{aligned} AB &= \sqrt{(3 - 1)^2 + (5 - 2)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(0 - 3)^2 + (3 - 5)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(0 - 1)^2 + (3 - 2)^2} \\ &= \sqrt{1 + 1} \\ &= \sqrt{2} \text{ units} \end{aligned}$$

Since $AB = BC$, triangle ABC is an isosceles triangle.
The triangle is not right angled.

Sometimes we may know a distance between two points and need to find an unknown coordinate.

Example 3   ◀) **Self Tutor**

Find a if $P(2, -1)$ and $Q(a, 3)$ are $2\sqrt{5}$ units apart.

Since $PQ = 2\sqrt{5}$,

$$\sqrt{(a - 2)^2 + (3 - -1)^2} = 2\sqrt{5}$$
$$\therefore \quad \sqrt{(a - 2)^2 + 16} = 2\sqrt{5}$$
$$\therefore \quad (a - 2)^2 + 16 = 20 \qquad \text{\{squaring both sides\}}$$
$$\therefore \quad (a - 2)^2 = 4$$
$$\therefore \quad a - 2 = \pm 2 \qquad \text{\{if } x^2 = k \text{ then } x = \pm\sqrt{k}\}$$
$$\therefore \quad a = 2 \pm 2$$
$$\therefore \quad a = 4 \text{ or } 0$$

Note: There are *two* answers in the above example. This is because Q lies on the horizontal line passing through $(0, 3)$. We can see from the graph why there must be two possible answers.
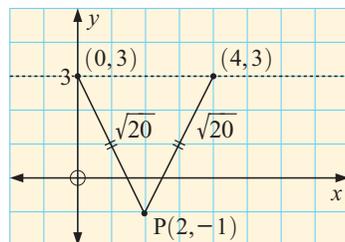
EXERCISE 6A.2

1 Find the distance between these points using the distance formula:

    a $O(0, 0)$ and $P(3, -1)$ **b** $A(2, 1)$ and $B(4, 4)$

    c $C(-2, 1)$ and $D(2, 5)$ **d** $E(4, 3)$ and $F(-1, -1)$

    e $G(0, -3)$ and $H(1, 4)$ **f** $I(0, 2)$ and $J(-3, 0)$

    g $K(2, 1)$ and $L(3, -2)$ **h** $M(-2, 5)$ and $N(-4, -1)$

    i $P(3, 9)$ and $Q(11, -1)$ **j** $R(-\sqrt{2}, 3)$ and $S(\sqrt{2}, -1)$.

2 Triangles can be constructed from the following sets of three points. By finding side lengths, classify each triangle as scalene, isosceles or equilateral:

    a $A(0, -2)$, $B(1, 2)$ and $C(-3, 1)$ **b** $X(1, 3)$, $Y(-1, 0)$ and $Z(3, -4)$

    c $P(0, \sqrt{2})$, $Q(\sqrt{6}, 0)$ and $R(0, -\sqrt{2})$ **d** $E(7, 1)$, $F(-1, 4)$ and $G(2, -1)$

    e $H(3, -2)$, $I(1, 4)$ and $J(-3, 0)$ **f** $W(2\sqrt{3}, 0)$, $X(0, 6)$ and $Y(-2\sqrt{3}, 0)$.

3 Triangle PQR has vertices $P(2, 1)$, $Q(5, 2)$ and $R(1, 4)$.

    a Find the lengths of [PQ], [QR] and [PR].

    b Is triangle PQR right angled, and if so, at what vertex is the right angle?

4 Show that A, B and C are the vertices of a right angled triangle, and in each case state the angle which is the right angle:

    a A(2, −1), B(2, 4), C(−5, 4)   **b** A(−4, 3), B(−5, −2), C(1, 2)

5 Consider triangle PQR with vertices P(1, 3), Q(−1, 0) and R(2, −2).
Classify the triangle using the lengths of its sides.

6 Find the value of the unknown if:

    a M(3, 2) and N(−1, a) are 4 units apart.

    b R(1, −1) is 5 units from S(−2, b)

    c R(3, c) and S(6, −1) are $3\sqrt{2}$ units apart.

B   MIDPOINTS

The **midpoint** of line segment [AB] is the point which lies midway between points A and B.
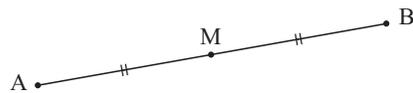
Consider the points A(1, −2) and B(4, 3). From the graph we see that the midpoint of [AB] is M($2\frac{1}{2}$, $\frac{1}{2}$).

Notice that:

the x-coordinate of M $= \dfrac{1+4}{2} = \dfrac{5}{2}$

the y-coordinate of M $= \dfrac{-2+3}{2} = \dfrac{1}{2}$.
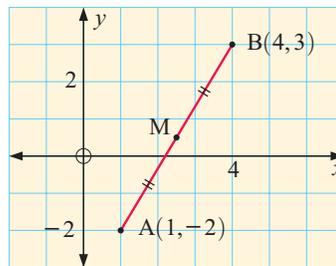
THE MIDPOINT FORMULA

If A(x_1, y_1) and B(x_2, y_2) are two points then the **midpoint** M of [AB] has coordinates

$$\left(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2} \right).$$

Example 4   ◀) **Self Tutor**

Use the midpoint formula to find the midpoint of [AB] for A(−2, 3) and B(4, 8).

A(−2, 3)   B(4, 8)   The midpoint is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

  $x_1 \ y_1$ $x_2 \ y_2$ which is $\left(\dfrac{-2+4}{2}, \dfrac{3+8}{2} \right)$ or $\left(\dfrac{2}{2}, \dfrac{11}{2} \right)$.

                                So, the midpoint is $(1, 5\frac{1}{2})$.

Example 5

🔊 **Self Tutor**

M is the midpoint of [PQ].
P is at $(3, 1)$ and M is at $(1, -3)$.
Find the coordinates of Q.

P$(3,1)$

M$(1,-3)$

Q

Suppose Q is at (a, b).

P$(3,1)$

M$(1,-3)$
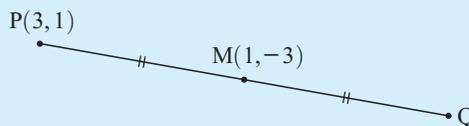
Q(a,b)
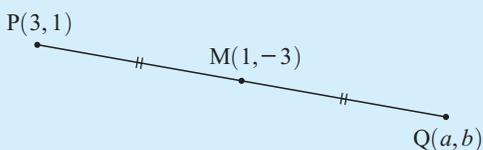
$\therefore$ M is $\left(\dfrac{3+a}{2}, \dfrac{1+b}{2}\right)$

But M is $(1, -3)$

$\therefore \quad \dfrac{3+a}{2} = 1 \quad$ and $\quad \dfrac{1+b}{2} = -3$

$\therefore \quad 3 + a = 2 \quad$ and $\quad 1 + b = -6$

$\therefore \quad a = -1 \quad$ and $\quad b = -7$

So, Q is at $(-1, -7)$.

EXERCISE 6B

1 Using the diagram *only*, find the coordinates of the midpoint of:

  a [BC]   **b** [AB]   **c** [CD]

  d [ED]   **e** [HE]   **f** [GE]

  g [BH]   **h** [AE]   **i** [GD]
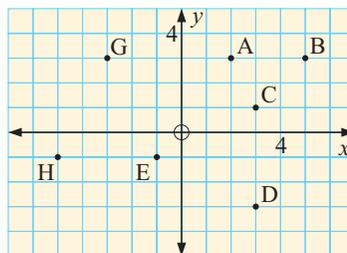
2 Use the midpoint formula to find the coordinates of the midpoint of [AB] given:

  a A$(5, 2)$ and B$(1, 8)$ **b** A$(4, 1)$ and B$(4, -1)$

  c A$(1, 0)$ and B$(-3, 2)$ **d** A$(6, 0)$ and B$(0, 3)$

  e A$(0, -1)$ and B$(-3, 5)$ **f** A$(-2, 4)$ and B$(4, -2)$

  g A$(-4, 3)$ and B$(9, 5)$ **h** A$(-5, 1)$ and B$(-2, 3)$

  i A$(a, 5)$ and B$(2, -1)$ **j** A(a, b) and B$(-a, 3b)$.

3 Suppose M is the midpoint of [PQ]. Find the coordinates of Q given:

  a P$(-1, 3)$ and M$(-1, 7)$ **b** P$(-1, 0)$ and M$(0, -5)$

  c M$(2, 1\frac{1}{2})$ and P$(-2, 3)$ **d** M$(-1, 3)$ and P$(-\frac{1}{2}, 0)$.

4 Suppose T is the midpoint of [AB]. Find the coordinates of A given:

  a T$(4, -3)$ and B$(-2, 3)$ **b** B$(0, 4)$ and T$(-3, -2)$.

5 Find the coordinates of:

  a B   **b** D

A$(1,3)$ B   C$(-1,7)$ D

6   A circle has diameter [AB]. If the centre of the circle is at $(1, 3)$ and B has coordinates $(4, -1)$, find:

    a   the coordinates of A

    b   the length of the diameter.

7
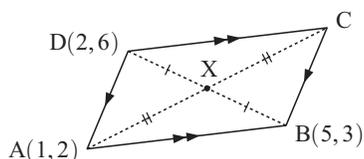
ABCD is a parallelogram. Diagonals [AC] and [BD] bisect each other at X. Find:

    a   the coordinates of X

    b   the coordinates of C.

8   In the triangle ABC, M is the midpoint of [AB] and N is the midpoint of [CM].
Find the coordinates of N.
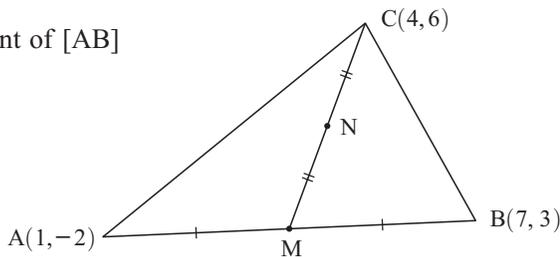
C   GRADIENT (OR SLOPE)

When looking at line segments drawn on a set of axes we notice that different line segments are inclined to the horizontal at different angles. Some appear *steeper* than others.

> The **gradient** or **slope** of a line is a measure of its steepness.

If we choose any two distinct (different) points on the line, a **horizontal step** and a **vertical step** may be determined.

Case 1:
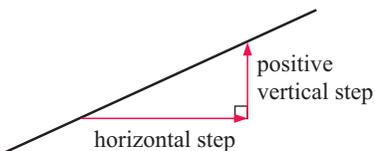
Case 2:
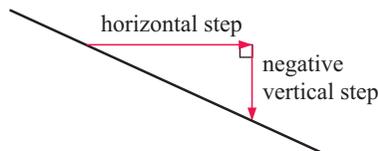
To measure the steepness of a line, we use

$$\textbf{gradient} = \frac{\textbf{vertical step}}{\textbf{horizontal step}} \quad \text{or} \quad \frac{\textbf{\textit{y}-step}}{\textbf{\textit{x}-step}}$$

In *Case 1*, both steps are positive, so   has a **positive** gradient.

In *Case 2*, the steps have opposite signs, so   has a **negative** gradient.

DISCUSSION

Why is the gradient formula $\dfrac{y\text{-step}}{x\text{-step}}$ and not $\dfrac{x\text{-step}}{y\text{-step}}$?

Example 6 ◀)) Self Tutor

Find the gradient of each line segment:

a   b   c   d

a
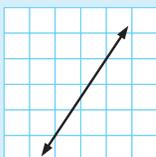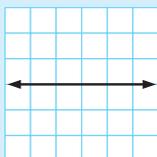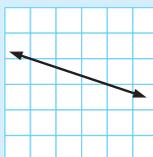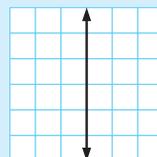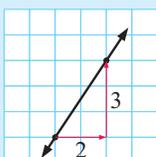
$$\text{gradient}$$
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{3}{2}$$

b
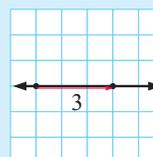
$$\text{gradient}$$
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{0}{3}$$
$$= 0$$

c
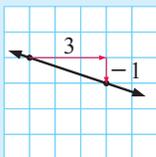
$$\text{gradient}$$
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{-1}{3}$$
$$= -\frac{1}{3}$$
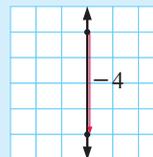
d

$$\text{gradient}$$
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{-4}{0}$$

which is undefined

Notice that:

- the gradient of all **horizontal** lines is **0**
- the gradient of all **vertical** lines is **undefined**.

Example 7 ◀)) Self Tutor

Draw lines with slope $\frac{3}{4}$ and -2 through the point $(2, 1)$.

$$\text{slope} = \frac{3}{4} \begin{array}{l} \leftarrow y\text{-step} \\ \leftarrow x\text{-step} \end{array}$$
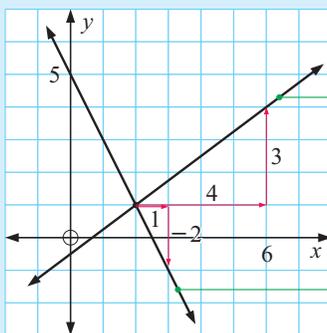
$$\text{slope} = -2$$
$$= \frac{-2}{1} \begin{array}{l} \leftarrow y\text{-step} \\ \leftarrow x\text{-step} \end{array}$$

EXERCISE 6C.1

1 Find the gradient of each line segment:

a

b

c

d

e

f

g

h
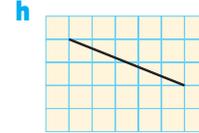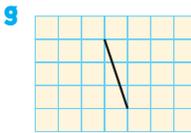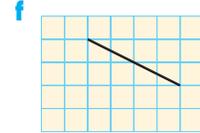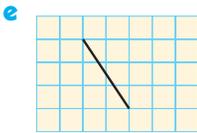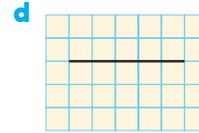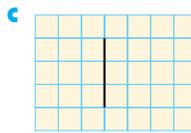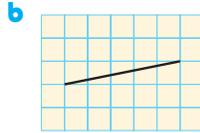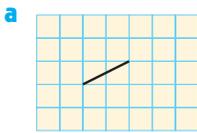
2 On grid paper, draw a line segment with gradient:

a $\frac{1}{2}$ **b** $\frac{2}{3}$ **c** 3   **d** $-\frac{2}{3}$ **e** -2 **f** $-\frac{4}{5}$ **g** 0

3 On the same set of axes, draw lines through $(0, 0)$ with gradients:

a $0, \frac{1}{3}, \frac{3}{4}, 1, \frac{3}{2}, \frac{5}{3}$ and 5   **b** $0, -\frac{1}{2}, -1, -\frac{3}{4}$ and -4.

THE GRADIENT FORMULA

If A is (x_1, y_1) and B is (x_2, y_2) then the **gradient** of [AB] is $\dfrac{y_2 - y_1}{x_2 - x_1}$.
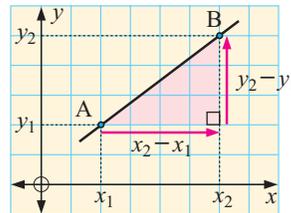
Example 8 ◀) **Self Tutor**

Find the gradient of the line through $(2, -4)$ and $(-1, 1)$.

A$(2, -4)$ B$(-1, 1)$ gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$x_1\ y_1 x_2\ y_2$

$= \dfrac{1 - -4}{-1 - 2}$

$= \dfrac{5}{-3}$

$= -\dfrac{5}{3}$

PARALLEL LINES

If [AB] is parallel to [CD] then we write   [AB] ∥ [CD].

We notice that:

- if two lines are **parallel** then they have **equal gradient**
- if two lines have **equal gradient** then they are **parallel**.
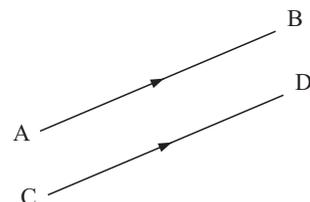
PERPENDICULAR LINES

Line (1) and line (2) are perpendicular.

Line (1) has gradient $\frac{3}{2}$.

Line (2) has gradient $-\frac{2}{3}$.

Notice that the gradient of line (2) is the negative reciprocal of the gradient of line (1).
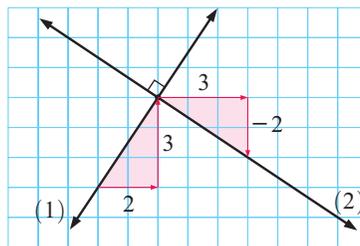
For two lines which are not horizontal or vertical:
- if the lines are perpendicular then their gradients are negative reciprocals
- if the lines have gradients which are negative reciprocals, then the lines are perpendicular.

If [AB] is perpendicular to [CD] then we write [AB] $\perp$ [CD].

EXERCISE 6C.2

1 Use the gradient formula to find the gradient of the line segment connecting:

  a $(4, 7)$ and $(3, 2)$ **b** $(6, 1)$ and $(7, 5)$ **c** $(6, 3)$ and $(-2, 1)$

  d $(0, 0)$ and $(4, -3)$ **e** $(5, -1)$ and $(5, 5)$ **f** $(-4, 3)$ and $(-1, 3)$

  g $(-3, -2)$ and $(-1, 5)$ **h** $(0, 2)$ and $(2, -5)$ **i** $(-2, -2)$ and $(-1, 0)$.

2 Find the gradient of a line which is perpendicular to a line with gradient:

  a $\frac{1}{3}$ **b** $-\frac{3}{7}$ **c** 2 **d** -11 **e** 0 **f** undefined   **g** $-1\frac{1}{2}$

3 A line has gradient $\frac{4}{5}$. Find the gradients of all lines which are:

  a parallel to it   **b** perpendicular to it.

4 A line has slope $\dfrac{a}{3}$. Find a given that the line is:

  a parallel to a line with slope $\frac{5}{6}$ **b** parallel to a line with slope $-\frac{2}{9}$

  c perpendicular to a line with slope 6 **d** perpendicular to a line with slope $\frac{6}{7}$.

5 A line has slope $\dfrac{4}{b}$. Find b given that the line is:

  a parallel to a line with slope -8 **b** perpendicular to a line with slope $\frac{3}{4}$.

6 $A(3, 1)$, $B(2, -4)$ and $C(7, -5)$ are three points in the Cartesian plane.

  a Find the gradient of: **i** [AB]   **ii** [BC].

  b What can be said about [AB] and [BC]?

  c Classify triangle ABC.

7 A line passes through points $A(1, 4)$ and $B(4, a)$.

  a Find the gradient of [AB].

  b Find a if [AB] is parallel to a line with gradient $\frac{1}{2}$

  c Find a if [AB] is perpendicular to a line with gradient -3.

8   A line passes through points P(-1, 5) and Q(b, 3).

    a   Find the gradient of [PQ].

    b   Find b if [PQ] is:

        i   parallel to a line with gradient $-\frac{2}{3}$

        ii   perpendicular to a line with gradient $\frac{4}{3}$.

9   For A(4, 1), B(0, -1), C(a, 2) and D(-1, 3), find a if:

    a   [AB] is parallel to [CD]　　　　　　**b**   [BC] is perpendicular to [AD].

D   USING GRADIENTS

In the previous exercise we considered the gradients of straight lines and the gradients between points. In real life gradients occur in many situations and have different meanings.

For example, the sign alongside would indicate to drivers that there is an uphill climb or uphill *gradient* ahead.

The graph alongside shows a car journey. The car travels at a constant speed for 8 hours, travelling a distance of 600 km.

Clearly, the gradient of the line $= \dfrac{\text{vertical step}}{\text{horizontal step}}$

$$= \dfrac{600}{8}$$

$$= 75$$
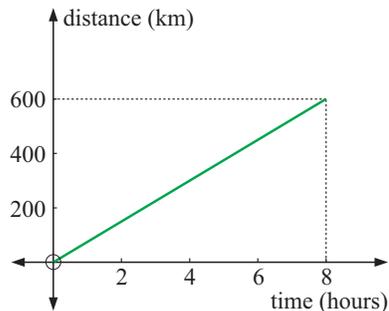
However, $\text{speed} = \dfrac{\text{distance}}{\text{time}} = \dfrac{600 \text{ km}}{8 \text{ hours}} = 75 \text{ km h}^{-1}.$

In a graph of distance against time, the *gradient* can be interpreted as the *speed*.

In general, gradients are a measure of the **rate of change** in one variable compared to another.

EXERCISE 6D

1
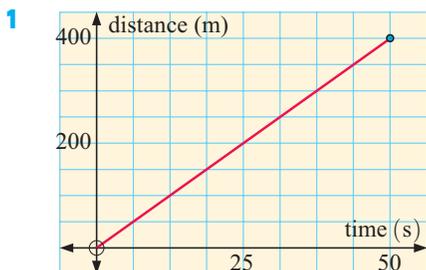
The graph alongside indicates the distance run by a sprinter in a number of seconds.

    a   Find the gradient of the line.

    b   Interpret the gradient found in **a**.

    c   Is the speed of the runner constant or variable? What evidence do you have for your answer?

2 The graph alongside indicates distances travelled by a truck driver. Determine:

  a the average speed for the whole trip

  b the average speed from

   i O to A   **ii** B to C

  c the time interval over which the average speed is greatest.

3
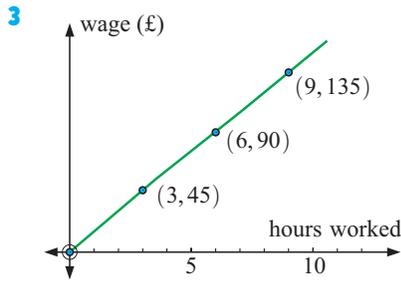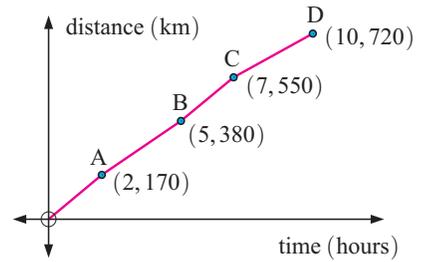
The graph alongside indicates the wages paid to sales assistants.

  a What does the intercept on the vertical axis mean?

  b Find the gradient of the line.   What does this gradient mean?

  c Determine the wages for working:

   i 6 hours   **ii** 18 hours.

4 The graphs alongside indicate the fuel consumption and distance travelled at speeds of 60 km h^{-1} (graph A) and 90 km h^{-1} (graph B).

  a Find the gradient of each line.

  b What do these gradients mean?

  c If fuel costs $1.24 per litre, how much more would it cost to travel 1000 km at 90 km h^{-1} than at 60 km h^{-1}?

5
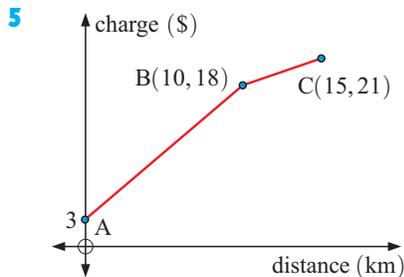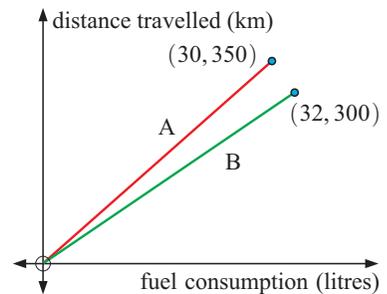
The graph alongside indicates the courier charge for carrying a parcel different distances.

  a What does the value at A indicate?

  b Find the gradients of the line segments [AB] and [BC]. What do these gradients indicate?

  c If a straight line segment was drawn from A to C, find its gradient. What would this gradient mean?

E   USING COORDINATE GEOMETRY

Coordinate geometry is a powerful tool which can be used:

- to **check** the truth of a geometrical fact
- to **prove** a geometrical fact by using general cases.

Example 9

Given the points A(1, 0), B(2, 4) and C(5, 1):

  a show that triangle ABC is isosceles

  b find the midpoint M of [BC]

  c use gradients to verify that [AM] and [BC] are perpendicular.

  d Illustrate **a**, **b** and **c** on a set of axes.

a

AB	BC	AC
$= \sqrt{(2-1)^2 + (4-0)^2}$	$= \sqrt{(5-2)^2 + (1-4)^2}$	$= \sqrt{(5-1)^2 + (1-0)^2}$
$= \sqrt{1+16}$	$= \sqrt{9+9}$	$= \sqrt{16+1}$
$= \sqrt{17}$ units	$= \sqrt{18}$ units	$= \sqrt{17}$ units

Since   AB $=$ AC, $\triangle$ABC is isosceles.

b M is $\left(\dfrac{2+5}{2}, \dfrac{4+1}{2}\right)$ which is $\left(\tfrac{7}{2}, \tfrac{5}{2}\right)$.

d

c gradient of [AM] $= \dfrac{\frac{5}{2}-0}{\frac{7}{2}-1} = \dfrac{\frac{5}{2}}{\frac{5}{2}} = 1$

gradient of [BC] $= \dfrac{1-4}{5-2} = -\dfrac{3}{3} = -1$

Since the gradients are negative reciprocals,
[AM] $\perp$ [BC].
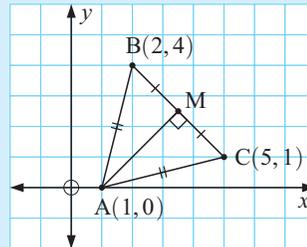
EXERCISE 6E

1 Given the points A(-1, 0), B(1, 4) and C(3, 2):

  a show that triangle ABC is isosceles

  b find the midpoint M of [BC]

  c use gradients to verify that [AM] and [BC] are perpendicular.

  d On grid paper, illustrate **a**, **b** and **c**.

2 Given the points A(7, 7), B(16, 7), C(1, -2) and D(-8, -2):

  a use gradients to show that ABCD is a parallelogram

  b use the distance formula to check that AB $=$ DC and BC $=$ AD

  c find the midpoints of diagonals: **i** [AC]   **ii** [BD].

  d What property of parallelograms has been verified in **c**?

3 Triangle ABC has vertices A(1, 3), B(5, 1) and C(3, -3).
  M is the midpoint of [AB] and N is the midpoint of [BC].

  a Use gradients to show that [MN] is parallel to [AC].

  b Show that [MN] is a half as long as [AC].

4 Given the points A$(-1, 4)$, B$(3, 6)$ and C$(1, 0)$:

    a show that triangle ABC is isosceles and right angled.

    b find the midpoint M of [BC]

    c use gradients to verify that [AM] and [BC] are perpendicular.

    d On grid paper, illustrate **a**, **b** and **c**.

5 Given the points A$(3, 4)$, B$(8, 4)$, C$(5, 0)$ and D$(0, 0)$:

    a use the distance formula to show that ABCD is a rhombus

    b use midpoints to verify that its diagonals bisect each other

    c use gradients to verify that its diagonals are at right angles.

    d On grid paper, illustrate **a**, **b** and **c**.

6
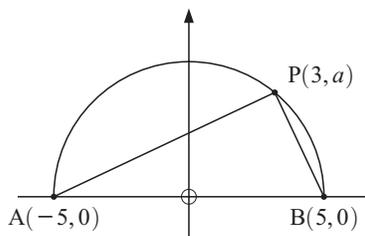
[AB] is the diameter of a semi-circle. P$(3, a)$ lies on the arc AB.

    a Find a.

    b Find the gradients of [AP] and [BP].

    c Verify that $A\widehat{P}B$ is a right angle.

F   VERTICAL AND HORIZONTAL LINES

Every point on the vertical line illustrated has an x-coordinate of 3.

Thus $x = 3$ is the equation of this line.
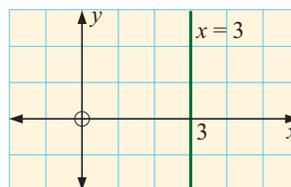
Every point on the horizontal line illustrated has a y-coordinate of 2.

Thus $y = 2$ is the equation of this line.
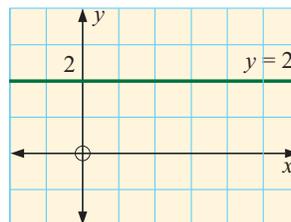
> All **vertical lines** have equations of the form $x = a$ where a is a constant.
> All **horizontal lines** have equations of the form $y = c$ where c is a constant.

Reminder: All horizontal lines have gradient 0.
All vertical lines have undefined gradient.

EXERCISE 6F

1 Sketch the graphs of the following lines and state their gradients:

    a $x = 3$ **b** $y = 3$ **c** $x = -3$

    d $y = 0$ **e** $x + 4 = 0$ **f** $2y - 1 = 0$

2 Find the equation of the line through:

    a $(1, 2)$ and $(3, 2)$ **b** $(3, 1)$ and $(3, -2)$ **c** $(2, 0)$ and $(-4, 0)$

    d $(2, -3)$ and $(7, -3)$ **e** $(-7, 0)$ and $(-7, 2)$ **f** $(2, -4)$ and $(-7, -4)$.

G   EQUATIONS OF STRAIGHT LINES

The **equation of a line** is an equation which connects the x and y values for every point on the line.

For example, a straight line can be drawn through the points $(0, 1)$, $(1, 3)$, $(2, 5)$ and $(3, 7)$ as shown.

We can see that the y-coordinate is always 'double the x-coordinate, plus one'.

This means that the line has equation

$$y = 2x + 1$$

  the y-coordinate is double the x-coordinate plus one
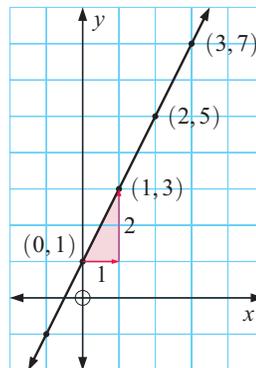
From previous years you should have found that:

> $y = mx + c$ is the **equation** of a straight line with **gradient** m and **y-intercept** c.
>
> This is called the **gradient-intercept** form of the equation of the line.

Notice in the graph above that the y-intercept is 1. Hence we know $c = 1$.

Also, the gradient $= \dfrac{y\text{-step}}{x\text{-step}} = \tfrac{2}{1} = 2$ and so $m = 2$.

So, the equation is $y = 2x + 1$.

Click on the demo icon which plots the graph of $y = 2x + 1$.

It first plots points with x values that are integers.

DEMO

It then plots points midway between, and then midway between these midpoints, and so on.

Eventually we have the entire line.

Example 10   ◀)) **Self Tutor**

Find the equation of the
line with graph:
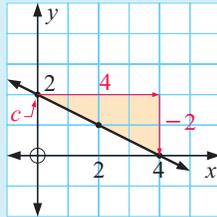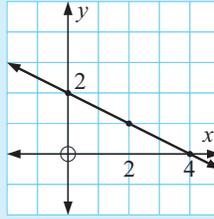
$m = \frac{-2}{4} = -\frac{1}{2}$ and $c = 2$

So, the equation of the line is

$y = -\frac{1}{2}x + 2.$

EXERCISE 6G.1

1 State the slope and y-intercept for these lines:

   a $y = 3x + 2$ **b** $y = 7x + 5$ **c** $y = -2x + 1$ **d** $y = \frac{1}{3}x + 6$

   e $y = -x + 6$ **f** $y = 3 - 2x$ **g** $y = 10 - x$ **h** $y = \dfrac{x + 2}{2}$

   i $y = \dfrac{3x + 4}{2}$ **j** $y = 0$ **k** $y = \dfrac{3 - x}{2}$ **l** $y = \dfrac{7 - 2x}{4}$

2 Find the equation of the line with graph:
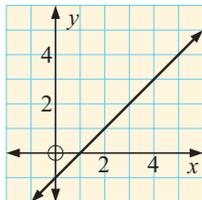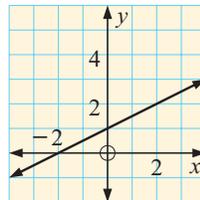
a **b** **c**
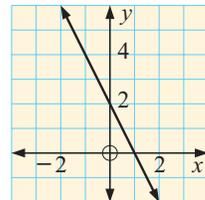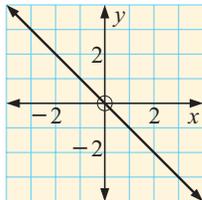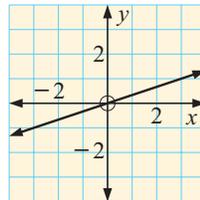
d **e** **f**
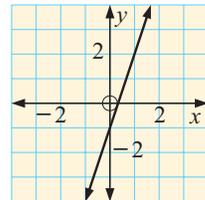
GRAPHING FROM THE GRADIENT-INTERCEPT FORM

Lines with equations given in the gradient-intercept form can be graphed by finding two points on the graph, one of which is the y-intercept. The other can be found by substitution or using the gradient.

Example 11

$\blacktriangleleft$ **Self Tutor**

Graph the line with equation $y = \frac{1}{2}x + 2$.

Method 1:

The y-intercept is 2

When $x = 2$, $y = 1 + 2 = 3$

$\therefore$ (2, 3) lies on the line.

Method 2:

The y-intercept is 2.

The gradient $= \frac{1}{2}$ $\xleftarrow{}$ y-step
$\xleftarrow{}$ x-step

So, we start at (0, 2) and move 2 units in the x-direction and 1 unit in the y-direction.
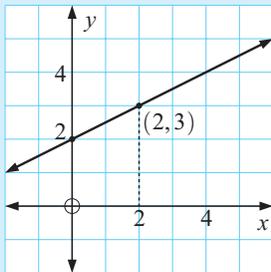
We could plot a third point to check our answer.
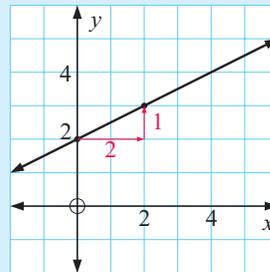
EXERCISE 6G.2

1 Graph the following by plotting at least two points:

a $y = 2x + 1$ **b** $y = 3x - 1$ **c** $y = \frac{2}{3}x$ **d** $y = \frac{4}{3}x - 2$

e $y = -x + 4$ **f** $y = -2x + 2$ **g** $y = -\frac{1}{2}x - 1$ **h** $y = -\frac{2}{3}x - 3$

2 Use the y-intercept and gradient method to graph:

GRAPHING PACKAGE

a $y = 2x + 1$ **b** $y = 4x - 2$ **c** $y = \frac{1}{2}x$

d $y = -x - 2$ **e** $y = -2x + 3$ **f** $y = -\frac{2}{3}x + 2$

FINDING THE EQUATION OF A LINE

If we know the gradient m and the y-intercept c we can write down the equation of the line immediately as $y = mx + c$.

For example, if $m = 2$ and $c = 3$ then the equation is $y = 2x + 3$.

However, if we only know the gradient and some other point not on the y-axis, then more work is required.

Example 12   ◀ۣ) **Self Tutor**

A line has gradient $\frac{2}{3}$ and passes through the point $(3, 1)$.
Find the equation of the line.

Method 1:

As the gradient is $\frac{2}{3}$, $m = \frac{2}{3}$

$\therefore$ $y = \frac{2}{3}x + c$ is the equation.

But when $x = 3$, $y = 1$

$\therefore$ $1 = \frac{2}{3}(3) + c$

$\therefore$ $1 = 2 + c$

$\therefore$ $c = -1$

So, the equation is $y = \frac{2}{3}x - 1$.

Method 2:

Let (x, y) be any point on the line.

Using the gradient formula,

$$\frac{y - 1}{x - 3} = \frac{2}{3}$$

$\therefore$ $y - 1 = \frac{2}{3}(x - 3)$

$\therefore$ $y - 1 = \frac{2}{3}x - 2$

$\therefore$ $y = \frac{2}{3}x - 1$

To find the equation of a straight line we need to know:
- the **gradient**
- the **coordinates** of any point on the line.

We can also find the equation of the line passing through two known points.

Example 13   ◀ۣ) **Self Tutor**

Find the equation of the line through $A(1, 3)$ and $B(-2, 5)$.
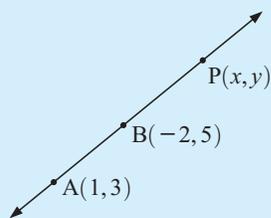
First we find the gradient of the line through A and B.

$$\text{gradient} = \frac{5 - 3}{-2 - 1} = \frac{2}{-3} = -\frac{2}{3}$$

$\therefore$ the equation of the line is $\dfrac{y - 3}{x - 1} = -\frac{2}{3}$

$\therefore$ $y - 3 = -\frac{2}{3}(x - 1)$

$\therefore$ $y = -\frac{2}{3}x + \frac{2}{3} + 3$

$\therefore$ $y = -\frac{2}{3}x + \frac{11}{3}$

EXERCISE 6G.3

1 Find the equation of the line through:

a $(1, 3)$ having a gradient of 2

b $(-1, 2)$ having a gradient of -1

c $(4, -2)$ having a gradient of -3

d $(-2, 1)$ having a gradient of $\frac{2}{3}$

e $(-1, 0)$ having a gradient of $-\frac{1}{2}$

f $(3, -3)$ having a gradient of 0

g $(-1, 5)$ having a gradient of $\frac{2}{3}$

h $(4, -2)$ having a gradient of $-\frac{4}{5}$

i $(6, -3)$ having a gradient of $-\frac{3}{4}$

j $(3, -2)$ having a gradient of $\frac{4}{7}$.

2 Find the equation of the line through:

    a $A(8, 4)$ and $B(5, 1)$ **b** $A(5, -1)$ and $B(4, 0)$

    c $A(-2, 4)$ and $B(-3, -2)$ **d** $P(0, 6)$ and $Q(1, -3)$

    e $M(-1, -2)$ and $N(5, -4)$ **f** $R(-1, -4)$ and $S(-3, 2)$.

3 Find the equation of the line:

    a which has gradient $-\frac{1}{2}$ and cuts the y-axis at 4

    b which is parallel to a line with slope 3, and passes through the point $(-1, -2)$

    c which cuts the x-axis at 4 and the y-axis at 3

    d which cuts the x-axis at -2, and passes through $(2, -5)$

    e which is perpendicular to a line with gradient $\frac{1}{2}$, and cuts the x-axis at -1

    f which is perpendicular to a line with gradient -3, and passes through $(4, -1)$.

H   THE GENERAL FORM OF A LINE

Consider the line $y = -\frac{2}{3}x + \frac{11}{3}$. Its equation is given in gradient-intercept form.

Equations in this form often contain fractions. We can remove them as follows:

If $y = -\frac{2}{3}x + \frac{11}{3}$ then

$3y = -2x + 11$ {multiplying each term by 3}

$\therefore \quad 2x + 3y = 11$

The equation is now in the form $Ax + By = C$, where $A = 2$, $B = 3$, $C = 11$.

> $Ax + By = C$ is called the **general form** of the equation of a line.
>
> A, B and C are constants, and x and y are variables.

Example 14   ◀) **Self Tutor**

Find, in *general form*, the equation of a line:

  a through $(2, 5)$ with slope $-\frac{1}{3}$ **b** through $(1, 3)$ and $(2, -1)$.

a The equation is

$\dfrac{y - 5}{x - 2} = \dfrac{-1}{3}$

$\therefore \quad 3(y - 5) = -1(x - 2)$

$\therefore \quad 3y - 15 = -x + 2$

$\therefore \quad x + 3y = 17$

b the gradient $= \dfrac{-1 - 3}{2 - 1} = -4$

$\therefore$ the equation is

$\dfrac{y - 3}{x - 1} = -4$

$\therefore \quad y - 3 = -4(x - 1)$

$\therefore \quad y - 3 = -4x + 4$

$\therefore \quad 4x + y = 7$

When the equation of a line is given in the general form, we can rearrange it to the form $y = mx + c$ so that we can determine its gradient.

Example 15   ◄) **Self Tutor**

Find the gradient of the line $3x + 4y = 10$.

$3x + 4y = 10$

$\therefore \quad 4y = -3x + 10$ {subtracting $3x$ from both sides}

$\therefore \quad y = -\dfrac{3x}{4} + \dfrac{10}{4}$ {dividing each term by 4}

$\therefore \quad y = -\frac{3}{4}x + \frac{5}{2}$ and so the slope is $-\frac{3}{4}$

EXERCISE 6H.1

1 Find, in *general form*, the equation of the line through:

  a $(1, 4)$ having gradient $\frac{1}{3}$ **b** $(-2, 1)$ having gradient $\frac{3}{5}$

  c $(6, 0)$ having gradient $-\frac{2}{3}$ **d** $(4, -1)$ having gradient $\frac{4}{5}$

  e $(-4, -2)$ having gradient 3 **f** $(3, -1)$ having gradient -2.

2 Find, in *general form*, the equation of the line through:

  a $A(8, 4)$ and $B(5, 1)$ **b** $C(5, -1)$ and $D(4, 0)$

  c $E(-2, 4)$ and $F(-2, -3)$ **d** $G(1, -3)$ and $H(0, 6)$

  e $I(-2, -1)$ and $J(-1, 2)$ **f** $K(-1, -4)$ and $L(-2, -3)$.

3 Find the gradient of the line with equation:

  a $y = 2x + 3$ **b** $y = 2$ **c** $y = -3x + 2$

  d $x = 5$ **e** $y = 2 - 4x$ **f** $y = 3 + \frac{2}{3}x$

  g $y = \dfrac{3x + 1}{4}$ **h** $y = \dfrac{2 - 3x}{5}$ **i** $3x + y = 4$

  j $2x + 3y = 8$ **k** $3x + 5y = 11$ **l** $4x + 7y = 20$

  m $x - 2y = 4$ **n** $3x - 4y = 12$ **o** $5x - 6y = 30$

GRAPHING FROM THE GENERAL FORM

The easiest method used to graph lines in the general form $Ax + By = C$ is to use axes intercepts.

The x-intercept is found by letting $y = 0$.
The y-intercept is found by letting $x = 0$.
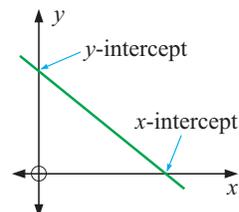
Example 16

📢 **Self Tutor**

Graph the line with equation $4x - 3y = 12$ using axes intercepts.

For $4x - 3y = 12$

when $x = 0$, $-3y = 12$
$$\therefore \quad y = -4$$

when $y = 0$, $4x = 12$
$$\therefore \quad x = 3$$
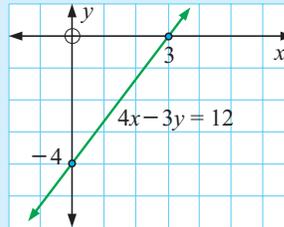
EXERCISE 6H.2

1 Use axes intercepts to sketch graphs of:

  a $x + 3y = 6$

  b $3x - 2y = 12$

  c $2x + 5y = 10$

  d $4x + 3y = 6$

  e $x + y = 5$

  f $x - y = -3$

  g $3x - y = -6$

  h $7x + 2y = 14$

  i $3x - 4y = -12$

I   POINTS ON LINES

A point lies on a line if its coordinates satisfy the equation of the line.

For example:

$(2, 3)$ lies on the line $3x + 4y = 18$ since $3 \times 2 + 4 \times 3 = 6 + 12 = 18$ ✓
$\quad (4, 1)$ does not lie on the line   since $3 \times 4 + 4 \times 1 = 12 + 4 = 16$.

EXERCISE 6I

1 **a** Does $(3, 4)$ lie on the line with equation $5x + 2y = 23$?

   b Does $(-1, 4)$ lie on the line with equation $3x - 2y = 11$?

   c Does $(5, -\frac{1}{2})$ lie on the line with equation $3x + 8y = 11$?

2 Find k if:

   a $(2, 5)$ lies on the line with equation $3x - 2y = k$

   b $(-1, 3)$ lies on the line with equation $5x + 2y = k$.

A point *satisfies* an equation if substitution of its coordinates makes the equation true.

3 Find a given that:

   a $(a, 3)$ lies on the line with equation $y = 2x - 11$

   b $(a, -5)$ lies on the line with equation $y = 4 - x$

   c $(4, a)$ lies on the line with equation $y = \frac{1}{2}x + 3$

   d $(-2, a)$ lies on the line with equation $y = 1 - 3x$.

4 Find b if:

   a $(2, b)$ lies on the line with equation $x + 2y = -4$

   b $(-1, b)$ lies on the line with equation $3x - 4y = 6$

   c $(b, 4)$ lies on the line with equation $5x + 2y = 1$

   d $(b, -3)$ lies on the line with equation $4x - y = 8$

   e $(b, 2)$ lies on the line with equation $3x + by = 15$

   f $(-3, b)$ lies on the line with equation $\dfrac{x}{3} + \dfrac{y}{2} = -b$.

J   WHERE LINES MEET

In this section we consider where lines meet by graphing the lines on the same set of axes. Points where the graphs meet are called **points of intersection**.

Remember:

A straight line can be graphed by finding the:

- x-intercept   (let $y = 0$)
- y-intercept   (let $x = 0$).

point of intersection
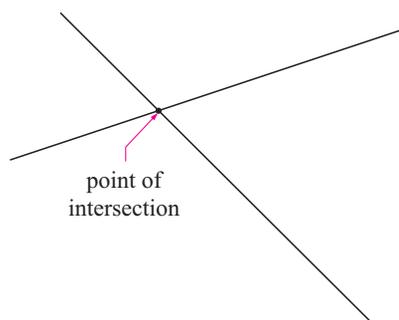
Example 17　　　　　　　　　　　　　　◀》 **Self Tutor**

Use graphical methods to find where the lines $x + y = 6$ and $2x - y = 6$ meet.

For $x + y = 6$:

when $x = 0$, $y = 6$
when $y = 0$, $x = 6$.

x	0	6
y	6	0

For $2x - y = 6$:

when $x = 0$, $-y = 6$,   so $y = -6$
when $y = 0$, $2x = 6$,   so $x = 3$.

x	0	3
y	-6	0

The graphs meet at $(4, 2)$.

Check: $4 + 2 = 6$ ✓ and $2 \times 4 - 2 = 6$ ✓
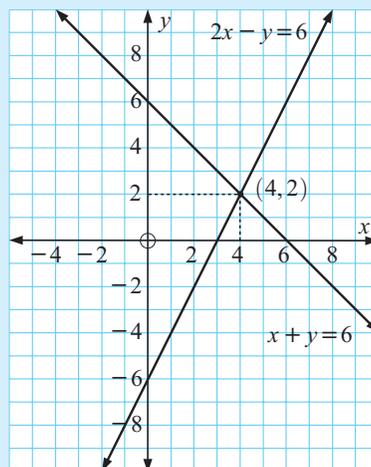
Observe that there are three possible situations which may occur. These are:

Case 1:
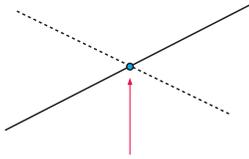
Case 2:

Case 3:

The lines meet in a single **point of intersection**.
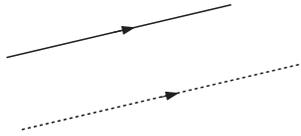
The lines are **parallel** and **never meet**. There are no points of intersection.

The lines are **coincident** or the same line. There are infinitely many points of intersection.

EXERCISE 6J

1 Use graphical methods to find the point of intersection of:

a $y = x + 3$
$y = 1 - x$

b $x + y = 6$
$y = 2x$

c $4x + 3y = 15$
$x - 2y = 1$

d $3x + y = -3$
$2x - 3y = -13$

e $3x + y = 6$
$3x - 2y = -12$

f $x + 2y = 3$
$2x - 3y = -8$

g $2x - y = 3$
$x + 2y = 4$

h $y = 2x - 3$
$2x - y = 2$

i $y = -x - 3$
$2x + 2y = -6$

2 How many points of intersection do the following pairs of lines have?
Explain, but **do not** graph them.

a $2x + y = 6$
$2x + y = 8$

b $3x + y = 2$
$6x + 2y = 4$

c $4x - y = 5$
$4x - y = k$ for some constant k.

Step 1: **Rearrange** each equation into the form $y = mx + c$.

$$4x + 3y = 10 \qquad \text{and} \qquad x - 2y = -3$$
$$\therefore \quad 3y = -4x + 10 \qquad\qquad \therefore \quad -2y = -x - 3$$
$$\therefore \quad y = -\tfrac{4}{3}x + \tfrac{10}{3} \qquad\qquad \therefore \quad y = \frac{x}{2} + \frac{3}{2}$$

Enter the functions $\text{Y}_1 = -4\text{X}/3 + 10/3$ and
$$\text{Y}_2 = \text{X}/2 + 3/2.$$

Step 2: Draw the **graphs** of the functions on the same set of axes. You may have to change the **viewing window**.
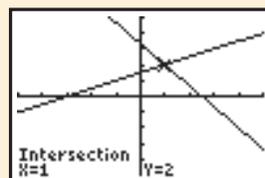
Step 3: Use the built in functions to calculate the point of **intersection**.

In this case, the point of intersection is $(1,\ 2)$.

What to do:

1 Use technology to find the point of intersection of:

a $y = x + 4$
  $5x - 3y = 0$

b $x + 2y = 8$
  $y = 7 - 2x$

c $x - y = 5$
  $2x + 3y = 4$

d $2x + y = 7$
  $3x - 2y = 1$

e $y = 3x - 1$
  $3x - y = 6$

f $y = -\dfrac{2x}{3} + 2$
  $2x + 3y = 6$

2 Comment on the use of technology to find the point(s) of intersection in **1e** and **1f**.

REVIEW SET 6A

1 For $A(3,\ -2)$ and $B(1,\ 6)$ find:

  a the distance from A to B
  b the midpoint of [AB]
  c the gradient of [AB]
  d the coordinates of C if B is the midpoint of [AC].

2 $P(3,\ 1)$ and $Q(-1,\ a)$ are 5 units apart. Find a.

3 Use the distance formula to help classify triangle PQR given the points $P(3,\ 2)$, $Q(-1,\ 4)$ and $R(-1,\ 0)$.

4 Two lines have gradients $-\dfrac{3}{5}$ and $\dfrac{a}{6}$. Find a if:

  a the lines are parallel
  b the lines are perpendicular.

5 ABCD is a parallelogram. Find:

  a the coordinates of M where its diagonals meet
  b the coordinates of A.
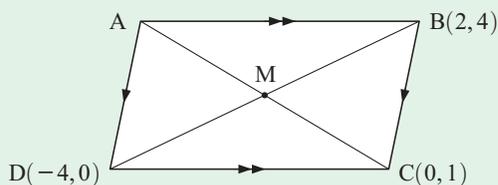
6 The line through $(2,\ 1)$ and $(-3,\ n)$ has a gradient of -4. Find n.

7 Find the gradient and y-intercept of the line with equation:

    a $y = \frac{2}{5}x + 3$ **b** $2x - 3y = 8$

8
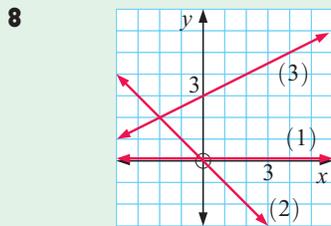
Find the equations of:

    a line (1),   the x-axis

    b line (2)

    c line (3).

9 Find the equation of the line:

    a with gradient 5 and y-intercept -2

    b with gradient $\frac{3}{4}$ passing through $(3,\ -1)$

    c passing through $(4,\ -3)$ and $(-1,\ 1)$.

10 Sketch the graph of the line with equation $\quad x - 2y = 6$.

11 Does the point $(-3,\ 4)$ lie on the line with equation $3x - y = 13$?

12 Find the point of intersection of $\quad 2x + y = 10\quad$ and $\quad 3x - y = 10\quad$ by graphing the two lines on the same set of axes.

13
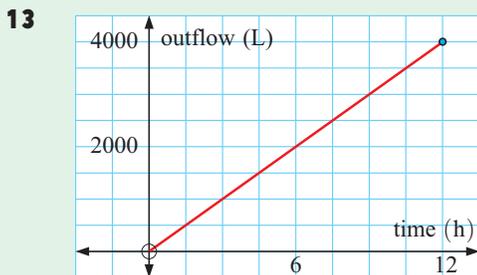
The graph alongside indicates the number of litres of water which run from a tank over a period of time.

    a Find the gradient of the line.

    b Interpret the gradient found in **a**.

    c Is the rate of outflow of water constant or variable? What evidence do you have for your answer?

REVIEW SET 6B

1 For $A(-1,\ 4)$ and $B(2,\ -3)$ find:

    a the distance from B to A

    b the midpoint of [AB]

    c the gradient of [AB]

    d the equation of the line through A and B.

2 Use distances to classify triangle PQR with $P(0,\ 1)$, $Q(-1,\ -2)$ and $R(3,\ -3)$.

3 $A(2,\ 4)$ and $B(k,\ -1)$ are $\sqrt{29}$ units apart. Find k.

4 Solve the **Opening Problem** on page **118**.

5
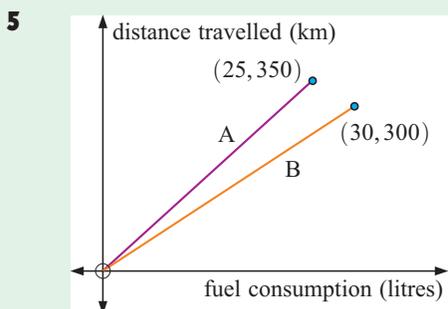
The graphs alongside indicate the distance travelled for different amounts of fuel consumed at speeds of 50 km h^{-1} (graph A) and 80 km h^{-1} (graph B).

a Find the gradient of each line.

b What do these gradients mean?

c If fuel costs \$1.17 per litre, how much more would it cost to travel 1000 km at 80 km h^{-1} compared with 50 km h^{-1}?

6 **a** Prove that OABC is a rhombus.

  b Use gradients to show that the diagonals [OB] and [AC] are perpendicular.
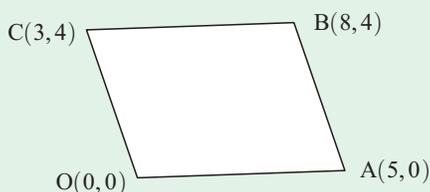
7 Two lines have gradients $\dfrac{4}{d}$ and $-\dfrac{2}{5}$.

  Find d if the lines are:

  a perpendicular   **b** parallel.

8 The line through $(a,\ 1)$ and $(2,\ -1)$ has a gradient of -2. Find a.

9 Find the gradient and y-intercept of the lines with equations:

  a $y = \dfrac{x+2}{3}$ **b** $3x + 2y = 8$

10 Find the equation of the line:

  a with gradient 5 and y-intercept -1

  b with x and y-intercepts 2 and -5 respectively

  c which passes through $(-1,\ 3)$ and $(2,\ 1)$.

11 Find k if $(2,\ k)$ lies on the line with equation $2x + 7y = 41$.

12 On the same set of axes, graph the lines with equations:

  a $x = 2$ **b** $3x + 4y = -12$

13 **a** On the same set of axes, graph the lines with equations $x + 3y = 9$ and $2x - y = 4$.

  b Find the simultaneous solution of $\begin{cases} x + 3y = 9 \\ 2x - y = 4. \end{cases}$

Chapter

7

Mensuration

Contents:

The measurement of **length**, **area**, **volume** and **capacity** is of great importance.

Constructing a tall building or a long bridge across a river, joining the circuits of a microchip, and rendezvousing in space to repair a satellite, all require the use of measurement with skill and precision.

Builders, architects, engineers and manufacturers need to measure the sizes of objects to considerable accuracy.

The most common system of measurement is the **Systeme International (SI)**.

Important units that you should be familiar with include:

Measurement of	Standard unit	What it means
Length	metre	How long or how far.
Mass	gram	How heavy an object is.
Capacity	litre	How much liquid or gas is contained.
Time	hours, minutes, seconds	How long it takes.
Temperature	degrees Celsius and Fahrenheit	How hot or cold.
Speed	metres per second ($m\,s^{-1}$)	How fast it is travelling.

The SI uses prefixes to indicate an increase or decrease in the size of a unit.

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
terra	T	1 000 000 000 000	centi	c	0.01
giga	G	1 000 000 000	milli	m	0.001
mega	M	1 000 000	micro	μ	0.000 001
kilo	k	1000	nano	n	0.000 000 001
hecto	h	100	pico	p	0.000 000 000 001

OPENING PROBLEM

Byron's house has a roof with dimensions shown. He knows that the average rainfall in his suburb is 50 cm per year. Byron would like to install a cylindrical rainwater tank to hold the water that runs off the roof. The tank is to be made of moulded plastic but Byron wants to minimise the amount of plastic required and hence the cost.

7 m   1 m   12 m
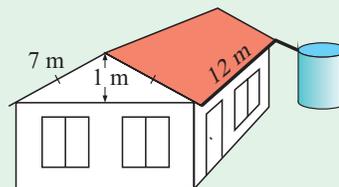
Can you help Byron answer the following questions?

1 On average, what volume of water will fall on the roof each year?

2 How many litres of water does the tank need to hold?

3 If the tank has base diameter 3 m, how high will it need to be?

4 What is the surface area of plastic required to build the tank in **3**?

5 Find the dimensions of the tank which minimise the amount of plastic needed.

A   ERROR

Whenever we take a measurement, there is always the possibility of error. Errors are caused by inaccuracies in the measuring device we use, and in rounding off the measurement we take. They can also be caused by human error, so we need to be careful when we take measurements.

DISCUSSION   ERRORS IN LENGTH MEASURING

Discuss the possible factors which may cause errors when we measure the length of an object.

TERMINOLOGY

- The **absolute error** due to rounding or approximation is the difference between the actual or true value and the measured value.
- The **percentage error** is the absolute value compared with the true value, expressed as a percentage.

Example 1   ◀◎ Self Tutor

The crowd at a tennis tournament was $14\,869$, but in the newspaper it was reported as $15\,000$. Find the absolute and percentage errors in this approximation.

$$\text{Absolute error} = 15\,000 - 14\,869$$
$$= 131$$
$$\text{Percentage error} = \frac{\text{absolute error}}{\text{true value}} \times 100\%$$
$$= \frac{131}{14\,869} \times 100\%$$
$$\approx 0.881\%$$

ACCURACY OF MEASUREMENT

When we take measurements, we are usually reading from some sort of scale. The scale of a ruler may have millimetres marked on it, but when we measure an object's length it is likely to lie between two marks.

So, when we round or estimate to the nearest millimetre, our answer may be inaccurate by up to a half a millimetre. We say that the ruler is accurate to the nearest half a millimetre.

A measurement is accurate to $\pm \frac{1}{2}$ of the smallest division on the scale.

Example 2
◀) **Self Tutor**

Rod's height was measured using a tape measure with centimetre graduations. It was recorded as 188 cm. For this measurement, state:

a the absolute error
b the percentage error.

The tape measure is accurate to $\pm\frac{1}{2}$ cm.

a The absolute error is 0.5 cm.

b The percentage error is $\frac{0.5}{188} \times 100\% \approx 0.266\%$.

EXERCISE 7A

1 Find the absolute error and percentage error in saying that:

a there were 300 people at the conference when there were actually 291

b 2.95 can be approximated by 3

c $31 823 can be rounded to $32 000

d π is about $3\frac{1}{7}$.

2 State the accuracy possible when using:

a a ruler marked in mm
b a set of scales marked in kg

c a tape measure marked in cm
d a jug marked with 100 mL increments.

3 Su-Lin's height was measured using a tape measure with centimetre markings. Her height was recorded as 154 cm.

a State the range of possible heights in which her true height lies.

b Find the absolute error in the measurement.

c Find the percentage error.

4 Charles measured the sides of his rectangular garden plot. He said that the length is 13.8 m and the width is 7.3 m.

a What are the smallest possible values of the length and width?

b Write the perimeter of the plot in the form $a \pm b$ where b is the absolute error.

c Find the percentage error for the perimeter.

5 Here is Ben's reasoning for finding the upper and lower boundaries in which the actual area of his garden plot lies.

Given measurements for its length l and width w, the actual length is $l \pm e$ and the actual width is $w \pm e$ where e is the absolute error in each length measurement.

So, the actual area lies between $(l-e)(w-e)$ and $(l+e)(w+e)$

i.e., between $lw - e(w+l) + e^2$ and $lw + e(w+l) + e^2$.

But e^2 is negligible compared with the other terms, so the actual area $= lw \pm (w+l)e$.

a Are the steps in Ben's argument valid?

b Use Ben's formula to calculate the boundaries in which the true area of Charles' garden plot in **4** lies.

B   LENGTH AND PERIMETER

The base unit of length in the SI is the **metre** (m). Other lengths are often measured in:

- **millimetres** (mm)   for example,   the length of a bee
- **centimetres** (cm)   for example,   the width of your desk
- **kilometres** (km)   for example,   the distance between two cities.

The table below summarises the connection between these units of length:

> 1 **kilometre** (km) = 1000 **metres** (m)
> 1 **metre** (m) = 100 **centimetres** (cm)
> 1 **centimetre** (cm) = 10 **millimetres** (mm)

LENGTH UNITS CONVERSIONS
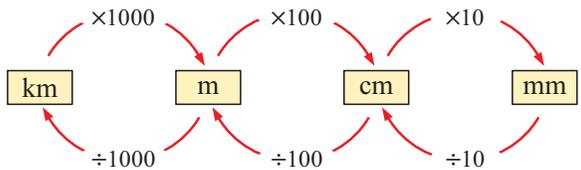
So, to convert cm into km we $\div$ 100 and then $\div$ 1000.

Notice that, when converting from:

- smaller units to larger units we **divide** by the conversion factor
- larger units to smaller units we **multiply** by the conversion factor.

Example 3   ◀) **Self Tutor**

Convert: **a** 6.32 km to m   **b** 2350 cm to m   **c** 32.8 m to mm

a   6.32 km
= (6.32×1000) m
= 6320 m

b   2350 cm
= $(2350 \div 100)$ m
= 23.5 m

c   32.8 m
= $(32.8 \times 100 \times 10)$ mm
= 32 800 mm

EXERCISE 7B.1

1 Estimate the following and then check by measuring:

  a the length of your pen   **b** the width of your desk

  c the height of your neighbour   **d** the depth of your classroom

  e the width of a football goal

2 Convert:

  a 2.61 km to m   **b** 4300 mm to m   **c** 865 cm to m

  d 700 mm to cm   **e** 11 500 m to km   **f** 3.675 m to cm

  g 381 mm to m   **h** 68.2 cm to mm   **i** 5.67 km to cm

  j 2860 cm to m   **k** 24 300 mm to m   **l** 0.328 km to mm

3 Claudia walked 7.92 km due east and Jasmin ran 5780 m due west. If they started at the same point, how far are they now apart:

    a in metres   **b** in kilometres   **c** in centimetres?

4 A nail has length 50 mm.

    a If 240 000 nails are placed end to end, how far would the line stretch in:

        i mm   **ii** m   **iii** km?

    b How many nails would need to be placed end to end to reach a distance of one kilometre?

5 Ami walks at a constant rate of 3.6 km every 60 minutes.

    a How many metres does Ami walk in a minute?

    b How many centimetres does Ami walk in 5 hours?

PERIMETER

> The distance around a closed figure is its **perimeter**.

For some shapes we can derive a formula for perimeter. The formulae for the most common shapes are given below:

Square	**Rectangle**	**Triangle**	**Circle**	**Arc**
$P = 4l$	$P = 2(l + w)$	$P = a + b + c$	$C = 2\pi r$ or $C = \pi d$	$l = \left(\dfrac{\theta}{360}\right) 2\pi r$

Example 4   🔊 **Self Tutor**

Find the perimeter of:

a
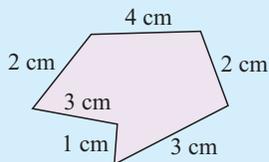
b
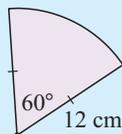
> The length of an arc is a fraction of the circumference of a circle!

a   Perimeter
$$= (2 + 3 + 1 + 3 + 2 + 4) \text{ cm}$$
$$= 15 \text{ cm}$$

b   Perimeter
$$= 12 + 12 + \text{length of arc}$$
$$= 24 + \left(\tfrac{60}{360}\right) \times 2 \times \pi \times 12$$
$$\approx 36.6 \text{ cm}$$

Example 5

◀) **Self Tutor**

Find the perimeter P of:
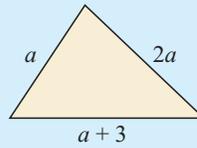
Perimeter $= a + 2a + (a + 3)$
$$= 4a + 3$$
$$\therefore \quad P = 4a + 3$$

EXERCISE 7B.2

1 Find the perimeter of:

   a a triangle with sides 8.3 cm, 9.0 cm and 11.9 cm

   b a square with sides 9.3 cm

   c a rhombus with sides 1.74 m

2   **a** Find the circumference of a circle of radius 13.4 cm.

   b Find the length of an arc of a circle of radius 8 cm and angle 120°.

   c Find the perimeter of a sector of a circle of radius 9 cm and sector angle 80°.

3 Find the perimeter of the following shapes. You may need to use Pythagoras' theorem.
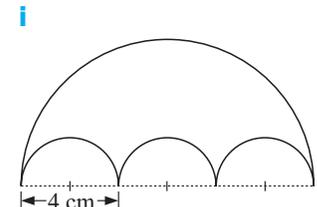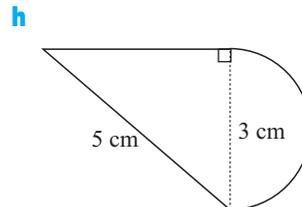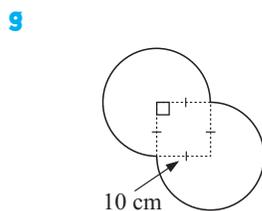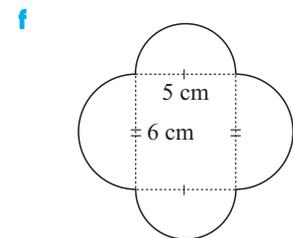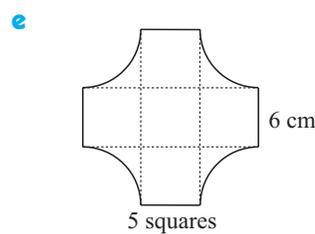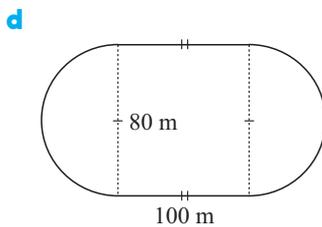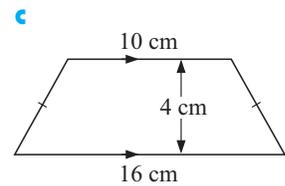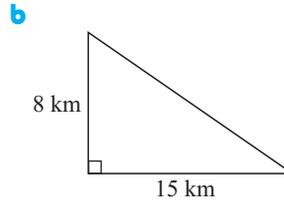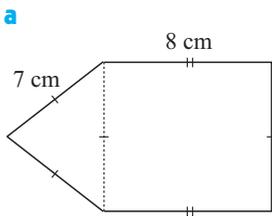
4 Find a formula for the perimeter P of the following:

a
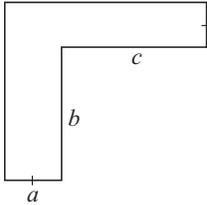
b
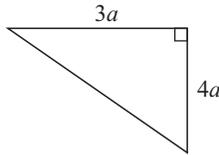
c
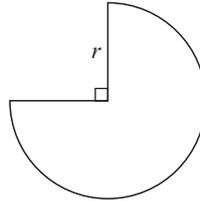
d
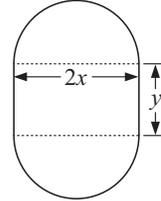
5 Determine the length of fencing around an 80 m by 170 m rectangular playing field if the fence is to be 25 m outside the edge of the playing field.

6 A cattle farmer has subdivided his property into six paddocks, as shown. Each paddock, and the property overall, is fenced by a single electric wire costing $0.23 per metre. Find the total cost of the electric fencing.
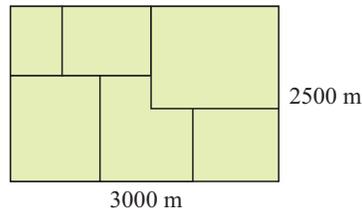
7
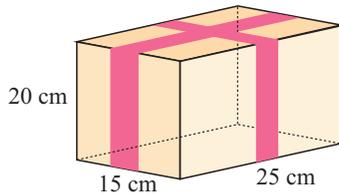
Over a period of 6 weeks a book company packed 568 boxes with books. Each box was 25 cm by 15 cm by 20 cm and had to be taped as shown. 5 cm of overlap was required on both tapes.

a Find the total length of tape used.

b How many rolls of tape, each 75 m long, were required to tape the boxes?

8 A soccer ball has a diameter of 24 cm. How many times must it be rolled over to travel from one end of a 100 metre soccer field to the other?

9
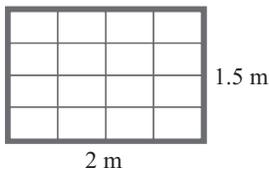
A new house is to have nine aluminium windows, each identical in shape to that shown in the diagram.

The outer framing costs €8.50 per metre and the inner slats cost €3.75 per metre.

Find the total cost for the framing and slats.

10 A soccer goal net has the shape shown. If the netting has 5 cm by 5 cm square gaps, what is the total length of cord needed to make the back rectangle of the net?
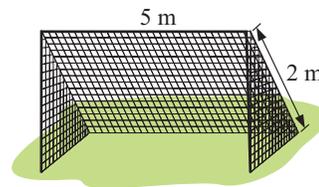
11
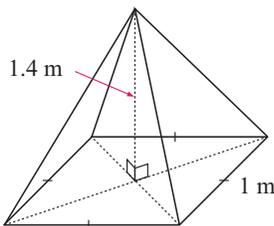
A square-based pyramid has base lengths of 1 m and a height of 1.4 m. It was made by joining 8 pieces of wire together to form the frame. Find the total length of wire in the frame.

12 Three plastic pipes, each of diameter 10 cm, are held together by straps. Find the length of each strap to the nearest centimetre.

Example 6   🔊 **Self Tutor**

Find the radius of a trundle wheel with circumference 1 m.

$1 \text{ m} = 100 \text{ cm}$

$C = 2\pi r$

$\therefore \quad 100 = 2\pi r$

$\therefore \quad \dfrac{100}{2\pi} = r$

$\therefore \quad r \approx 15.92$

The radius is approximately 15.9 cm.

---
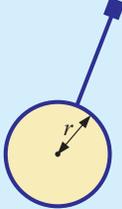
13 If a circular plate has a circumference of 80 cm, what are the internal measurements of the smallest box into which it would fit?

Example 7   🔊 **Self Tutor**

The door in the diagram is made from a semi-circle of radius r, and a square with sides of length $2r$.

The perimeter of the door is 6 m long.
Find, to the nearest mm, the width of the door.
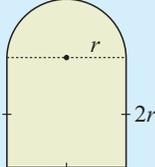
The *perimeter* consists of the arc of a semi-circle and three sides of a square.

$P = \frac{1}{2}(2\pi r) + 3 \times 2r$

$\therefore \quad 6 = \pi r + 6r$

$\therefore \quad 6 = (\pi + 6)r$ $\{r \text{ is a common factor}\}$

$\therefore \quad \dfrac{6}{(\pi + 6)} = r$ $\{\text{dividing both sides by } (\pi + 6)\}$

$\therefore \quad r \approx 0.656\,34 \text{ m}$ $\{6 \boxed{\div} \boxed{(}\ \boxed{\pi}\ \boxed{+}\ 6\ \boxed{)}\ \boxed{\text{ENTER}}\}$

$\therefore \quad r \approx 0.656\,34 \times 1000 \text{ mm}$

$\therefore \quad r \approx 656.34 \text{ mm}$

$\therefore \quad 2r \approx 1312.68 \text{ mm}$

The width of the door is 1313 mm (to the nearest mm).

---

14 An ideal athletics track is 400 m long, with two 'straights' and semi-circular ends of diameter 80 m. Find:

  a the length of each straight to the nearest cm

  b the staggered distance the athlete in the second lane must start in front of the athlete in the innermost lane so that they both run 400 m. Assume that each lane is 1 m wide.
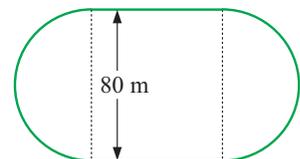
15 A rocket has a circular orbit 1000 km above the surface of the Earth. If the rocket travels at a constant speed of 18 000 km per hour, how long will it take to complete 48 orbits? Assume that the Earth has a radius of 6400 km.

16 A lighting company produces conical lampshades from sectors of circles as illustrated. When the lampshades are made, lace is stitched around the circular base. Determine the total cost of the lace if the manager decides to make 1500 lampshades and the lace costs $0.75 per metre.

17 A cyclist used a bicycle with a ratio of pedal revolutions to wheel revolutions of $1 : 6$. If the diameter of a wheel is 70 cm and the cyclist averaged 32 pedal revolutions per minute, how long would it take to travel 40 km?
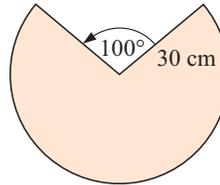
INVESTIGATION 1   CONSTRUCTING A LAMPSHADE

Your company has received an order for a large number of lampshades in the shape illustrated. A pattern is required from which the lampshades can be mass-produced.
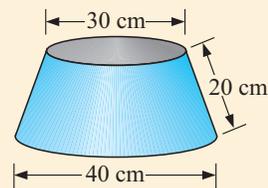
What to do:

1 On a piece of paper, draw two concentric circles. Cut out the *annulus* or washer shape shown.

2 Make two cuts [AB] and [CD] in line with the circle's centre.

3 For each shape join [AB] to [CD]. You have made two lampshades of different sizes.

4 To obtain the shape of the special lampshades you are to mass-produce, you must first do some calculations.
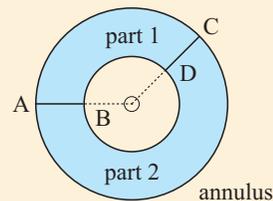
Let $\stackrel{\frown}{BOD}$ be θ^o and $OB = OD = x$ cm.

a What is the length of [CD] for the lampshade shown?

b Show that:

    i arc $BD = \dfrac{\theta \pi x}{180}$ **ii** arc $AC = \dfrac{\theta \pi (x + 20)}{180}$.

c For these special lampshades, arc $BD = 30\pi$, and arc $AC = 40\pi$. Explain why this should be so.

d Hence use **b** and **c** to find x and θ.
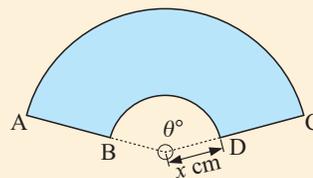
5 Make a half-sized pattern of the lampshade using a large sheet of paper.

INVESTIGATION 2   FINDING A MINIMUM PERIMETER

Consider the collection of all rectangles with area 1000 m². They have a variety of shapes, such as:

Which rectangle has least perimeter?

Consider the following method of solution:

Let one side of the rectangle be x m.
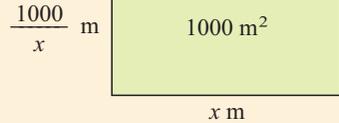
The other side is therefore $\dfrac{1000}{x}$ m.

$\dfrac{1000}{x}$ m   1000 m² x m

If P is the perimeter in metres, then $P = 2x + \dfrac{2000}{x}$.

Our task is to find the value of x that minimises the perimeter P.

To solve this problem we can use a graphing package or a graphics calculator. To use the graphing package, click on the icon, plot the graph, and find its minimum.

GRAPHING PACKAGE

Otherwise, follow these steps for a graphics calculator. If you need further instructions, see pages **21** to **24**.

Step 1:

Graph the function   Y= 2X+2000/X.

Change the **window** settings to show X values between 0 and 50 and Y values between 0 and 250.

```
Plot1  Plot2  Plot3
\Y1☐2X+2000/X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```
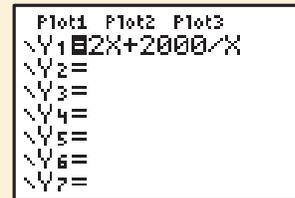
Step 2:

Use **trace** to estimate the value of X which gives the minimum value of Y.

Notice that it is difficult to *visually* identify the minimum value of Y!

Y1=2X+2000/X

X=31.383 ⸱⸱⸱⸱⸱Y=126.495 ⸱⸱⸱⸱⸱
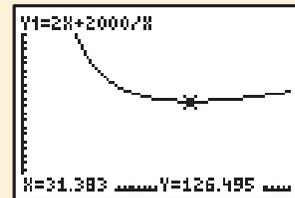
Step 3:

Check your estimation by using the built-in functions to calculate the value of X that gives the **minimum** value of Y.

The minimum value of Y is 126.491 when X is 31.623. Thus the minimum perimeter is 126.491 m when one side is 31.623 m. The other side is thus $1000 \div 31.623$ m = 31.623 m also!

Minimum
X=31.623 ⸱⸱⸱⸱⸱Y=126.491 ⸱⸱⸱⸱⸱
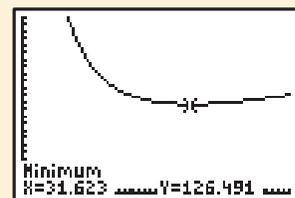
What to do:

Use the method above to solve these problems:

1 Find the minimum perimeter of a rectangle of area 2000 m^2.

2 Nadine wishes to grow vegetables on her property. She wants a rectangular garden of area 1600 m^2 and needs to build a fence around it to stop her goats eating the vegetables.

Help her decide the dimensions of the rectangular garden of area 1600 m^2 which minimises her fencing costs.

3 Maximise the area of a right angled triangle which has variable legs and a fixed hypotenuse of 20 cm.

C   AREA

The **area** of a closed figure is the number of square units it encloses.

UNITS OF AREA

Area can be measured in square millimetres, square centimetres, square metres and square kilometres; there is also another unit called a hectare (ha).

$$1\ \textbf{mm}^2 = 1\ \text{mm} \times 1\ \text{mm}$$
$$1\ \textbf{cm}^2 = 10\ \text{mm} \times 10\ \text{mm} = 100\ \text{mm}^2$$
$$1\ \textbf{m}^2 = 100\ \text{cm} \times 100\ \text{cm} = 10\,000\ \text{cm}^2$$
$$1\ \textbf{ha} = 100\ \text{m} \times 100\ \text{m} = 10\,000\ \text{m}^2$$
$$1\ \textbf{km}^2 = 1000\ \text{m} \times 1000\ \text{m} = 1\,000\,000\ \text{m}^2\ \text{or}\ 100\ \text{ha}$$

AREA UNITS CONVERSIONS

	$\times 100$		$\times 10\,000$		$\times 10\,000$		$\times 100$	
km^2		ha		m^2		cm^2		mm^2
	$\div 100$		$\div 10\,000$		$\div 10\,000$		$\div 100$	
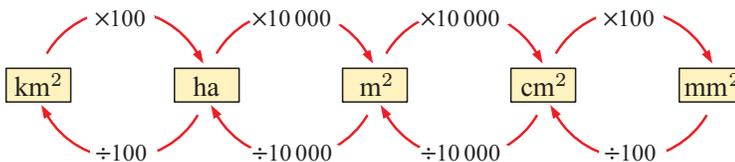
> To convert m to cm we multiply by 100.
> So, to convert m^2 to cm^2, we multiply by 100^2.

Example 8   ◀⁎ Self Tutor

Convert the following:

a 4.8 m^2 to cm^2 **b** 4500 m^2 to ha

a   4.8 m^2 to cm^2
   $= (4.8 \times 10\,000)$ cm^2
   $= 48\,000$ cm^2

b   4500 m^2 to ha
   $= 4500 \div 10\,000$ ha
   $= 0.45$ ha

AREA FORMULAE

Shape	Figure	Formula
Rectangle	width / length	$\textbf{Area} = \textbf{length} \times \textbf{width}$
Triangle	height, base, base	$\textbf{Area} = \frac{1}{2} \times \textbf{base} \times \textbf{height}$
Parallelogram	height / base	$\textbf{Area} = \textbf{base} \times \textbf{height}$
Trapezium	a / h / b	$\textbf{Area} = \left(\dfrac{a+b}{2}\right) \times h$
Circle	r	$\textbf{Area} = \pi r^2$
Sector	θ / r	$\textbf{Area} = \left(\dfrac{\theta}{360}\right) \times \pi r^2$
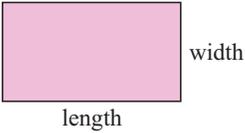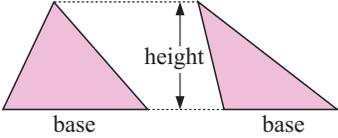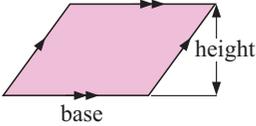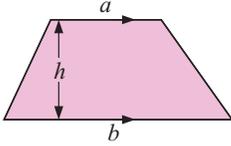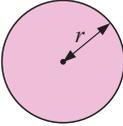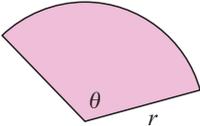
Example 9   ◀)) **Self Tutor**

Find the area of each of the following figures:

a 7 cm, 10 cm

b 60°, 8 cm

> The area of a sector is a fraction of the area of a circle!
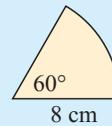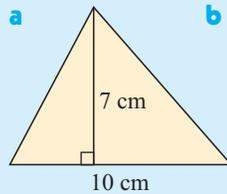
a Area $= \frac{1}{2}(\text{base} \times \text{height})$
   $= \frac{1}{2} \times 10 \times 7$
   $= 35$ cm^2

b Area $= \frac{\theta}{360} \times \pi r^2$
   $= \frac{60}{360} \times \pi \times 8^2$
   ≈ 33.5 cm^2

Example 10   ◀) **Self Tutor**

Find the green shaded area in the following figures:

a

6 cm
12 cm
10 cm
18 cm

b

5 cm   3 cm

a	Area	**b**	Area
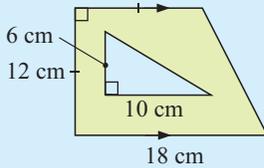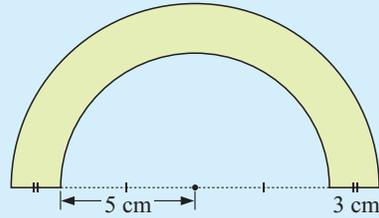
a   Area

$= \text{trapezium area} - \text{triangle area}$

$= \left(\dfrac{12 + 18}{2}\right) \times 12 - \frac{1}{2} \times 10 \times 6$

$= 15 \times 12 - 5 \times 6$

$= 150 \text{ cm}^2$

b   Area

$= \text{area of large semi-circle}$

$\quad - \text{area of small semi-circle}$

$= \frac{1}{2} \times (\pi \times 8^2) - \frac{1}{2} \times (\pi \times 5^2)$

$\approx 61.3 \text{ cm}^2$

EXERCISE 7C.1

1 Convert the following:

   a $38\,400 \text{ m}^2$ to ha   **b** 25.7 ha to m^2 **c** 3.5 m^2 to cm^2

   d $500\,000 \text{ cm}^2$ to m^2 **e** 18 cm^2 to mm^2 **f** 460 mm^2 to cm^2

   g 9 km^2 to m^2 **h** 35 km^2 to ha   **i** 500 ha to km^2

2 Estimate the area of the following and then check by measuring and calculating:

   a the area of the front of this book   **b** the area of the classroom floor

   c the area of a netball court 'goal circle'   **d** the area of a football pitch

   e the area of a basketball court 'key'.

3 Find the areas of the following figures:

a

7 cm
18 cm

b

7 cm

c

6 cm
5 cm
14 cm

d

4 cm   10 cm

e

8 cm   17 cm
6 cm

f

50 m
100 m
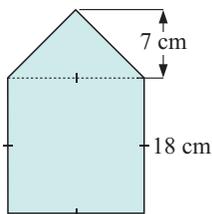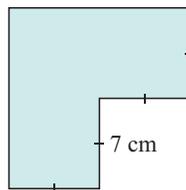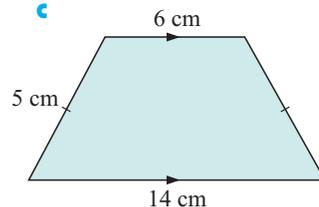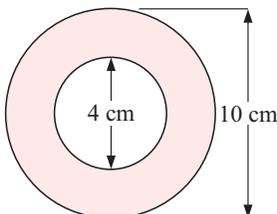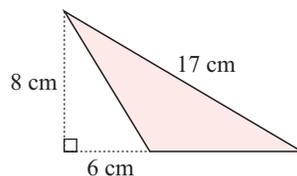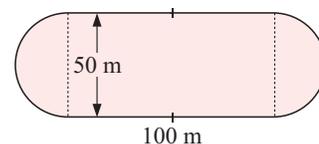
4 Find the area of:

    a a sector of radius 10 cm and angle 100°

    b an annulus of radii 2 m and 2.4 m

   **c** a rhombus with sides of length 10 cm and one diagonal of length 8 cm.
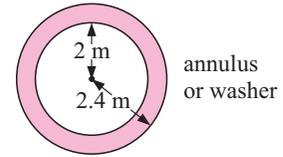
5 A door is in the shape of a rectangle surmounted by a semi-circle. The width of the door is 1.2 m and the height of the door is 2.5 m. Find the total area of the door.
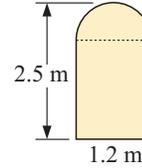
6 A restaurant uses square tables with sides 1.3 m and round tablecloths of diameter 2 m. Determine the percentage of each tablecloth which overhangs its table.

7 What is the cost of laying artificial grass over an 80 m by 120 m rectangular playing field if the grass comes in 6 m wide strips and costs £85 for 10 m?

8 A cropduster can carry 240 kg of fertiliser at a time. It is necessary to spread 50 kg of fertiliser per hectare. How many flights are necessary to fertilise a 1.2 km by 450 m rectangular property?

9 A chess board consists of 5 cm squares of blackwood for the black squares and maple for the white squares. The squares are surrounded by an 8 cm wide blackwood border. Determine the percentage of the board which is made of maple.

10 A circular portrait photograph has diameter 18 cm and is to be placed in a 20 cm square frame. Determine the ratio of the area of the photograph to the area of the frame.
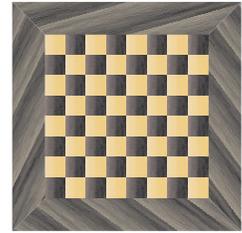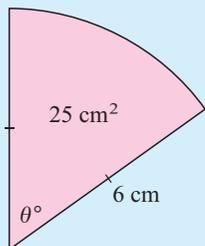
11 A metal washer has an external diameter of 2 cm and an internal diameter of 1 cm.

    a How many washers can be cut from a sheet of steel 3 m by 1 m?

    b If the metal left over was remelted and cast into a sheet of the same thickness and width 1 m, how many additional washers could be cut?

Example 11   🔊 **Self Tutor**

A sector has area 25 cm^2 and radius 6 cm. Find the angle subtended at the centre.

The area of a sector is $\frac{\theta}{360}$ of the area of a circle of the same radius.

$$\text{Area} = \left(\frac{\theta}{360}\right)\pi r^2$$

$$\therefore \quad 25 = \frac{\theta}{360} \times \pi \times 6^2$$

$$\therefore \quad 25 = \frac{\theta\pi}{10}$$

$$\therefore \quad \frac{250}{\pi} = \theta$$

$$\therefore \quad \theta \approx 79.58$$

∴ the angle measures 79.6^o.

12 Find the angle of a sector with area 30 cm^2 and radius 12 cm.

13 If a circle's area is to be doubled, by what factor must its radius be multiplied?

Example 12   ◄ⁱ) **Self Tutor**
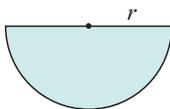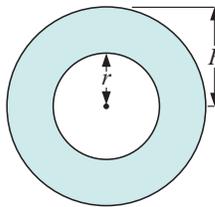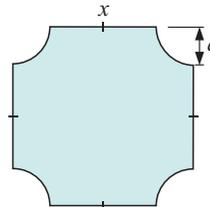
Find a formula for the area A of:
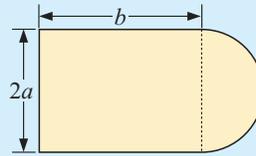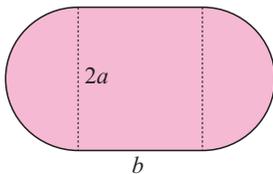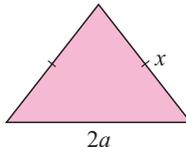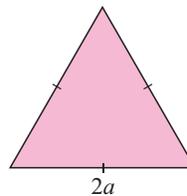
Area = area of rectangle + area of semi-circle

∴ $A = 2a \times b + \frac{1}{2} \times \pi \times a^2$ {as the radius of the semi-circle is a units}

∴ $A = \left(2ab + \dfrac{\pi a^2}{2}\right)$ units^2

---

14 Find a formula for the area A of the following regions:

a

b

c

d

e

f

You may need to use Pythagoras' theorem!

HERON'S OR HERO'S FORMULA

Heron or **Hero** of Alexandria was an important geometer and engineer of the first century AD. He invented machines such as the steam turbine, but is most famous for devising a formula for calculating the area of a triangle given the lengths of its three sides:
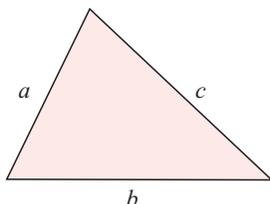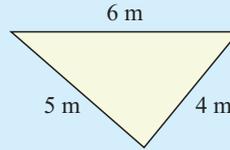
Area $A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \dfrac{a+b+c}{2}$.

s is known as the semi-perimeter of the triangle.

Example 13

Use Heron's formula to find, correct to 2 decimal places, the area of:

6 m

5 m   4 m

$$s = \frac{a+b+c}{2}$$

$$\therefore \quad s = \frac{4+5+6}{2}$$

$$\therefore \quad s = 7.5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

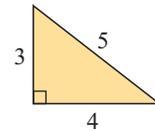$$\therefore \quad A = \sqrt{7.5(7.5-4)(7.5-5)(7.5-6)}$$

$$\therefore \quad A = \sqrt{7.5 \times 3.5 \times 2.5 \times 1.5}$$

$$\therefore \quad A \approx 9.92 \text{ m}^2$$
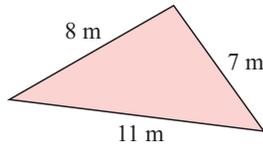
EXERCISE 7C.2

1 Find the area of the triangle shown without using Heron's formula.
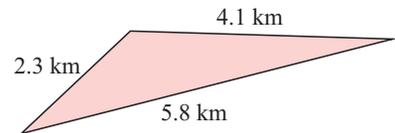
Use Heron's formula to check your answer.

3   5
4

2 Use Heron's formula to find, correct to 1 decimal place, the area of:

a

8 m
7 m
11 m

b

4.1 km
2.3 km
5.8 km

3 Find the area of the lawn with dimensions shown.

A   4.2 m   B
4.9 m
8.3 m
5.1 m
D
7.6 m   C
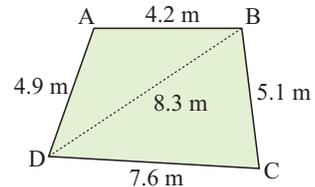
INVESTIGATION 3

MAXIMISING AREA

In **Investigation 2**, we considered the problem of finding the minimum perimeter surrounding a fixed area. In this investigation we will consider a similar problem, but this time we have a fixed amount of material to build a fence, and need to choose how it should be used to maximise the area enclosed.

The turkey problem

Maxine has 40 m of fencing. She wishes to form a rectangular enclosure in which she will keep turkeys. To help maximise the area she can enclose, she uses an existing fence. The 40 m of fencing forms the other three sides of the rectangle. Your task is to determine the rectangular shape which encloses the maximum area of ground.

existing fence

existing fence

x m x m

y m
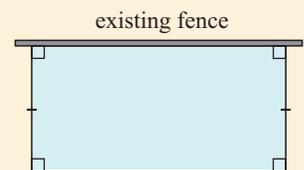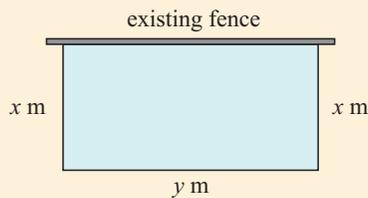
Let the sides adjacent to the wall have length x m, and the side opposite the wall have length y m.

Notice that $x + x + y = 40$ {40 m of fencing}

$$\therefore \quad y = 40 - 2x$$

Now   area $= x \times y$

$$\therefore \quad \text{area} = x \times (40 - 2x)$$

$$\therefore \quad \text{area} = 40x - 2x^2$$

What to do:

You can use a graphing package to find the value of x that maximises the area, or else follow the graphics calculator steps below. You may need the graphics calculator instructions on pages **21** to **24**.

GRAPHING PACKAGE

1 Use a graphics calculator to graph the **function** $Y = 40X - 2X^2$. Change the **window** settings to show X values between 0 and 20 and Y values between 0 and 250.

```
Plot1  Plot2  Plot3
\Y1■40X-2X²
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

2 Use **trace** to estimate the value of X which gives the maximum value of Y. The maximum value of Y seems to be 200 when X = 10.

```
Y1=40X-2X²



X=10          Y=200
```

3 Check this estimate by using the built-in functions to calculate the value of X that gives the **maximum** value of Y.

You should find that if Maxine makes her enclosure 10 m by 20 m, then the maximum area of 200 m^2 is obtained.

```
Maximum
X=10          Y=200
```

4 Investigate the situation where 50 m of fencing is available.

5 Investigate the situation where a right angled triangle enclosure is required.   What shape encloses maximum area using the 40 m of fencing on the two shorter sides?

existing fence
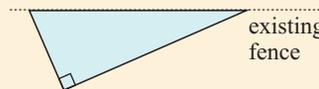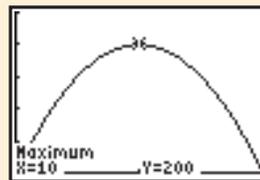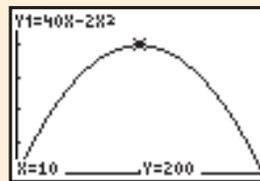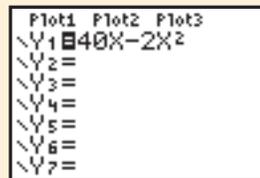
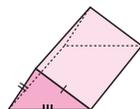# D   SURFACE AREA

SOLIDS WITH PLANE FACES

The **surface area** of a three-dimensional solid with plane faces is the sum of the areas of the faces.

This means that the surface area is the same as the area of the **net** required to make the figure.

For example, the surface area of   = the area of
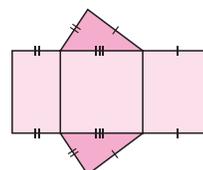
Example 14   ◀ **Self Tutor**

Find the surface area of the
rectangular prism:

3 cm

4 cm

6 cm
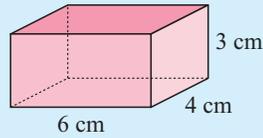
> The surface area of any
> solid can be found by
> adding the areas of all
> its surfaces.

The figure has   2 faces of   6 cm × 4 cm
2 faces of   6 cm × 3 cm
and   2 faces of   3 cm × 4 cm.

Total surface area $= (2 \times 6 \times 4 + 2 \times 6 \times 3 + 2 \times 3 \times 4)$
$= (48 + 36 + 24)$
$= 108$ cm^2

Example 15   ◀ **Self Tutor**

Find the surface area of the
square-based pyramid:

5 cm

8 cm
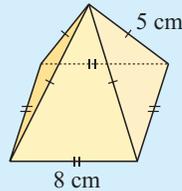
The figure has:

• 1 square base

8 cm

8 cm

• 4 triangular faces

5 cm

h cm

4 cm   4 cm
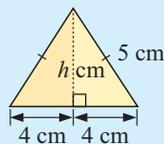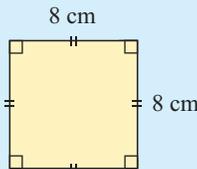
$h^2 + 4^2 = 5^2$ {Pythagoras}
∴ $h^2 + 16 = 25$
∴ $h^2 = 9$
∴ $h = 3$ {as $h > 0$}

Total surface area $= 8 \times 8 \ + \ 4 \times (\frac{1}{2} \times 8 \times 3)$
$= 64 + 48$
$= 112$ cm^2

EXERCISE 7D.1

1 Find the surface area of the following rectangular prisms:

a

2 cm

6 cm

5 cm

b

45 cm

20 cm   20 cm

c

2.5 m

3.5 m   2 m
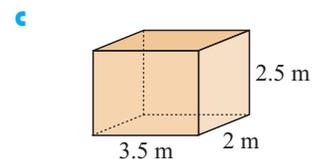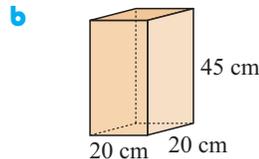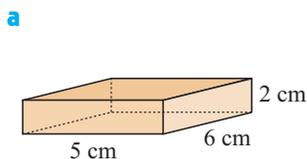
2 Find the surface area of the following square-based pyramids:

a

9 cm

12 cm

b

13 m

24 m

c

60 cm
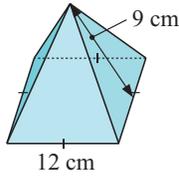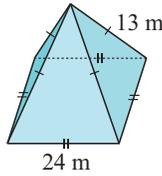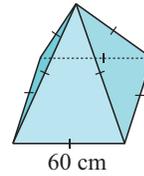
3 Find a formula for the total surface area A of a rectangular box a units long, b units wide and c units deep.

c

a   *b*
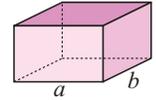
4 A shadehouse with the dimensions illustrated is to be covered with shadecloth. The cloth costs $6.75 per square metre.

a Find the area of each end of the shadehouse.

b Find the total area to be covered with cloth.

c Find the total cost of the cloth needed given that 5% more than the calculated amount is necessary to attach it to the frame.

3 m

5 m

3 m   end

8 m

20 m
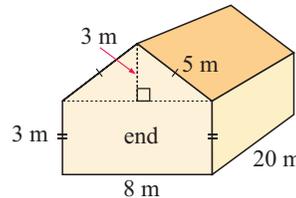
OBJECTS WITH CURVED SURFACES

Formulae can be derived for the surface areas of cylinders, cones and spheres. We include a proof of the formula for the area of surface of a cone, but the proof for a sphere is beyond the level of this course.

CYLINDERS

Object	Figure	Outer surface area	
Hollow cylinder	hollow h r hollow	$A = 2\pi rh$	(no ends)
Hollow can	hollow h r solid	$A = 2\pi rh + \pi r^2$	(one end)
Solid cylinder	solid h r solid	$A = 2\pi rh + 2\pi r^2$	(two ends)

bottom

CONES

The curved surface of a cone is made from a sector of a circle with radius equal to the slant height of the cone. The circumference of the base equals the arc length of the sector.
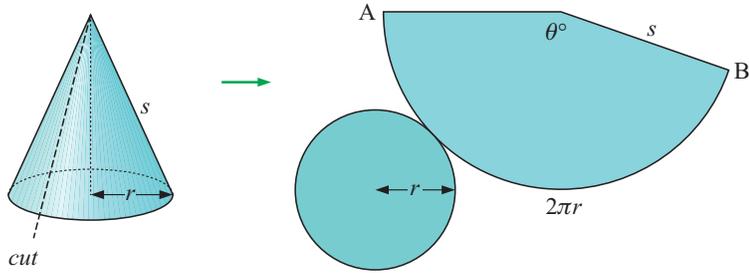
$$\text{arc AB} = \left(\frac{\theta}{360}\right) 2\pi s$$

But $\text{arcAB} = 2\pi r$

$$\therefore \quad \left(\frac{\theta}{360}\right) 2\pi s = 2\pi r$$

$$\therefore \quad \frac{\theta}{360} = \frac{r}{s}$$

The area of curved surface = area of sector

$$= \left(\frac{\theta}{360}\right) \pi s^2$$

$$= \left(\frac{r}{s}\right) \pi s^2$$

$$= \pi r s$$

The area of the base $= \pi r^2$

$$\therefore \quad \text{the total area} = \pi r s + \pi r^2$$

Object	Figure	Outer surface area	
Hollow cone		$A = \pi r s$	(no base)
Solid cone		$A = \pi r s + \pi r^2$	(solid)

SPHERES
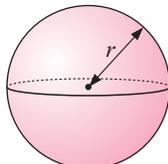
Surface area $A = 4\pi r^2$

Example 16   ◀) **Self Tutor**

Find the surface area of a solid cone of base radius 5 cm and height 12 cm.

Let the slant height be s cm.
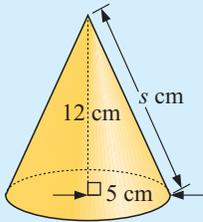
$$s^2 = 5^2 + 12^2 \qquad \text{\{Pythagoras\}}$$
$$\therefore \quad s^2 = 169$$
$$\therefore \quad s = \sqrt{169} = 13 \qquad \text{\{as } s > 0\text{\}}$$
$$\text{Now} \quad A = \pi r^2 + \pi rs$$
$$\therefore \quad A = \pi \times 5^2 + \pi \times 5 \times 13$$
$$\therefore \quad A \approx 282.7$$

Calculator: $\boxed{\pi}$ $\boxed{\times}$ 5 $\boxed{x^2}$ $\boxed{+}$ $\boxed{\pi}$ $\boxed{\times}$ 5 $\boxed{\times}$ 13 $\boxed{\text{ENTER}}$

Thus the surface area is approximately 283 cm^2.

EXERCISE 7D.2

1 Find the total surface area of:

    a a cylinder of base radius 9 cm and height 20 cm

    b a cone of base radius and perpendicular height both 10 cm

    c a sphere of radius 6 cm

    d a hemisphere of base radius 10 m

    e a cone of base radius 8 cm and vertical angle 60°.
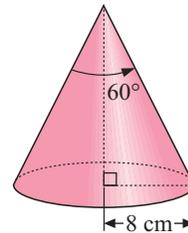
2 The cost of manufacturing a hollow hemispherical glass dome is given by $C = \$(5200 + 35A)$, where A is its outer surface area in square metres. Find the cost of making a glass hemispherical dome of diameter 10 m.

3 A cylindrical wheat silo is 40 m high and 20 m in diameter. Determine the cost of painting the exterior walls and top of the silo given that each litre of paint costs €7.25 and covers 8 m^2.
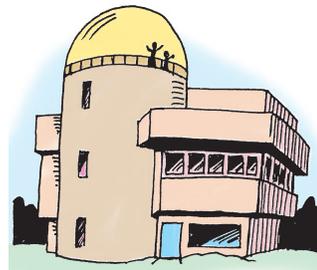
4 How many spheres of 15 cm diameter can be covered by 10 m^2 of material?

5 Determine the total area of leather needed to cover 20 dozen cricket balls, each with diameter 7 cm.

6 Find the cost of making 125 cylindrical tennis ball containers closed at one end, if the diameter is 7 cm and height is 21 cm, and the metal costs $4.50 per square metre.

7 Find: **a** the radius of a sphere of surface area 400 m^2

        b the height of a solid cylinder of radius 10 cm and surface area 2000 cm^2

        c the slant height of a solid cone of base radius 8 m and surface area 850 m^2.

E   VOLUME AND CAPACITY

The **volume** of a solid is the amount of space it occupies.

UNITS OF VOLUME

Volume can be measured in cubic millimetres, cubic centimetres or cubic metres.

1 cm^3
$= 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm}$
$= 1000 \text{ mm}^3$

1 m^3
$= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm}$
$= 1\,000\,000 \text{ cm}^3$
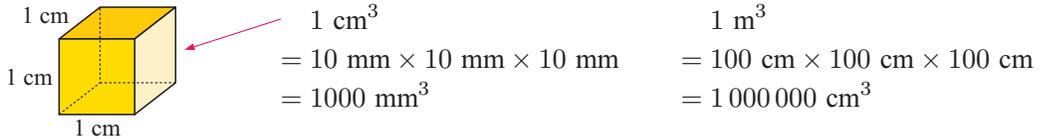
VOLUME UNITS CONVERSIONS

$\times 1\,000\,000\,000$ $\times 1\,000\,000$ $\times 1000$

km³   m³   cm³   mm³

$\div 1\,000\,000\,000$ $\div 1\,000\,000$ $\div 1000$
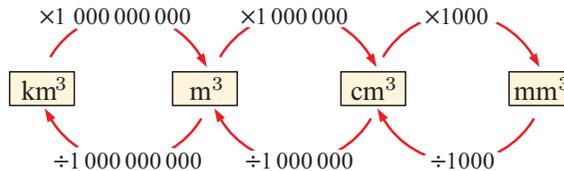
The **capacity** of a container is the quantity of fluid (liquid or gas) that it may contain.

UNITS OF CAPACITY

The basic unit of capacity is the **litre (L)**.

1 litre (L) = 1000 millilitres (mL)
1 kilolitre (kL) = 1000 litres (L)
1 megalitre (ML) = 1000 kilolitres (kL)

CONNECTING VOLUME AND CAPACITY

1 millilitre (mL) of fluid fills a container of size 1 cm^3.

We say: $1 \text{ mL} \equiv 1 \text{ cm}^3$, $1 \text{ L} \equiv 1000 \text{ cm}^3$ and $1 \text{ kL} = 1000 \text{ L} \equiv 1 \text{ m}^3$.

CAPACITY UNITS CONVERSION

$\times 1000$ $\times 1000$ $\times 1000$

ML   kL   L   mL
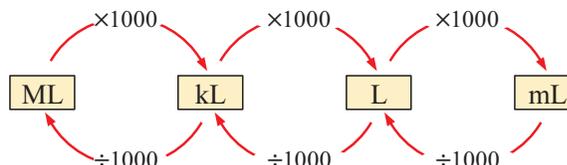
$\div 1000$ $\div 1000$ $\div 1000$

Example 17   ◀) **Self Tutor**

Convert the following:

a $300\,000$ cm^3 to m^3 **b** 0.38 L to mL   **c** 5.6 L to cm^3

a $300\,000$ cm^3
$= (300\,000 \div 100^3)$ m^3
$= (300\,000 \div 1\,000\,000)$ m^3
$= 0.3$ m^3

b 0.38 L to mL
$= (0.38 \times 1000)$ mL
$= 380$ mL

c 5.6 L to cm^3
$= (5.6 \times 1000)$ cm^3
$= 5600$ cm^3

EXERCISE 7E.1

1 Convert the following:

    a 1.9 m^3 to cm^3 **b** $34\,000$ cm^3 to m^3 **c** 2000 mm^3 to cm^3

    d 0.84 kL to L   **e** 4120 mL to L   **f** 6250 mL to kL

    g 180 L to cm^3 **h** 6500 cm^3 to kL   **i** 4800 mL to m^3

2 How many 750 mL bottles can be filled from a vat containing 42 L of wine?

3 A household used 44.1 kL of water over a 180 day period. On average, how many litres of water were used per day?

VOLUME FORMULAE

Object	Figure	Formula
Solids of uniform cross-section		**Volume of uniform solid** $=$ **area of end** $\times$ **height**
Pyramids and cones		**Volume of a pyramid or cone** $= \frac{1}{3}$**(area of base** $\times$ **height)**
Spheres		**Volume of a sphere** $= \frac{4}{3}\pi r^3$

Example 18

◀)) **Self Tutor**

Find the volume of the following:

a

4.5 cm
16.8 cm
8 cm

b
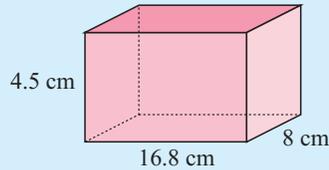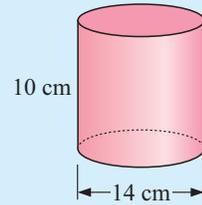
10 cm
◀—14 cm—▶

a Volume = area of end × height
= length × width × height
= $16.8 \times 8 \times 4.5$
≈ 605 cm³

b Volume = area of end × height
= $\pi \times 7^2 \times 10$
≈ 1540 cm³

EXERCISE 7E.2

1 Find the volume of:

a a rectangular box 12 cm by 15 cm by 10 cm

b a cone of radius 10 cm and slant height 18 cm

c a cylinder of height 3 m and base diameter 80 cm

d an equilateral triangular prism with height 12 cm and triangles of side length 2 cm.

2 The average depth of water in a lake is 1.7 m. It is estimated that the total surface area of the lake is 135 ha.

a Convert 135 ha to m².

b How many kilolitres of water does the lake contain?

3

1.2 m
25 m
2.4 m
8 m
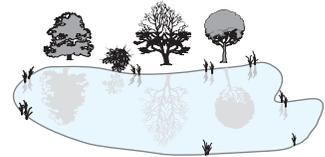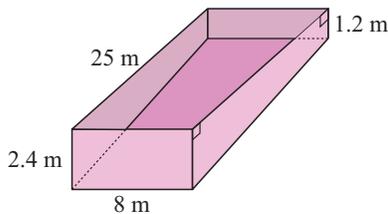
A swimming pool has the dimensions shown.

a Find the area of a trapezium-shaped side.

b Find the capacity of the pool.

4 A concrete contractor is considering his next job. He estimates the surface area of concrete to be laid is 85 m². The customer has a choice of 8 cm thick concrete or 12 cm thick concrete. How much extra would he have to pay for the thicker concrete if the concrete costs €175 per cubic metre?

5 Water enters a cylindrical rainwater tank at 80 L per minute. The base diameter of the tank is 2.4 m and the height is 4 m.

a Find the capacity of the full tank.

b Convert 80 L per min into m³ per min.

c How long will it take to fill the tank?
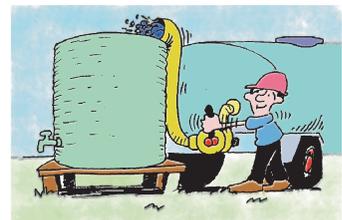
6 A gold ingot 10 cm by 5 cm by 2 cm is melted down and cast into small spheres of radius 1 cm.

    a What is the volume of the ingot?

    b What is the volume of one small sphere?

    c How many spheres can be cast?

    d What volume of gold is left over?

7 A motor car has a rectangular prism petrol tank 48 cm by 56 cm by 20 cm. If the car consumes petrol at an average rate of 8.7 litres per 100 km, how far could it travel on a full tank of petrol?

8 A concrete path 90 cm wide and 10 cm deep is placed around a circular garden bed of diameter 15 m.

    a Draw a plan view of the situation.   **b** Find the surface area of the concrete.

    c Find the volume of concrete required to lay the path.

Example 19   ◀) Self Tutor

A rectangular tank is 3 m by 2.4 m by 2 m and is full of water. The water is pumped into an empty cylindrical tank of base radius 2 m. How high up the tank will the water level rise?

Volume of rectangular tank $=$ length $\times$ width $\times$ depth

$$= 3 \times 2.4 \times 2$$
$$= 14.4 \text{ m}^3$$
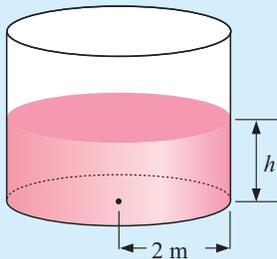
If the water in the cylindrical tank is h m deep,

its volume $= \pi \times 2^2 \times h$ $\{V = \pi r^2 h\}$
$$= 4\pi h \text{ m}^3$$

$\therefore \quad 4\pi h = 14.4$ {equating volumes}

$\therefore \quad h = \dfrac{14.4}{4\pi} \approx 1.146$

$\therefore$ the water level will rise to 1.15 m.

---

9 37 mm of rain fell overnight. Determine the diameter of a 3 m high cylindrical tank required to catch the water running off an 18 m by 12 m rectangular roof.

Example 20   ◀) Self Tutor

A concrete tank has an external diameter of 10 m and an internal height of 3 m. If the walls and bottom of the tank are 30 cm thick, how many cubic metres of concrete are required to make the tank?

30 cm

3 m

30 cm

10 m
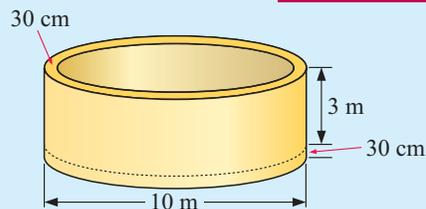
The tank's walls form a hollow cylinder with outer radius 5 m and inner radius 4.7 m. Its bottom is a cylinder with radius 5 m and height 30 cm.

walls of tank: volume = base area × height
$$= [\pi \times 5^2 - \pi \times (4.7)^2] \times 3$$
$$\approx 27.43 \text{ m}^3$$

bottom of tank: volume = base area × height
$$= \pi \times 5^2 \times 0.3$$
$$\approx 23.56 \text{ m}^3$$

Total volume of concrete required $\approx (27.43 + 23.56)$ m$^3 \approx 51.0$ m^3.

10 A concrete tank has an external diameter of 5 m and an internal height of 3 m. The walls and base of the tank are 20 cm thick.

20 cm

3 m

20 cm

5 m

   a Find the volume of concrete required to make the base.

   b Find the volume of concrete required to make the walls.

   c Find the total volume of concrete required.

   d Find the cost of the concrete at \$142 per m^3.
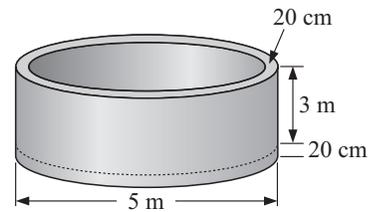
11 Find a formula for the volume V of the following illustrated solids:

a $3x$, x, $2x$

b b, a, c

c $4r$, $2r$

d $3x$, $2x$

e h, a

f $4x$, semi-circular, x

g semi-circular, b, $2a$

h x, $2x$

i $2x$
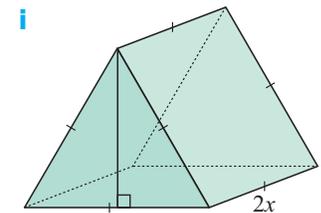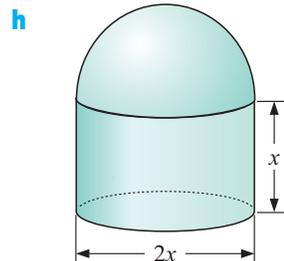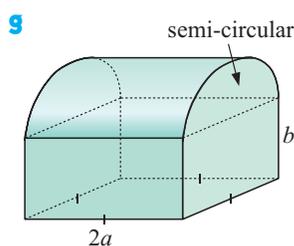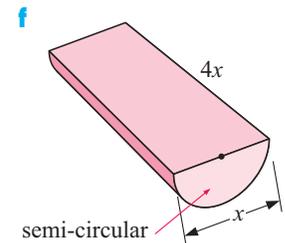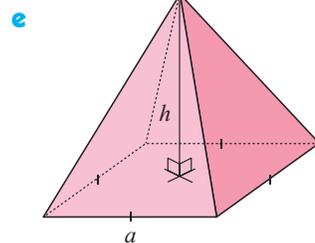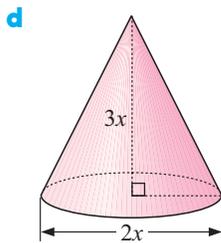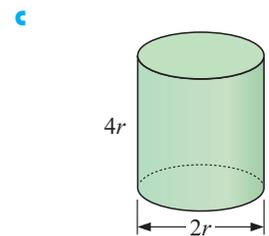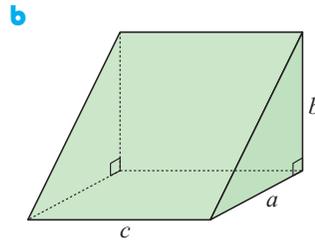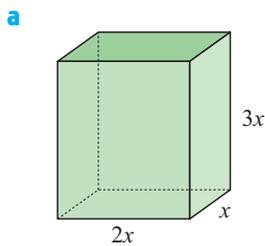
Example 21

🔊 **Self Tutor**

A sphere has volume $60\,000$ cm^3. Find its radius.

$$\text{Volume} = \tfrac{4}{3}\pi r^3$$

$$\therefore \quad \tfrac{4}{3}\pi r^3 = 60\,000$$

$$\therefore \quad \tfrac{3}{4} \times \tfrac{4}{3}\pi r^3 = 60\,000 \times \tfrac{3}{4}$$

$$\therefore \quad \pi r^3 = 45\,000$$

$$\therefore \quad r^3 = \frac{45\,000}{\pi}$$

$$\therefore \quad r = \sqrt[3]{\frac{45\,000}{\pi}}$$

$$\therefore \quad r \approx 24.3 \quad \text{(1 d.p.)} \qquad \text{so the radius is} \approx 24.3 \text{ cm.}$$

$\sqrt[3]{}$ reads "the cube root of".

12 **a** A sphere has surface area 100 cm^2. Find its: **i** radius **ii** volume.

   b A sphere has volume 27 m^3. Find its: **i** radius **ii** surface area.

13 The height of a cylinder is four times its diameter. Determine the height of the cylinder if its capacity is 60 litres.

14 A cubic metre of brass is melted down and cast into solid door handles with the shape shown. How many handles can be cast?

2 cm
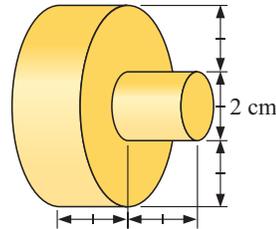
INVESTIGATION 4   MAKING CYLINDRICAL BINS

Your business has won a contract to make $40\,000$ cylindrical bins, each to contain $\frac{1}{20}$ m^3.

To minimise costs (and therefore maximise profits) you need to design the bin of minimum surface area.

What to do:

1 Find the formula for the volume V and the outer surface area A in terms of the base radius x and the height h.

2 Convert $\frac{1}{20}$ m^3 into cm^3.

3 Show that the surface area can be written as $A = \pi x^2 + \dfrac{100\,000}{x}$ cm^2.

4 Use a graphing package to find the value of x that minimises the surface area, or else follow the graphics calculator steps below:

no top

h

$2x$
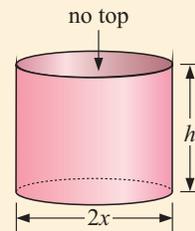
GRAPHING PACKAGE

5 Enter the **function** $Y_1 = \pi X^2 + 100\,000/X$.
Set up a **table** that calculates the value of Y for X values starting at 1 with increments of 1.
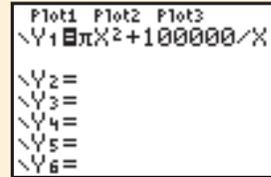
6 View the **table** you have created. Scroll down and find the value of X that seems to make Y a minimum.
You should find the minimum value of Y is 5963.5 when X = 25.000

7 Change the **window** settings based on your observations in **6** and **graph** $Y_1 = \pi X^2 + 100\,000/X$

8 Use **trace** to estimate the value of X that produces the minimum value of Y.
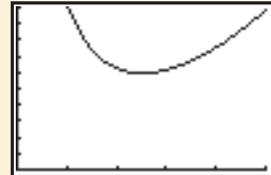
9 Check your estimation by using the built-in functions to calculate the **minimum** value of Y.
What radius will minimise the surface area of the bin to be manufactured?
State the dimensions of the bin and calculate the surface area of materials required to fill the contract.
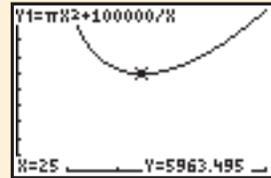
10 Investigate the dimensions of a cylindrical can which will hold 375 mL of drink and has minimum surface area. How do these dimensions compare with cans used by drink manufacturers?
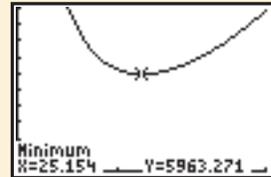
11 Revisit the **Opening Problem** on page **146** and suggest the dimensions of Byron's tank so that it has minimum surface area.

INVESTIGATION 5   THE LARGEST BAKING DISH PROBLEM

A baking dish is made from a rectangular sheet of tin plate. Squares are first removed from its corners. It is then bent into the required shape and soldered along the joins.
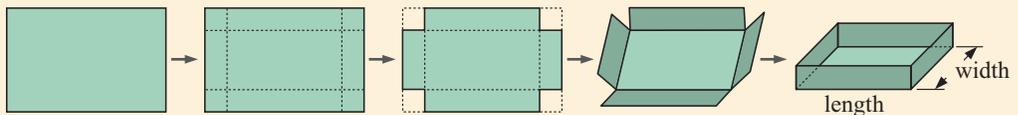
Your task is to determine the size of the squares which should be cut from each corner of a 24 cm by 18 cm sheet of tin plate so that the final dish has **maximum capacity**.

What to do:

1   Make a dish from a 24 cm by 18 cm sheet of paper or cardboard with 3 cm by 3 cm squares cut from its corners. Calculate the length, width and depth of the dish, and hence its capacity.

3 cm

2   A spreadsheet can be used to calculate the length, width, depth, and capacity of dishes with various sized squares cut from each corner of the sheet of tin-plate. We suppose the squares cut from each corner are x cm by x cm.

Open a new spreadsheet and enter the following:

SPREADSHEET

	A	B	C	D	E
1	x	length	width	depth	capacity
2	1	=24-2*A2	=18-2*A2	=A2	=B2*C2*D2
3	=A2+0.5				
4					
5					
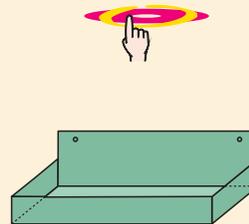
3   Fill the formulae shown down until you find the maximum capacity. What sized squares cut out will achieve a dish of maximum capacity?

4   An 'odds and ends' tray is to be made from a 30 cm by 20 cm piece of tin plate. Investigate the shape of the tray with maximum capacity.

2x cm

square cut out

x cm
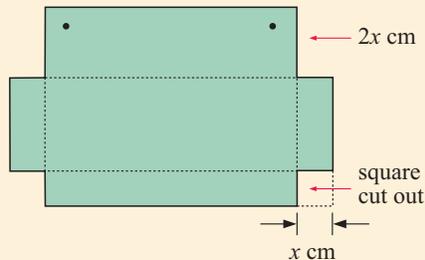
PROJECT   **OXYGEN AROUND THE SCHOOL**

Click on the icon to produce a printable set of pages for this project.

PRINTABLE PAGES

WHAT SHAPE CONTAINER SHOULD WE USE?

LINKS
click here

Areas of interaction:
Approaches to learning/The environment

REVIEW SET 7A

1   Convert:

   a 5.3 km to m   **b** 20 000 cm^2 to m^2 **c** 5 m^3 to cm^3 **d** 0.48 L to cm^3

2   Find the:

   a circumference of a circle of radius 7.5 cm

   b perimeter of a rhombus with sides 23.2 m

c perimeter of an isosceles triangle with base 8 cm and height 3 cm.

3 Find the area of:

    a a sector of radius 10 cm and angle 120°

    b a right angled triangle with base 5 cm and hypotenuse 13 cm

    c a rhombus with sides of length 5 cm and one diagonal of length 4 cm.

4 Determine the angle of a sector with arc length 32 cm and radius 7 cm.

5 Find the percentage error in a height measurement of 136 cm.

6 A rectangular prism has dimensions 5 cm $\times$ 8 cm $\times$ 10 cm. Find its:

    a volume   **b** surface area.

7 Find the total surface area of a cone of radius 5 cm and perpendicular height 8 cm.

8 How many cylindrical cans of diameter 12 cm and height 15 cm can be filled from a 3 m by 2 m by 1.5 m rectangular tank?

9 Find the formula for the area A of:

    a   **b**
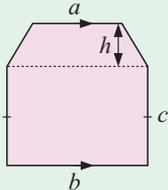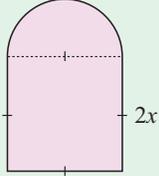
10 A cone without a base is made from a sector of a circle. If the cone is 10 cm high and has a base of diameter 8 cm, what was the radius of the sector?

REVIEW SET 7B

1 Convert:

    a 3200 mm to m   **b** 15 ha to m^2 **c** 3600 cm^3 to mm^3 **d** 4.5 kL to m^3

2 **a** Determine the length of fencing around a circular playing field of radius 80 metres if the fence is 10 metres from the edge of the field.

    b If the fencing cost £12.25 per metre of fence, what was the total cost?

3 A triangle has sides measured as 23 cm, 28 cm and 33 cm. Find, in the form $a \pm b$, the boundaries within which the true perimeter lies.

4 A sector of a circle has radius 12 cm and angle 135°.

    a What fraction of a whole circle is this sector?

    b Find the perimeter of the sector.

    c Find the area of the sector.

5 What is the cost of laying 'instant lawn' over a 60 m by 160 m playing field if the lawn strips are $\frac{1}{2}$ m wide and cost \$22.70 for each 5 m length?

6 Cans used for canned soup have a base diameter of 7 cm and a height of 10 cm.

    a How many such cans can be filled from a vat containing 2000 litres of soup?

    b Calculate the total surface area of metal required to make the cans in **a**.

7 Determine the total volume of steel required to construct 450 m of steel piping with internal diameter 1.46 m and external diameter 1.50 m.

8 What area of leather is required to manufacture 10 dozen soccer balls, each with a diameter of 24 cm?

9 If both the height and radius of a cone are doubled, by what factor is the volume increased?

HISTORICAL NOTE   PIERRE DE FERMAT 1601 - 1665

Pierre de Fermat was born in Beaumont-de-Lomagne in France, near the border of Spain, in 1601. He studied Latin and Greek literature, ancient science, mathematics and modern languages at the University of Toulouse, but his main purpose was to study law.

In 1629 Pierre studied the work of Apollonius, a geometer of ancient Greece, and discovered for himself that loci or sets of points could be studied using coordinates and algebra. His work 'Introduction to Loci' was not published for another fifty years, and together with 'La Geometrie' by Descartes, formed the basis of Cartesian geometry.

In 1631 Pierre received his degree in law and was awarded the position of 'Commissioner of Requests' in Toulouse. He was promoted to councillor, then lawyer, and was awarded the status of a minor nobleman. In 1648 he became King's Councillor.

Pierre was a man of great integrity who worked hard. He remained aloof from matters outside his own jurisdiction, and pursued his great interest in mathematics. He worked with Pascal on the Theory of Probability. He worked on a variety of equations and curves and the Archimedean spiral. In 1657 he wrote 'Concerning the Comparison of Curved Lines with Straight Lines' which was published during his lifetime.

Fermat died in 1665. He was the acknowledged master of mathematics in France at the time, but his fame would have been greater if he had published more of his work while he was alive. He became known as the founder of the modern theory of numbers.

In mid-1993, one of the most famous unsolved problems in mathematics, **Fermat's Last Theorem** was solved by **Andrew Wiles** of Princeton University (USA). Wiles made the final breakthrough after 350 years of searching by many famous mathematicians (both amateur and professional).

Fermat's Last Theorem is a simple assertion which he wrote in the margin of a mathematics book, but which he never proved, although he claimed he could.

The theorem is: There exist no positive integers, x, y and z which satisfy all equations of the form $x^n + y^n = z^n$ for integers $n \geqslant 3$.

Wiles' work establishes a whole new mathematical theory, proposed and developed over the last 60 years by the finest mathematical minds of the 20th century.

Chapter 8

Quadratic factorisation

Contents:

A **quadratic expression** in x is an expression of the form $ax^2 + bx + c$ where x is the variable, and a, b and c are constants with $a \neq 0$.

$$ax^2 \qquad + \qquad bx \qquad + \qquad c$$

the x^2 term   the x term   the constant term

For example: $x^2 + 5x + 6$, $4x^2 - 9$ and $9x^2 + 6x + 1$ are quadratic expressions.

In **Chapter 3** we studied the expansion of algebraic factors, many of which resulted in quadratic expressions. In this chapter we will consider **factorisation**, which is the reverse process of expansion. We will find later that factorisation is critical in the solution of problems that convert to quadratic equations.

Factorisation is the process of writing an expression as a *product* of *factors*.

For example:

expansion

$$(x + 2)\,(x + 3) = x^2 + 5x + 6$$

factorisation

Since $x^2 + 5x + 6 = (x + 2)(x + 3)$, we say that

$(x + 2)$ and $(x + 3)$ are *factors* of $x^2 + 5x + 6$.

You should remember the following expansion rules from **Chapter 3**:

$(x + p)(x + q) = x^2 + (p + q)x + pq$	**sum and product** expansion
$(x + a)^2 = x^2 + 2ax + a^2$	**perfect square** expansion
$(x + a)(x - a) = x^2 - a^2$	**difference of two squares** expansion

These statements are called **identities** because they are true for all values of the variable x.

Notice that the RHS of each identity is a quadratic expression which has been formed by **expanding** the LHS.

The LHS of the identities above can be obtained by **factorising** the RHS.

A   FACTORISATION BY REMOVAL OF COMMON FACTORS

Some quadratic expressions can be factorised by removing the Highest Common Factor (HCF) of the terms in the expression. In fact, we should always look to remove the HCF before proceeding with any other factorisation.

Example 1   ◀)) **Self Tutor**

Factorise by removing a common factor:

    a $2x^2 + 3x$ **b** $-2x^2 - 6x$

    a $2x^2 + 3x$ has HCF x **b** $-2x^2 - 6x$ has HCF $-2x$

        $\therefore \quad 2x^2 + 3x = x(2x + 3)$ $\therefore \quad -2x^2 - 6x = -2x(x + 3)$

Example 2   ◀)) **Self Tutor**

Fully factorise by removing a common factor:

    a $(x - 5)^2 - 2(x - 5)$ **b** $(x + 2)^2 + 2x + 4$

    a $(x - 5)^2 - 2(x - 5)$

        $= (x - 5)(x - 5) - 2(x - 5)$ $\{\text{HCF} = (x - 5)\}$

        $= (x - 5)[(x - 5) - 2]$

        $= (x - 5)(x - 7)$ $\{\text{simplifying}\}$

    b $(x + 2)^2 + 2x + 4$

        $= (x + 2)(x + 2) + 2(x + 2)$ $\{\text{HCF} = (x + 2)\}$

        $= (x + 2)[(x + 2) + 2]$

        $= (x + 2)(x + 4)$

> Check your factorisations by expansion! Notice the use of the square brackets.

EXERCISE 8A

1 Fully factorise by first removing a common factor:

    a $3x^2 + 5x$ **b** $2x^2 - 7x$ **c** $3x^2 + 6x$

    d $4x^2 - 8x$ **e** $-2x^2 + 9x$ **f** $-3x^2 - 15x$

    g $-4x + 8x^2$ **h** $-5x - 10x^2$ **i** $12x - 4x^2$

    j $x^3 + x^2 + x$ **k** $2x^3 + 11x^2 + 4x$ **l** $ab + ac + ad$

    m $ax^2 + 2ax$ **n** $ab^2 + a^2b$ **o** $ax^3 + ax^2$

2 Fully factorise by removing a common factor:

    a $(x + 2)^2 - 5(x + 2)$ **b** $(x - 1)^2 - 3(x - 1)$ **c** $(x + 1)^2 + 2(x + 1)$

    d $(x - 2)^2 + 3x - 6$ **e** $x + 3 + (x + 3)^2$ **f** $(x + 4)^2 + 8 + 2x$

    g $(x - 3)^2 - x + 3$ **h** $(x + 4)^2 - 2x - 8$ **i** $(x - 4)^2 - 5x + 20$

    j $3x + 6 + (x + 2)^2$ **k** $(x + 1)^3 + (x + 1)^2$ **l** $(a + b)^3 + a + b$

    m $2(x + 1)^2 + x + 1$ **n** $3(x - 2)^2 - (x - 2)$ **o** $4(a + b)^2 - 2a - 2b$

B   DIFFERENCE OF TWO SQUARES FACTORISATION

We know the **expansion** of $(a+b)(a-b)$ is $a^2 - b^2$.

Thus, the **factorisation** of $a^2 - b^2$ is $(a+b)(a-b)$.

$$a^2 - b^2 = (a+b)(a-b)$$

> The **difference** between a^2 and b^2 is $a^2 - b^2$ which is the difference of two squares.

Note: The **sum of two squares** does not factorise into two real linear factors.

Example 3   ◀⑴ Self Tutor

Use the rule $a^2 - b^2 = (a+b)(a-b)$ to factorise fully:

a $9 - x^2$ **b** $4x^2 - 25$

a $9 - x^2$
$= 3^2 - x^2$ {difference of squares}
$= (3+x)(3-x)$

b $4x^2 - 25$
$= (2x)^2 - 5^2$ {difference of squares}
$= (2x+5)(2x-5)$

Example 4   ◀⑴ Self Tutor

Fully factorise: **a** $2x^2 - 8$ **b** $-3x^2 + 48$

a $2x^2 - 8$
$= 2(x^2 - 4)$ {HCF is 2}
$= 2(x^2 - 2^2)$ {difference of squares}
$= 2(x+2)(x-2)$

b $-3x^2 + 48$
$= -3(x^2 - 16)$ {HCF is -3}
$= -3(x^2 - 4^2)$ {difference of squares}
$= -3(x+4)(x-4)$

> Always look to remove a common factor first.

We notice that $x^2 - 9$ is the difference of two squares and therefore we can factorise it using $a^2 - b^2 = (a + b)(a - b)$.

Even though 7 is not a perfect square, we can still factorise $x^2 - 7$ by writing $7 = (\sqrt{7})^2$.

$$\text{So,} \quad x^2 - 7 = x^2 - (\sqrt{7})^2 = (x + \sqrt{7})(x - \sqrt{7}).$$

We say that $x + \sqrt{7}$ and $x - \sqrt{7}$ are the **linear factors** of $x^2 - 7$.

Example 5   ◀ঠ **Self Tutor**

Factorise into linear factors: **a** $x^2 - 11$ **b** $(x + 3)^2 - 5$

a $\quad x^2 - 11$
$\quad = x^2 - (\sqrt{11})^2$
$\quad = (x + \sqrt{11})(x - \sqrt{11})$

b $\quad (x + 3)^2 - 5$
$\quad = (x + 3)^2 - (\sqrt{5})^2$
$\quad = [(x + 3) + \sqrt{5}][(x + 3) - \sqrt{5}]$
$\quad = [x + 3 + \sqrt{5}][x + 3 - \sqrt{5}]$

Example 6   ◀ঠ **Self Tutor**

Factorise using the difference between two squares:
a $(3x + 2)^2 - 9$
b $(x + 2)^2 - (x - 1)^2$

a $\quad (3x + 2)^2 - 9$
$\quad = (3x + 2)^2 - 3^2$
$\quad = [(3x + 2) + 3][(3x + 2) - 3]$
$\quad = [3x + 5][3x - 1]$

b $\quad (x + 2)^2 - (x - 1)^2$
$\quad = [(x + 2) + (x - 1)][(x + 2) - (x - 1)]$
$\quad = [x + 2 + x - 1][x + 2 - x + 1]$
$\quad = [2x + 1][3]$
$\quad = 3(2x + 1)$

EXERCISE 8B

1 Use the rule $a^2 - b^2 = (a + b)(a - b)$ to fully factorise:

a $x^2 - 4$ **b** $4 - x^2$ **c** $x^2 - 81$ **d** $25 - x^2$

e $4x^2 - 1$ **f** $9x^2 - 16$ **g** $4x^2 - 9$ **h** $36 - 49x^2$

2 Fully factorise:

a $3x^2 - 27$ **b** $-2x^2 + 8$ **c** $3x^2 - 75$

d $-5x^2 + 5$ **e** $8x^2 - 18$ **f** $-27x^2 + 75$

3 If possible, factorise into linear factors:

a $x^2 - 3$ **b** $x^2 + 4$ **c** $x^2 - 15$

d $3x^2 - 15$ **e** $(x + 1)^2 - 6$ **f** $(x + 2)^2 + 6$

g $(x - 2)^2 - 7$ **h** $(x + 3)^2 - 17$ **i** $(x - 4)^2 + 9$

4   Factorise using the difference of two squares:

a $(x+1)^2 - 4$

b $(2x+1)^2 - 9$

c $(1-x)^2 - 16$

d $(x+3)^2 - 4x^2$

e $4x^2 - (x+2)^2$

f $9x^2 - (3-x)^2$

g $(2x+1)^2 - (x-2)^2$

h $(3x-1)^2 - (x+1)^2$

i $4x^2 - (2x+3)^2$

C   PERFECT SQUARE FACTORISATION

We know the **expansion** of $(x+a)^2$ is $x^2 + 2ax + a^2$,

so the **factorisation** of $x^2 + 2ax + a^2$ is $(x+a)^2$.

$$x^2 + 2ax + a^2 = (x+a)^2$$

> $(x+a)^2$ and $(x-a)^2$ are **perfect squares**!

Notice that
$$(x-a)^2$$
$$= (x + (-a))^2$$
$$= x^2 + 2(-a)x + (-a)^2$$
$$= x^2 - 2ax + a^2$$

So,
$$x^2 - 2ax + a^2 = (x-a)^2$$

Example 7   ◄) **Self Tutor**

Use perfect square rules to fully factorise:

a $x^2 + 10x + 25$

b $x^2 - 14x + 49$

a $x^2 + 10x + 25$
$$= x^2 + 2 \times x \times 5 + 5^2$$
$$= (x+5)^2$$

b $x^2 - 14x + 49$
$$= x^2 - 2 \times x \times 7 + 7^2$$
$$= (x-7)^2$$

Example 8   ◄) **Self Tutor**

Fully factorise:

a $9x^2 - 6x + 1$

b $-8x^2 - 24x - 18$

a $9x^2 - 6x + 1$
$$= (3x)^2 - 2 \times 3x \times 1 + 1^2$$
$$= (3x-1)^2$$

b $-8x^2 - 24x - 18$
$$= -2(4x^2 + 12x + 9) \quad \{\text{HCF} = -2\}$$
$$= -2([2x]^2 + 2 \times 2x \times 3 + 3^2)$$
$$= -2(2x+3)^2$$

EXERCISE 8C

1 Use perfect square rules to fully factorise:

a $x^2 + 6x + 9$ **b** $x^2 + 8x + 16$ **c** $x^2 - 6x + 9$

d $x^2 - 8x + 16$ **e** $x^2 + 2x + 1$ **f** $x^2 - 10x + 25$

g $y^2 + 18y + 81$ **h** $m^2 - 20m + 100$ **i** $t^2 + 12t + 36$

2 Fully factorise:

a $9x^2 + 6x + 1$ **b** $4x^2 - 4x + 1$ **c** $9x^2 + 12x + 4$

d $25x^2 - 10x + 1$ **e** $16x^2 + 24x + 9$ **f** $25x^2 - 20x + 4$

g $-x^2 + 2x - 1$ **h** $-2x^2 - 8x - 8$ **i** $-3x^2 - 30x - 75$

D   FACTORISING EXPRESSIONS WITH FOUR TERMS

Sometimes we can factorise an expression containing four terms by grouping them in two pairs.

For example, $ax^2 + 2x + 2 + ax$ can be rewritten as

$$\underbrace{ax^2 + ax}_{} + \underbrace{2x + 2}_{}$$
$$= ax(x + 1) + 2(x + 1) \qquad \text{\{factorising each pair\}}$$
$$= (x + 1)(ax + 2) \qquad \text{\{}(x + 1)\text{ is a common factor\}}$$

Example 9   ◀) Self Tutor

Fully factorise: **a** $ax + by + bx + ay$ **b** $2x^2 - 15 + 3x - 10x$

a $ax + by + bx + ay$

$= \underbrace{ax + ay}_{} + \underbrace{bx + by}_{}$ {putting terms containing a together}

$= a(x + y) + b(x + y)$ {factorising each pair}

$= (x + y)(a + b)$ {$(x + y)$ is a common factor}

b $2x^2 - 15 + 3x - 10x$

$= \underbrace{2x^2 - 10x}_{} + \underbrace{3x - 15}_{}$ {splitting into two pairs}

$= 2x(x - 5) + 3(x - 5)$ {factorising each pair}

$= (x - 5)(2x + 3)$ {$(x - 5)$ is a common factor}

EXERCISE 8D

1 Fully factorise:

- **a** $bx + cx + by + cy$
- **b** $2px + 3q + 2qx + 3p$
- **c** $6ax + 3bx + 2b + 4a$
- **d** $am - bn - an + bm$
- **e** $3dr + r - 3ds - s$
- **f** $2ac - 5a + 2bc - 5b$
- **g** $x^2 + 5x + 7x + 35$
- **h** $x^2 - 2x - 6x + 12$
- **i** $x^2 + 3x + 9 + 3x$
- **j** $x^2 + 8 + 8x + x$
- **k** $2x^2 + 3x + 3 + 2x$
- **l** $3x^2 + x - 3x - 1$
- **m** $2x^2 + 3x + 10x + 15$
- **n** $6x^2 - 3x - 2 + 4x$
- **o** $4x^2 + x + 8x + 2$
- **p** $6x^2 + 3x + 10x + 5$
- **q** $6x^2 - 4x + 9x - 6$
- **r** $4x^2 + x - 8x - 2$
- **s** $3x^2 + 4x + 33x + 44$
- **t** $18x^2 + 3x - 12x - 2$
- **u** $10x^2 + 4x - 35x - 14$

2 Fully factorise:

- **a** $x^2 - 2x + 1 - a^2$
- **b** $x^2 - a^2 + x + a$
- **c** $b^2 - x^2 - 4x - 4$
- **d** $c^2 - x^2 + 6x - 9$
- **e** $x^2 - y^2 + y - x$
- **f** $a^2 + 2ab + a^2 - 4b^2$
- **g** $x^2 + 4x + 4 - m^2$
- **h** $x^2 + 2ax + a^2 - b^2$
- **i** $x^2 - y^2 - 3x - 3y$

E   QUADRATIC TRINOMIAL FACTORISATION

A **quadratic trinomial** is an expression of the form $ax^2 + bx + c$ where x is a variable and a, b, c are constants, $a \neq 0$.

For example: $x^2 + 7x + 6$ and $3x^2 - 13x - 10$ are both quadratic trinomials.

Consider the expansion of the product $(x + 1)(x + 6)$:

$$(x + 1)(x + 6) = x^2 + 6x + x + 1 \times 6 \qquad \{\text{using FOIL}\}$$
$$= x^2 + [6 + 1]x + [1 \times 6]$$
$$= x^2 + [\textbf{sum of 1 and 6}]x + [\textbf{product of 1 and 6}]$$
$$= x^2 + 7x + 6$$

More generally, $(x + p)(x + q) = x^2 + qx + px + pq$
$$= x^2 + (p + q)x + pq$$

and so

$$x^2 + (p + q)x + pq = (x + p)(x + q)$$

the coefficient of x is the sum of p and q

the constant term is the product of p and q

So, if we are asked to factorise $x^2 + 7x + 6$,   we need to look for two numbers with a product of 6 and a sum of 7. These numbers are 1 and 6, and so $x^2 + 7x + 6 = (x + 1)(x + 6)$.

We call this the **sum and product method**.

Example 10   ◀)) **Self Tutor**

Use the sum and product method to fully factorise:

   a $x^2 + 5x + 4$ **b** $x^2 - x - 12$

---

  a $x^2 + 5x + 4$ has $p + q = 5$ and $pq = 4$.

      $\therefore$ p and q are 1 and 4.

      $\therefore$ $x^2 + 5x + 4 = (x + 1)(x + 4)$

  b $x^2 - x - 12$ has $p + q = -1$ and $pq = -12$.

      $\therefore$ p and q are -4 and 3.

      $\therefore$ $x^2 - x - 12 = (x - 4)(x + 3)$

> The sum of the numbers is the coefficient of x. The product of the numbers is the constant term.

Example 11   ◀)) **Self Tutor**

Fully factorise by first removing a common factor:

   a $3x^2 - 9x + 6$ **b** $-2x^2 + 2x + 12$

---

  a $3x^2 - 9x + 6$

      $= 3(x^2 - 3x + 2)$ {removing 3 as a common factor}

      $= 3(x - 2)(x - 1)$ {sum $= -3$ and product $= 2$

                         $\therefore$ the numbers are -2 and -1}

  b $-2x^2 + 2x + 12$

      $= -2(x^2 - x - 6)$ {removing -2 as a common factor}

      $= -2(x - 3)(x + 2)$ {sum $= -1$ and product $= -6$

                         $\therefore$ the numbers are -3 and 2}

EXERCISE 8E

1 Use the $x^2 + (p + q)x + pq = (x + p)(x + q)$ factorisation to fully factorise:

  a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 - x - 6$

  d $x^2 + 3x - 10$ **e** $x^2 + 4x - 21$ **f** $x^2 + 8x + 16$

  g $x^2 - 14x + 49$ **h** $x^2 + 3x - 28$ **i** $x^2 + 7x + 10$

  j $x^2 - 11x + 24$ **k** $x^2 + 15x + 44$ **l** $x^2 + x - 42$

  m $x^2 - x - 56$ **n** $x^2 - 18x + 81$ **o** $x^2 - 4x - 32$

2 Fully factorise by first removing a common factor:

  a $2x^2 - 6x - 8$ **b** $3x^2 + 9x - 12$ **c** $5x^2 + 10x - 15$

  d $4x^2 + 4x - 80$ **e** $2x^2 - 4x - 30$ **f** $3x^2 + 12x - 63$

  g $-2x^2 + 2x + 40$ **h** $-3x^2 + 12x - 12$ **i** $-7x^2 - 21x + 28$

  j $-x^2 - 3x - 2$ **k** $-x^2 + 5x - 6$ **l** $-x^2 + 9x - 18$

  m $5x^2 + 15x - 50$ **n** $-2x^2 - 8x + 42$ **o** $4x - x^2 + 32$

F   MISCELLANEOUS FACTORISATION

Use the following steps in order to **factorise quadratic expressions**:

> *Step 1:*   Look carefully at the quadratic expression to be factorised.
>
> *Step 2:*   If there is a **common factor**, take it out.
>
> *Step 3:*   Look for a **perfect square** factorisation: $x^2 + 2ax + a^2 = (x + a)^2$
> $$\text{or} \quad x^2 - 2ax + a^2 = (x - a)^2$$
>
> *Step 4:*   Look for the **difference of two squares**: $x^2 - a^2 = (x + a)(x - a)$
>
> *Step 5:*   Look for the **sum and product** type: $x^2 + (p + q)x + pq$
> $$= (x + p)(x + q)$$

EXERCISE 8F

1   Where possible, fully factorise the following expressions:

a $3x^2 + 9x$

b $4x^2 - 1$

c $5x^2 - 15$

d $3x - 5x^2$

e $x^2 + 3x - 40$

f $2x^2 - 32$

g $x^2 + 9$

h $x^2 + 10x + 25$

i $x^2 - x - 6$

j $x^2 - 16x + 39$

k $x^2 - 7x - 60$

l $x^2 - 2x - 8$

m $x^2 + 11x + 30$

n $x^2 + 6x - 16$

o $x^2 - 5x - 24$

p $3x^2 + 6x - 72$

q $4x^2 - 8x - 60$

r $3x^2 - 42x + 99$

s $-x^2 + 9x - 14$

t $-x^2 - 13x - 36$

u $-2x^2 - 14x + 36$

G   FACTORISATION OF $ax^2 + bx + c$, $a \neq 1$

In the previous section we revised techniques for factorising quadratic expressions in the form $ax^2 + bx + c$ where:

- $a = 1$
  For example: $x^2 + 5x + 6$
  $$= (x + 3)(x + 2)$$

- a was a common factor
  For example: $2x^2 + 10x + 12$
  $$= 2(x^2 + 5x + 6)$$
  $$= 2(x + 3)(x + 2)$$

- we had a perfect square or difference of two squares type
  For example: $4x^2 - 9 = (2x)^2 - 3^2$
  $$= (2x + 3)(2x - 3)$$

Factorising a quadratic expression such as $3x^2 + 11x + 6$ appears to be more complicated because it does not fall into any of these categories.

We need to develop a method for factorising this type of quadratic expression.

Two methods for factorising $ax^2 + bx + c$ where $a \neq 1$ are commonly used:

- trial and error
- 'splitting' the x-term

FACTORISATION BY TRIAL AND ERROR

Consider the quadratic $3x^2 + 13x + 4$.

Since 3 is a prime number, $3x^2 + 13x + 4 = (3x\quad)(x\quad)$

with arrows labelled *outers* and *inners*.

To fill the gaps we need two numbers with a product of 4 and so the sum of the inner and outer terms is $13x$.

As the product is 4 we will try   2 and 2,   4 and 1,   and   1 and 4.

$$(3x + 2)(x + 2) = 3x^2 + 6x + 2x + 4 \quad \text{fails}$$
$$(3x + 4)(x + 1) = 3x^2 + 3x + 4x + 4 \quad \text{fails}$$
$$(3x + 1)(x + 4) = 3x^2 + 12x + x + 4 \quad \text{is successful}$$

So, $3x^2 + 13x + 4 = (3x + 1)(x + 4)$

We could set these trials out in table form:

$3x$	2	4	1
x	2	1	4
	$8x$	$7x$	$13x$

This entry is $3x \times 2 + x \times 2$

For the general case $ax^2 + bx + c$ where a and c are not prime, there can be many possibilities.

For example,   consider $8x^2 + 22x + 15$.

By using trial and error, the possible factorisations are:

$(8x + 5)(x + 3)$	$\times$	$(4x + 5)(2x + 3)$	$\checkmark$	this is correct
$(8x + 3)(x + 5)$	$\times$	$(4x + 3)(2x + 5)$	$\times$	
$(8x + 1)(x + 15)$	$\times$	$(4x + 15)(2x + 1)$	$\times$	
$(8x + 15)(x + 1)$	$\times$	$(4x + 1)(2x + 15)$	$\times$	

We could set these trials out in table form:

$8x$	5	3	1	15
x	3	5	15	1
	$29x$	$43x$	$121x$	$23x$

or

$4x$	5	3	1	15
$2x$	3	5	15	1
	$22x$	$26x$	$62x$	$34x$

As you can see, this process can be very tedious and time consuming.

FACTORISATION BY 'SPLITTING' THE x-TERM

Using the FOIL rule, we see that

$$(2x + 3)(4x + 5)$$
$$= 8x^2 + 10x + 12x + 15$$
$$= 8x^2 + 22x + 15$$

We will now *reverse* the process to factorise the quadratic expression $8x^2 + 22x + 15$.

Notice that: $8x^2 + 22x + 15$

$= 8x^2 + 10x + 12x + 15$	{'splitting' the middle term}
$= (8x^2 + 10x) + (12x + 15)$	{grouping in pairs}
$= 2x(4x + 5) + 3(4x + 5)$	{factorising each pair separately}
$= (4x + 5)(2x + 3)$	{completing the factorisation}

But how do we correctly 'split' the middle term? How do we determine that $22x$ must be written as $+10x + 12x$?

When looking at $8x^2 + 10x + 12x + 15$ we notice that $8 \times 15 = 120$ and $10 \times 12 = 120$
and also $10 + 12 = 22$.

So, for $8x^2 + 22x + 15$,   we need two numbers whose *sum* is 22 and whose *product* is $8 \times 15 = 120$.   These numbers are 10 and 12.

Likewise, for $6x^2 + 19x + 15$ we would need two numbers with sum 19 and product $6 \times 15 = 90$.

These numbers are 10 and 9, so

$$6x^2 + 19x + 15$$
$$= 6x^2 + 10x + 9x + 15$$
$$= (6x^2 + 10x) + (9x + 15)$$
$$= 2x(3x + 5) + 3(3x + 5)$$
$$= (3x + 5)(2x + 3)$$

The following procedure is recommended for factorising $ax^2 + bx + c$ by 'splitting' the x-term:

Step 1:   Find ac and then the factors of ac which add to b.

Step 2:   If these factors are p and q, replace bx by $px + qx$.

Step 3:   Complete the factorisation.

Example 12
🔊 **Self Tutor**

Show how to split the middle term of the following so that factorisation can occur:

a $3x^2 + 7x + 2$ **b** $10x^2 - 23x - 5$

a In $3x^2 + 7x + 2$, $ac = 3 \times 2 = 6$ and $b = 7$.
We need two numbers with a product of 6 and a sum of 7. These are 1 and 6.
So, the split is $7x = x + 6x$.

b In $10x^2 - 23x - 5$, $ac = 10 \times -5 = -50$ and $b = -23$.
We need two numbers with a product of -50 and a sum of -23. These are -25 and 2.
So, the split is $-23x = -25x + 2x$.

Example 13 ◀) **Self Tutor**

Factorise by 'splitting' the x-term:

a $6x^2 + 19x + 10$ **b** $3x^2 - x - 10$

a $6x^2 + 19x + 10$ has $ac = 60$ and $b = 19$. We need two numbers with a product of 60 and a sum of 19.
Searching amongst the factors of 60, only 4 and 15 have a sum of 19.

$\therefore \quad 6x^2 + 19x + 10$
$= 6x^2 + 4x + 15x + 10$ {splitting the x-term}
$= 2x(3x + 2) + 5(3x + 2)$ {factorising in pairs}
$= (3x + 2)(2x + 5)$ {taking out the common factor}

b $3x^2 - x - 10$ has $ac = -30$ and $b = -1$. We need two numbers with a product of -30 and a sum of -1.
Searching amongst the factors of -30, only 5 and -6 have a sum of -1.

$\therefore \quad 3x^2 - x - 10$
$= 3x^2 + 5x - 6x - 10$ {splitting the x-term}
$= x(3x + 5) - 2(3x + 5)$ {factorising in pairs}
$= (3x + 5)(x - 2)$ {taking out the common factor}

Remember to check your factorisations by expansion!

EXERCISE 8G

1 Fully factorise:

  a $2x^2 + 5x + 3$ **b** $2x^2 + 7x + 5$ **c** $7x^2 + 9x + 2$

  d $3x^2 + 7x + 4$ **e** $3x^2 + 13x + 4$ **f** $3x^2 + 8x + 4$

  g $8x^2 + 14x + 3$ **h** $21x^2 + 17x + 2$ **i** $6x^2 + 5x + 1$

  j $6x^2 + 19x + 3$ **k** $10x^2 + 17x + 3$ **l** $14x^2 + 37x + 5$

2 Fully factorise:

  a $2x^2 - 9x - 5$ **b** $3x^2 + 5x - 2$ **c** $3x^2 - 5x - 2$

  d $2x^2 + 3x - 2$ **e** $2x^2 + 3x - 5$ **f** $5x^2 - 14x - 3$

  g $5x^2 - 8x + 3$ **h** $11x^2 - 9x - 2$ **i** $3x^2 - 7x - 6$

  j $2x^2 - 3x - 9$ **k** $3x^2 - 17x + 10$ **l** $5x^2 - 13x - 6$

m $3x^2 + 10x - 8$	**n** $2x^2 + 17x - 9$	**o** $2x^2 + 9x - 18$
p $2x^2 + 11x - 21$	**q** $15x^2 + x - 2$	**r** $21x^2 - 62x - 3$
s $9x^2 - 12x + 4$	**t** $12x^2 + 17x - 40$	**u** $16x^2 + 34x - 15$

Example 14
◀ϑ **Self Tutor**

Fully factorise: $-5x^2 - 7x + 6$

We remove -1 as a common factor first.

$-5x^2 - 7x + 6$
$= -1[5x^2 + 7x - 6]$ ◀——————
$= -[5x^2 + 10x - 3x - 6]$
$= -[5x(x + 2) - 3(x + 2)]$
$= -[(x + 2)(5x - 3)]$
$= -(x + 2)(5x - 3)$

Here, $ac = -30$ and $b = 7$. We need two numbers with a product of -30 and a sum of 7. These are 10 and -3.

3 Fully factorise by first removing -1 as a common factor:

a $-3x^2 - x + 14$	**b** $-5x^2 + 11x - 2$	**c** $-4x^2 - 9x + 9$
d $-9x^2 + 12x - 4$	**e** $-8x^2 - 14x - 3$	**f** $-12x^2 + 16x + 3$

INVESTIGATION   ANOTHER FACTORISATION TECHNIQUE

What to do:

1 By expanding the brackets, show that
$$\frac{(ax + p)(ax + q)}{a} = ax^2 + [p + q]x + \left[\frac{pq}{a}\right].$$

2 If $ax^2 + bx + c = \dfrac{(ax + p)(ax + q)}{a}$,   show that $p + q = b$ and $pq = ac$.

3 Using **2** on $8x^2 + 22x + 15$,   we have

$$8x^2 + 22x + 15 = \frac{(8x + p)(8x + q)}{8} \quad \text{where} \quad \begin{cases} p + q = 22 \\ pq = 8 \times 15 = 120. \end{cases}$$

So, $p = 12$ and $q = 10$ (or vice versa)

$$\therefore \quad 8x^2 + 22x + 15 = \frac{(8x + 12)(8x + 10)}{8}$$
$$= \frac{\cancel{4}(2x + 3)\cancel{2}(4x + 5)}{\cancel{8}}$$
$$= (2x + 3)(4x + 5)$$

a Use the method shown to factorise:
i $3x^2 + 14x + 8$ **ii** $12x^2 + 17x + 6$ **iii** $15x^2 + 14x - 8$

b Check your answers to **a** using expansion.

LINKS
click here

THE GOLDEN RATIO

Areas of interaction:
Human ingenuity

REVIEW SET 8A

1 Fully factorise:

    a $3x^2 + 12x$ **b** $15x - 3x^2$ **c** $(x+3)^2 - 4(x+3)$

    d $4x^2 - 9$ **e** $6x^2 - 24y^2$ **f** $x^2 - 13$

    g $(x+1)^2 - 6$ **h** $x^2 + 8x + 16$ **i** $x^2 - 10x + 25$

2 Fully factorise:

    a $2x^2 + 8x + 6$ **b** $5x^2 - 10x + 5$

    c $ax + 2a + 2b + bx$ **d** $2cx - 2dx + d - c$

    e $3x^2 + 2x + 8 + 12x$ **f** $6x^2 + 9x - 2x - 3$

    g $x^2 - 8x + 24 - 3x$ **h** $x^2 + 4x + 4 - a^2$

3 Fully factorise:

    a $2x^2 + 17x + 8$ **b** $2x^2 + 15x - 8$ **c** $2x^2 - 17x + 8$

    d $6x^2 - 11x - 10$ **e** $12x^2 + 5x - 2$ **f** $12x^2 - 8x - 15$

REVIEW SET 8B

1 Fully factorise:

    a $4x^2 - 8x$ **b** $16x - 8x^2$ **c** $(2x-1)^2 + 2x - 1$

    d $9 - 25x^2$ **e** $18 - 2a^2$ **f** $x^2 - 23$

    g $(x+2)^2 - 3$ **h** $x^2 - 12x + 36$ **i** $2x^2 + 8x + 8$

2 Fully factorise:

    a $3x^2 - 6x - 9$ **b** $7x^2 + 28x + 28$ **c** $mx + nx - my - ny$

    d $3a^2 + ab - 2b^2 - 6ab$ **e** $3x + 2x^2 + 8x + 12$ **f** $6x + 4x^2 - 2x - 3$

3 Fully factorise:

    a $3x^2 - 17x - 6$ **b** $3x^2 - 19x + 6$ **c** $3x^2 + 17x - 6$

    d $12x^2 + 7x + 1$ **e** $12x^2 - 23x - 2$ **f** $9x^2 + 12x + 4$

HISTORICAL NOTE   SRINIVASA RAMANUJAN 1887 - 1920

Ramanujan was born in India in 1887. His parents were poor, but were able to send him to school. He was fascinated by mathematics.

Early attempts to study at University failed because he was required to study other subjects as well as mathematics, and mathematics was the only subject at which he excelled. Also, he was extremely poor.

He taught himself from books and worked at home on mathematical research. He was always meticulous in the recording of his work and his results, but only rarely did he work on proofs of his theories.

Fortunately, he was able to obtain a position at the Madras Port Trust Office, a job that paid a small wage and left him with enough time to continue his research. He was able to take away used wrapping paper on which to write his mathematics. Eventually, Ramanujan obtained a grant from Madras University which enabled him to have access to, and time to use, the library and research facilities, and he was able to undertake his studies and research in a logical way. As a result, Ramanujan was awarded a scholarship to Cambridge University in England. Some of his work revealed amazing discoveries, but it also revealed a lack of background knowledge and Ramanujan spent most of his time improving his basic knowledge and establishing proofs for some of his discoveries.

The English climate and food did not agree with Ramanujan, but he continued working on mathematics and he published 32 important papers between 1914 and 1921 even though he was ill with tuberculosis.

In 1918 Ramanujan was made a Fellow of the Royal Society and was awarded Fellowship of Trinity College. He was too ill to accept the position of professor of mathematics at Madras University. He returned to India and died in 1920.

The famous English mathematician Godfrey Hardy wrote of Ramanujan:

"One gift he has which no-one can deny: profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain."

Chapter **9**

Statistics

Contents:

In today's world vast quantities of information are recorded, such as the population of countries and where they live, the number of children in families, the number of people unemployed, how much wheat is produced, daily temperatures, and much more.

Many groups in our society use this information to help them discover facts, make decisions, and predict outcomes. Government departments, businesses and scientific research bodies are all groups in our society which use statistics.

The word **statistics** can be used in three different contexts:

- **Statistics** is a branch of mathematics that is concerned with how information is collected, organised, presented, summarised and then analysed so that conclusions may be drawn from the information.
- **Statistics** may be defined as a collection of facts or data about a group or population.
- The singular **statistic** refers to a quantity calculated from sample data. For example, the mean is a statistic, the range is a statistic.

HISTORICAL NOTE

The earliest statistical recordings include:

- ancient **Babylonians** recorded their crop yields on clay tablets.
- ancient **Egyptian pharaohs** recorded their wealth on stone walls.

More recently:

- **William Playfair** (1759-1823) developed the histogram to display data.
- **Florence Nightingale** (1820-1910) kept records of the dead and injured during the Crimean War. She developed and used graphs extensively.
- **John Tukey** (1915-2000) invented the stemplot in 1972 and the boxplot in 1977.

OPENING PROBLEM

The heights of 1432 year 10 girls were measured to the nearest centimetre. The results were recorded in classes in the frequency table alongside.

Height	Frequency
120 - 129	1
130 - 139	0
140 - 149	34
150 - 159	139
160 - 169	478
170 - 179	642
180 - 189	117
190 - 199	20
200 - 209	1
Total	1432

Things to think about:

- Is the data categorical or quantitative (numerical)?
- Why was the data collected like this?
- Is the data discrete or continuous, and what are your reasons for making your decision?
- What does the height 140 - 149 actually mean?
- How should the data be displayed?
- How can the shape of the distribution be described?
- Are there any outliers in the data and how should they be treated?
- What is the best way of measuring the centre of the height distribution?
- What measure of the distribution's spread is appropriate?

A   DISCRETE NUMERICAL DATA

A **variable** is a quantity that can have a value recorded for it or to which we can assign an attribute or quality. Data is made up of individual observations of a **variable**.

The variables that we commonly deal with are:

- **categorical variables** which come in types or categories
- **quantitative** or **numerical variables** which could be **discrete** or **continuous**.

In previous courses we have examined how to collect, organise, graph and analyse categorical data. In this section we consider only **discrete numerical data**.

DISCRETE NUMERICAL DATA

A **discrete numerical variable** can only take distinct values which we find by **counting**. We record it as a **number**.

Examples of discrete numerical variables are:

The number of children in a family: the variable can take the values 0, 1, 2, 3, ...

The score for a test, out of 30 marks: the variable can take the values 0, 1, 2, 3, ..., 29, 30.

ORGANISING DISCRETE NUMERICAL DATA

Discrete numerical data can be organised:

- in a **tally and frequency table**
- using a **dot plot**
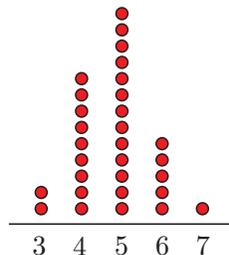- using a **stem-and-leaf plot** or **stemplot**.

Stemplots are used when there are many possible data values. The stemplot is a form of grouping of the data which displays frequencies but retains the actual data values.

Examples:

- **frequency table**

Number	Tally	Freq.
3	||	2
4	⊬⊬ ||||	9
5	⊬⊬ ⊬⊬ |||	13
6	⊬⊬	5
7	|	1

- **dot plot**

- **stemplot**

Stem	Leaf
0	9
1	7 1
2	8 3 6 7 6 4
3	9 3 5 5 6 8 2 1
4	7 9 3 4 2
5	1

As data is collected, it can be entered directly into a carefully prepared dot plot or stemplot.

Example 1   ◄) Self Tutor

The score for a test out of 50   25, 36, 38, 49, 23, 46, 47, 15, 28, 38, 34, 9, 30,
was recorded for 36 students.   24, 27, 27, 42, 16, 28, 31, 24, 46, 25, 31, 37, 35,
                                       32, 39, 43, 40, 50, 47, 29, 36, 35, 33

  a Organise the data using a stem-and-leaf plot.

  b What percentage of students scored 40 or more marks?

The stems will be 0, 1, 2, 3, 4, 5 to account for numbers from 0 to 50.

  a **Unordered stemplot**   **Ordered stemplot**

Stem	Leaf		Stem	Leaf
0	9		0	9
1	5 6		1	5 6
2	5 3 8 4 7 7 8 4 5 9		2	3 4 4 5 5 7 7 8 8 9
3	6 8 8 4 0 1 1 7 5 2 9 6 5 3		3	0 1 1 2 3 4 5 5 6 6 7 8 8 9
4	9 6 7 2 6 3 0 7		4	0 2 3 6 6 7 7 9
5	0 *Scale:* 2 | 4 is 24 marks		5	0

 **b** 9 students scored 40 or more marks, and $\frac{9}{36} \times 100\% = 25\%$.

Notice in the **ordered stemplot** in **Example 1** that:

- all of the actual data values are shown
- the minimum or smallest data value is easy to find (9 in this case)
- the maximum or largest data value is easy to find (50 in this case)
- the range of values that occurs most often is easy to see (30 - 39 in this case)
- the shape of the distribution of the data is easy to see.

DESCRIBING THE DISTRIBUTION OF THE DATA SET

Many data sets show **symmetry** or **partial symmetry** about the mean.

If we place a curve over the column graph alongside we see that this curve shows symmetry. We have a **symmetrical distribution** of the data.
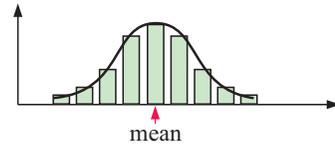
This distribution is said to be **negatively skewed** since, if we compare it with the symmetrical distribution above, it has been 'stretched' on the left or negative side of the mode.

So, we have:

symmetrical   negative side is stretched   negatively skewed   positive side is stretched   positively skewed
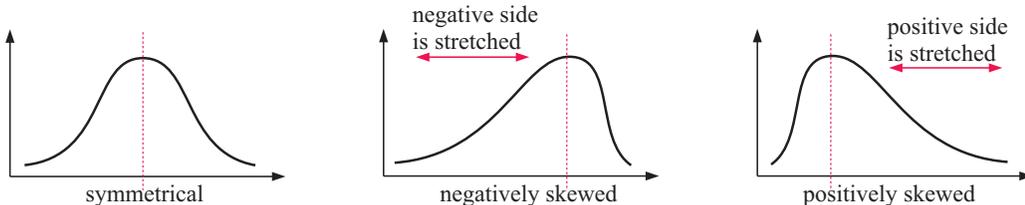
OUTLIERS

Outliers are data values that are either much larger or much smaller than the general body of data. Outliers appear separated from the body of data on a frequency graph.

For example, suppose we are examining the number of peas in a pod. We find in a sample one pod which contains 13 peas. It is much larger than the other data in the sample, and appears separated on the column graph. We consider the value 13 to be an outlier.
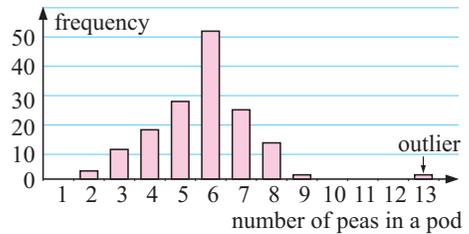
EXERCISE 9A

1 Classify the following data as categorical, discrete numerical, or continuous numerical:
   a the number of pages in a daily newspaper
   b the maximum daily temperature in the city
   c the manufacturer of a car
   d the preferred football code
   e the position taken by a player on a hockey field
   f the time it takes 15-year-olds to run one kilometre
   g the length of feet
   h the number of goals shot by a netballer
   i the amount spent weekly at the supermarket.

2 A sample of lamp posts were surveyed for the following data. Classify the data as categorical, discrete numerical, or continuous numerical:
   a the diameter of the lamp post in centimetres, measured 1 metre from its base
   b the material from which the lamp post is made
   c the location of the lamp post (inner, outer, North, South, East, or West)
   d the height of the lamp post in metres
   e the time in months since the last inspection
   f the number of inspections since installation
   g the condition of the lamp post (very good, good, fair, or unsatisfactory).

3 a Construct a vertical column graph for the given data.
   b Classify the given data set.
   c Classify the shape of the distribution.

STATISTICS PACKAGE

Number of tablets in a box	Frequency
29	11
30	28
31	7
32	3
33	0
34	1

4 **a** Construct a vertical column graph for the given data:

b Classify the given data set.

c Classify the shape of the distribution.

STATISTICS PACKAGE

Number of toothpicks in a box	Frequency
33	1
34	5
35	7
36	13
37	12
38	8
39	2

5 An ice hockey player has recorded the number of goals he has scored in each of his last 30 matches:

1   1   3   2   0   0   4   2   2   4   3   1   0   1   0
2   1   5   1   3   7   2   2   2   4   3   1   1   0   3

a Construct a dot plot for the raw data.

b Comment on the distribution of the data, noting any outliers.

6 The following marks were scored for a test out of 50 marks:

47   32   32   29   36   39   40   46   43   39
44   18   38   45   35   46   7   44   27   48

a Construct an ordered stemplot for the data.

b What percentage of the students scored 40 or more marks?

c What percentage of the students scored less than 30 marks?

d If a score of 25 or more is a pass, what percentage of the students passed?

e Describe the distribution of the data.

7 The number of peanuts in a jar varies slightly from jar to jar. A manufacturer has taken a sample of sixty jars of peanuts and recorded the number of peanuts in each:

901   904   913   924   921   893   894   895   878   885   896   910
901   903   907   907   904   892   888   905   907   901   915   901
909   917   889   891   894   894   898   895   904   908   913   924
927   885   898   903   903   913   916   931   882   893   894   903
900   906   910   928   901   896   886   897   899   908   904   889

a Complete an ordered stemplot for the data. The 'stems' are to be split to give a better display of the distribution of numbers. Use the stem 87 for numbers from 870 to 874, 87* for numbers from 875 to 879, and so on.

b What percentage of the jars had 900 peanuts or more?

c What percentage of the jars had less than 890 peanuts?

d Describe the distribution of the data.

e The manufacturer would like at least 95% of his jars to have within 20 peanuts of the stated number which is 900. Is this the case for this sample?

B   CONTINUOUS NUMERICAL DATA

A **continuous numerical variable** can theoretically take any value on the number line. A continuous variable often has to be **measured** so that data can be recorded.

Examples of continuous numerical variables are:

The height of Year 9 students:	the variable can take any value from about 115 cm to 190 cm.
The speed of cars on a highway:	the variable can take any value from 0 $km\,h^{-1}$ to the fastest speed that a car can travel, but is most likely to be in the range 50 $km\,h^{-1}$ to 120 $km\,h^{-1}$.

ORGANISATION AND DISPLAY OF CONTINUOUS DATA

When data is recorded for a continuous variable there are likely to be many different values. We therefore organise the data by grouping it into **class intervals**. We use a special type of graph called a **frequency histogram** to display the data.

A frequency histogram is similar to a column graph, but to account for the continuous nature of the variable, a number line is used for the horizontal axis and the 'columns' are joined together.

An example is given alongside:

The **modal class** or class of values that appears most often, is easy to identify from the frequency histogram.

Frequency histogram
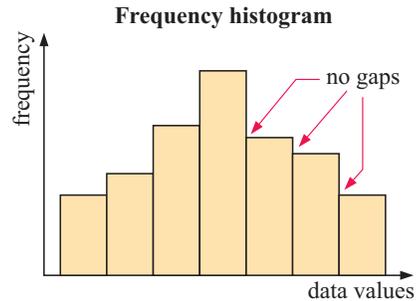
If the class intervals are the same size then the frequency is represented by the height of the 'columns'.

Example 2   ◀) Self Tutor

The weight of pumpkins harvested by Salvi from his garden was recorded in kilograms:

   2.1, 3.0, 0.6, 1.5, 1.9, 2.4, 3.2, 4.2, 2.6, 3.1, 1.8, 1.7, 3.9, 2.4, 0.3, 1.5, 1.2

Organise the data using a frequency table, and hence graph the data.

The data is *continuous* because the weight could be any value from 0.1 kg up to 10 kg.

The lowest weight recorded was 0.3 kg and the heaviest was 4.2 kg, so we use class intervals of 1 kg. The class interval 1 - < 2 would include all weights from 1 kg up to, but not including 2 kg.

Weight (kg)	Frequency
0 - < 1	2
1 - < 2	6
2 - < 3	4
3 - < 4	4
4 - < 5	1

A frequency histogram is used to graph this continuous data.
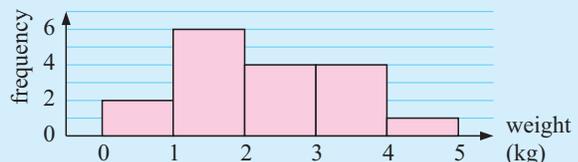
A stemplot could also be used to organise the data: The modal class is (1 - < 2) kg as this occurred the most frequently.	*Stem*	*Leaf*
	0	3 6
	1	2 5 5 7 8 9
	2	1 4 4 6
	3	0 1 2 9
	4	2 *Scale:* 1 | 2 means 1.2 kg.

EXERCISE 9B

1 A frequency table for the heights of a basketball squad is given below.

a Explain why 'height' is a continuous variable.

b Construct a histogram for the data. The axes should be carefully marked and labelled, and you should include a heading for the graph.

c What is the modal class? Explain what this means.

d Describe the distribution of the data.

Height (cm)	Frequency
170 - < 175	1
175 - < 180	8
180 - < 185	9
185 - < 190	11
190 - < 195	9
195 - < 200	3
200 - < 205	3

2 The local fitness centre is trying to help 60 overweight people lose weight. They encourage this through sensible exercise and improved eating habits. After 20 weeks they measure the people's weights and calculate the amount that each has lost. The weights lost are shown below to the nearest kilogram:

12	15	16	8	10	17	25	34	42	18	24	18	45	33	38
45	40	3	20	12	10	10	27	16	37	45	15	16	26	32
35	8	14	18	15	27	19	32	6	12	14	20	10	16	14
28	31	21	25	8	32	46	14	15	20	18	8	10	25	22

a Is weight loss a discrete or continuous variable?

b Construct an ordered stemplot for the data using stems 0, 1, 2, ...

c Describe the distribution of the data.

d Copy and complete:
"The modal weight loss was between ..... and ..... kilograms."

3 The speeds of vehicles travelling along a section of highway have been recorded and displayed in the frequency histogram alongside.

a How many vehicles were included in this survey?

b What percentage of the vehicles were travelling at speeds between 100 and 110 $km\,h^{-1}$?

c What percentage of the vehicles were travelling at speeds equal to or greater than 100 $km\,h^{-1}$?
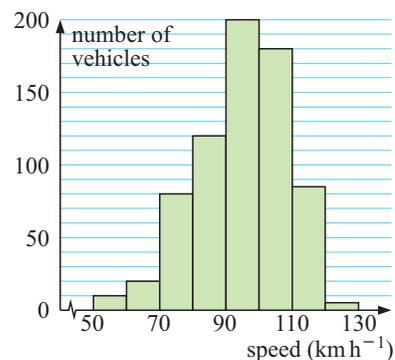
d What percentage of the vehicles were travelling at speeds less than 80 $km\,h^{-1}$?

e If the owners of the vehicles travelling at 110 $km\,h^{-1}$ or more were fined $165 each, what amount would be collected in fines?

C   MEASURING THE MIDDLE OF A DATA SET

We can gain a better understanding of a data set by locating the **middle** or **centre** of the data, and measuring its **spread**. Knowing one of these without the other is often of little use.

In this course we consider two statistics that are commonly used to measure the **centre** of a data set. These are the **mean** and the **median**.

THE MEAN

The **mean** $\overline{x}$ of a data set is the statistical name for its *arithmetic average*. It can be found by dividing the sum of the data values by the number of data values.

$$\textbf{mean} = \frac{\textbf{sum of all data values}}{\textbf{the number of data values}}$$

$\overline{x}$ is read 'x bar'.

The mean gives us a single number which measures the centre of the data set. However, it does not necessarily have to equal one of the data values.

For example, a mean test mark of 68% tells us that there are some marks below 68% and some above it with 68% at the centre. However, it does not necessarily mean that one of the students scored 68%.

Example 3   ◀ **Self Tutor**

Julie and Andrea both play goalshooter for their respective netball teams. Their performances for the season were as follows:

Julie played 11 games and scored 14, 22, 17, 31, 15, 19, 24, 28, 26, 35, and 29 goals.

Andrea played 8 games and scored 17, 21, 36, 19, 16, 28, 26, and 32 goals.

Who had the higher mean, Julie or Andrea?

Julie's mean $= \dfrac{14 + 22 + 17 + 31 + 15 + 19 + 24 + 28 + 26 + 35 + 29}{11}$

$\qquad\qquad = \frac{260}{11}$

$\qquad\qquad \approx 23.6$ goals

Andrea's mean $= \dfrac{17 + 21 + 36 + 19 + 16 + 28 + 26 + 32}{8}$

$\qquad\qquad = \frac{195}{8}$

$\qquad\qquad \approx 24.4$ goals

Andrea had the higher mean number of goals.

Example 4

The table below shows the numbers of aces served by tennis players in their first set of a tournament.

Number of aces	1	2	3	4	5	6
Frequency	4	11	18	13	7	2

Determine the mean number of aces for these sets.

No. of aces	Frequency	Product
1	4	4
2	11	22
3	18	54
4	13	52
5	7	35
6	2	12
Total	55	179

2 aces occurred 11 times. Instead of adding $2 + 2 + ... + 2$, we simply calculate 11×2.

$$\overline{x} = \frac{\text{sum of all data values}}{\text{the number of data values}}$$

$$= \frac{179}{55}$$

$$\approx 3.25 \text{ aces}$$

THE MEDIAN

The **median** is the *middle value* of an ordered data set.

A data set is ordered by listing the data from smallest to largest. The median splits the data in two halves. Half the data are less than or equal to the median and half are greater than or equal to it.

For example, if the median mark for a test is 68% then you know that half the class scored less than or equal to 68% and half scored greater than or equal to 68%.

For an **odd number** of data, the median is one of the data.

For an **even number** of data, the median is the average of the two middlevalues and may not be one of the original data.

Here is a **rule for finding the median**:

If there are n data values, find the value of $\dfrac{n+1}{2}$.

The median is the $\left(\dfrac{n+1}{2}\right)$th data value.

When $n = 23$, $\quad \dfrac{23+1}{2} = 12 \quad \therefore \quad$ the median is the 12th ordered data value.

When $n = 28$, $\quad \dfrac{28+1}{2} = 14.5 \quad \therefore \quad$ the median is the average of the 14th and 15th ordered data values.

Example 5   ◀ⁿ) **Self Tutor**

The following sets of data show the number of peas in a randomly selected sample of pods. Find the median for each set.

a 3, 6, 5, 7, 7, 4, 6, 5, 6, 7, 6, 8, 10, 7, 8

b 3, 6, 5, 7, 7, 4, 6, 5, 6, 7, 6, 8, 10, 7, 8, 9

a The ordered data set is:

$\quad\quad$ ~~3~~ ~~4~~ ~~5~~ ~~5~~ ~~6~~ ~~6~~ ~~6~~ 6 ~~7~~ ~~7~~ ~~7~~ ~~7~~ ~~8~~ ~~8~~ ~~10~~   (15 of them)

$\quad$ Since $\quad n = 15$, $\quad \dfrac{n+1}{2} = 8$ $\quad\therefore\quad$ the median is the 8th data value.

$\quad \therefore\quad$ the median $= 6$ peas

b The ordered data set is:

$\quad\quad$ ~~3~~ ~~4~~ ~~5~~ ~~5~~ ~~6~~ ~~6~~ ~~6~~ 6 7 ~~7~~ ~~7~~ ~~7~~ ~~8~~ ~~8~~ ~~9~~ ~~10~~   (16 of them)

$\quad$ Since $\quad n = 16$, $\quad \dfrac{n+1}{2} = 8.5$ $\quad\therefore\quad$ the median is the average of the 8th and 9th data values.

$\quad \therefore\quad$ the median $= \dfrac{6+7}{2} = 6.5$ peas

---

Example 6   ◀ⁿ) **Self Tutor**

The data in the table below shows the number of people on each table at a restaurant. Find the median of this data.

Number of people	5	6	7	8	9	10	11	12
Frequency	1	0	3	9	12	7	4	2

The total number of data values is the number of tables in the restaurant. It is the sum of the frequencies, which is $\quad n = 38$.

$\dfrac{n+1}{2} = \dfrac{39}{2} = 19.5,$ so the median is the average of the 19th and 20th data values.

Number of people	5	6	7	8	9	10	11	12
Frequency	1	0	3	9	12	7	4	2

$\quad\quad\quad\quad\quad\quad$ 13 data values$\quad\quad\quad$ the 14th to the
$\quad\quad\quad\quad\quad\quad$ of 8 or less$\quad\quad\quad\quad$ 25th are all 9s

$\therefore\quad$ the median $= \dfrac{9+9}{2} = 9$ people

EXERCISE 9C

1 Below are the points scored by two basketball teams over a 12 match series:

       Team A: 91, 76, 104, 88, 73, 55, 121, 98, 102, 91, 114, 82

       Team B: 87, 104, 112, 82, 64, 48, 99, 119, 112, 77, 89, 108

Which team had the higher mean score?

2 Calculate the median value for each of the following data sets:

  a 21, 23, 24, 25, 29, 31, 34, 37, 41

  b 105, 106, 107, 107, 107, 107, 109, 120, 124, 132

  c 173, 146, 128, 132, 116, 129, 141, 163, 187, 153, 162, 184

3 A survey of 50 students revealed the following number of siblings per student:

      1, 1, 3, 2, 2, 2, 0, 0, 3, 2, 0, 0, 1, 3, 3, 4, 0, 0, 5, 3, 3, 0, 1, 4, 5,

      1, 3, 2, 2, 0, 0, 1, 1, 5, 1, 0, 0, 1, 2, 2, 1, 3, 2, 1, 4, 2, 0, 0, 1, 2

  a What is the mean number of siblings per student?

  b What is the median number of siblings per student?

4 The following table shows the average monthly rainfall for Kuala Lumpur.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Ave. rainfall (mm)	157	202	258	291	222	127	98	161	220	249	259	190

Calculate the mean average monthly rainfall for this city.

5 The selling prices of the last 10 houses sold in Wulverhampton were:

£146 400,   £127 600,   £211 000,   £192 500,

£256 400,   £132 400,   £148 000,   £129 500,

£131 400,   £162 500

  a Calculate the mean and median selling prices of these houses and comment on the results.

  b Which measure would you use if you were:

    i a vendor wanting to sell your house

    ii looking to buy a house in the district?

6 A hardware store maintains that its packets on sale contain 60 nails. A quality control inspector tested 100 packets and found the distribution alongside.

  a Find the mean and median number of nails per packet.

  b Comment on the results in relation to the store's claim.

  c Which of the two measures is most reliable? Comment on your answer.

Number of nails	Frequency
56	8
57	11
58	14
59	18
60	21
61	8
62	12
63	8
Total	100

7 51 packets of chocolate almonds were opened and their contents counted. The table gives the distribution of the number of chocolates per packet sampled.

Find the mean and median of the distribution.

Number in packet	Frequency
32	6
33	8
34	9
35	13
36	10
37	3
38	2

8

Guinea Pig	Mass (g) at birth	Mass (g) at 2 weeks
A	75	210
B	70	200
C	80	200
D	70	220
E	74	215
F	60	200
G	55	206
H	83	230

A sample of eight guinea pigs were weighed at birth and after 2 weeks. The results are shown in the table opposite.

a What was the mean birth mass?

b What was the mean mass after two weeks?

c What was the mean increase over the two weeks?

9 Towards the end of the season, a netballer had played 14 matches and had an average of 16.5 goals per game. In the final two matches of the season the netballer threw 21 goals and 24 goals. Find the netballer's new average.

10 A sample of 12 measurements has a mean of 16.5 and a sample of 15 measurements has a mean of 18.6. Find the mean of all 27 measurements.

11 15 of 31 measurements are below 10 cm and 12 measurements are above 11 cm. Find the median if the other 4 measurements are 10.1 cm, 10.4 cm, 10.7 cm and 10.9 cm.

12 The mean and median of a set of 9 measurements are both 12. If 7 of the measurements are 7, 9, 11, 13, 14, 17 and 19, find the other two measurements.

INVESTIGATION   THE EFFECT OF OUTLIERS

In a set of data, an **outlier** or **extreme value** is a value which is much greater than or much less than the other values. In this investigation we will examine the effect of an outlier on the two measures of central tendency.

What to do:

1 Consider the set of data: 4, 5, 6, 6, 6, 7, 7, 8, 9, 10.   Calculate:

   a the mean   **b** the median.

2 Introduce the extreme value 100 to the data set. It is now   4, 5, 6, 6, 6, 7, 7, 8, 9, 10, 100.   Calculate:

   a the mean   **b** the median.

3 Comment on the effect that this extreme value has on:

   a the mean   **b** the median.

4 Which of the measures of central tendency is most affected by the inclusion of an outlier? Discuss your findings with your class.

CHOOSING THE APPROPRIATE MEASURE

The mean and median can both be used to indicate the centre of a set of numbers. Which of these is the more appropriate measure to use will depend upon the type of data under consideration.

For example, when reporting on shoe size stocked by a shoe store, the average or mean size would be a useless measure of the stock. In real estate values the median is used.

When selecting which of the measures of central tendency to use, you should keep the following **advantages and disadvantages** of each measure in mind.

- **Mean**
  - ▸ The mean's main advantage is that it is commonly used, easy to understand, and easy to calculate.
  - ▸ Its main disadvantage is that it is affected by extreme values within a data set, and so may give a distorted impression of the data.
    For example, consider the data: 4, 6, 7, 8, 19, 111.   The total of these 6 numbers is 155, and so the mean is approximately 25.8.   The **outlier** 111 has distorted the mean so it is no longer representative of the data.

- **Median**
  - ▸ The median's main advantage is that it is very easy to find.
  - ▸ Unlike the mean, it is not affected by extreme values.
  - ▸ The main disadvantage is that it ignores all values outside the middle range.

D   MEASURING THE SPREAD OF DATA

To accurately describe a data set, we need not only a measure of its centre, but also a measure of its spread.

For example,   1, 4, 5, 5, 6, 7, 8, 9, 9   has a mean value of 6 and so does
4, 4, 5, 6, 6, 7, 7, 7, 8.

However, the first data set is more widely spread than the second one.

Two commonly used statistics that indicate the spread of a set of data are:

- the range
- the interquartile range.

THE RANGE

The **range** is the difference between the **maximum** or largest data value and the **minimum** or smallest data value.

range = maximum data value − minimum data value

Example 7   ◀) **Self Tutor**

Find the range of the data set: 4, 7, 5, 3, 4, 3, 6, 5, 7, 5, 3, 8, 9, 3, 6, 5, 6

Searching through the data set we find: minimum value $= 3$

maximum value $= 9$

$\therefore$ range $= 9 - 3 = 6$

THE QUARTILES AND THE INTERQUARTILE RANGE

We have already seen how the median divides an ordered data set into two halves. These halves are divided in half again by the **quartiles**.

The middle value of the lower half is called the **lower quartile** or Q_1. One quarter or 25% of the data have a value less than or equal to the lower quartile. 75% of the data have values greater than or equal to the lower quartile.

The middle value of the upper half is called the **upper quartile** or Q_3. One quarter or 25% of the data have a value greater than or equal to the upper quartile. 75% of the data have values less than or equal to the upper quartile.

The **interquartile range** is the range of the middle half of the data.

> **interquartile range = upper quartile − lower quartile**

The data set is thus divided into quarters by the lower quartile Q_1, the median Q_2, and the upper quartile Q_3.

So, the **interquartile range** $IQR = Q_3 - Q_1.$

Example 8   ◀) **Self Tutor**

For the data set   6, 7, 3, 7, 9, 8, 5, 5, 4, 6, 6, 8, 7, 6, 6, 5, 4, 5, 6   find the:
  a median   **b** lower and upper quartiles   **c** interquartile range.

The ordered data set is:

   3 4 4 5 5 5 5 6 6 6 6 6 6 7 7 7 8 8 9   (19 of them)

a The median $= \left(\dfrac{19+1}{2}\right)$ th score $= $ 10th score $= 6$

b As the median is a data value, we ignore it and split the remaining data into two groups.

   3 4 4 5 5 5 6 6   6 6 6 7 7 7 8 8 9

   Q_1 = median of lower half Q_3 = median of upper half
        $= 5$ $= 7$

c IQR $= Q_3 - Q_1 = 2$

Example 9

◆) Self Tutor

For the data set   9, 8, 2, 3, 7, 6, 5, 4, 5, 4, 6, 8, 9, 5, 5, 5, 4, 6, 6, 8   find the:

a median   **b** lower quartile

c upper quartile   **d** interquartile range.

The ordered data set is:

2 3 4 4 4 5 5 5 5 5 6 6 6 6 7 8 8 8 9 9   (20 of them)

a As $n = 20$, $\dfrac{n+1}{2} = \dfrac{21}{2} = 10.5$

$\therefore$ median $= \dfrac{10\text{th value} + 11\text{th value}}{2} = \dfrac{5+6}{2} = 5.5$

b As the median is not a data value, we split the original data into two equal groups of 10.

2 3 4 4 4 5 5 5 5 5   6 6 6 6 7 8 8 8 9 9

$\therefore$ $Q_1 = 4.5$ $\therefore$ $Q_3 = 7.5$

c IQR $= Q_3 - Q_1 = 3$

EXERCISE 9D

1 For each of the following sets of data, find:

   i the upper quartile   **ii** the lower quartile

   iii the interquartile range   **iv** the range.

a 2, 3, 4, 7, 8, 10, 11, 13, 14, 15, 15

b 35, 41, 43, 48, 48, 49, 50, 51, 52, 52, 52, 56

c

Stem	Leaf
1	3 5 7 7 9
2	0 1 3 4 6 7 8 9
3	0 1 2 7
4	2 6
5	1 *Scale:* 4	2 means 42

d

Score	0	1	2	3	4	5
Frequency	1	4	7	3	3	1

2 The time spent by 24 people in a queue at a bank, waiting to be attended by a teller, has been recorded in minutes as follows:

   0   3.2   0   2.4   3.2   0   1.3   0
   1.6   2.8   1.4   2.9   0   3.2   4.8   1.7
   3.0   0.9   3.7   5.6   1.4   2.6   3.1   1.6

   a Find the median waiting time and the upper and lower quartiles.

   b Find the range and interquartile range of the waiting time.

c Copy and complete the following statements:

    i "50% of the waiting times were greater than .... minutes."

    ii "75% of the waiting times were less than ...... minutes."

    iii "The minimum waiting time was .... minutes and the maximum waiting time was ..... minutes. The waiting times were spread over ...... minutes."

3 For the data set given, find:

a the minimum value	**b** the maximum value
c the median	**d** the lower quartile
e the upper quartile	**f** the range
g the interquartile range	

Stem	Leaf
6	0 3 8
7	0 1 5 6 7 7
8	1 1 2 4 4 8 9 9
9	0 4 7 9
10	1

Scale: 7 | 5 means 7.5

E   BOX-AND-WHISKER PLOTS

A **box-and-whisker plot** (or simply a **boxplot**) is a visual display of some of the descriptive statistics of a data set. It shows:

- the minimum value (Min_x)
- the lower quartile (Q_1)
- the median (Q_2)
- the upper quartile (Q_3)
- the maximum value (Max_x)

These five numbers form the **five-number summary** of a data set.

For **Example 9**, the **five-number summary** and corresponding boxplot are:

$$minimum = 2$$
$$Q_1 = 4.5$$
$$median = 5.5$$
$$Q_3 = 7.5$$
$$maximum = 9$$
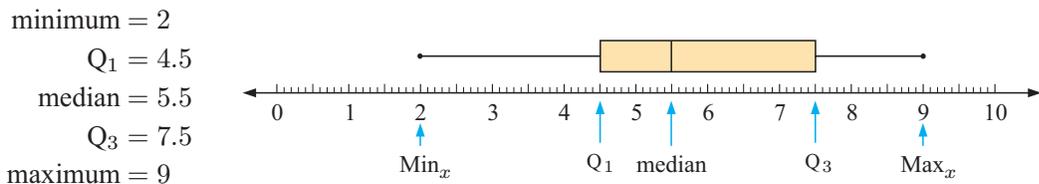
Notice that:

- the rectangular box represents the 'middle' half of the data set
- the lower whisker represents the 25% of the data with smallest values
- the upper whisker represents the 25% of the data with greatest values.

Example 10   ◀) **Self Tutor**

For the data set: 5 6 7 6 2 8 9 8 4 6 7 4 5 4 3 6 6

a construct the five-number summary   **b** draw a boxplot

c find the   **i** range   **ii** interquartile range

d find the percentage of data values below 7.

a The ordered data set is:

  2   3   4   4   4   5   5   6   6   6   6   6   7   7   8   8   9   (17 of them)

  $Q_1 = 4$ median $= 6$ $Q_3 = 7$

So, the **5-number summary** is:
$$\begin{cases} \text{Min}_x = 2 & Q_1 = 4 \\ \text{median} = 6 & Q_3 = 7 \\ \text{Max}_x = 9 \end{cases}$$

b

  0   1   2   3   4   5   6   7   8   9   10
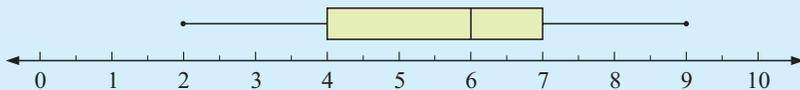
c   **i** range $= \text{Max}_x - \text{Min}_x = 9 - 2 = 7$ **ii** IQR $= Q_3 - Q_1 = 7 - 4 = 3$

d 75% of the data values are less than or equal to 7.

BOXPLOTS AND OUTLIERS

We have already seen that **outliers** are extraordinary data that are either much larger or much smaller than most of the data. A common test used to identify outliers involves the calculation of 'boundaries':

- **upper boundary = upper quartile + 1.5 × IQR**
  Any data larger than the upper boundary is an outlier.
- **lower boundary = lower quartile − 1.5 × IQR**
  Any data smaller than the lower boundary is an outlier.

Outliers are marked on a boxplot with an asterisk. It is possible to have more than one outlier at either end. The whiskers extend to the last value that is not an outlier.

Example 11   ◀) Self Tutor

Draw a boxplot for the following data, marking any outliers with an asterisk:

  3, 7, 8, 8, 5, 9, 10, 12, 14, 7, 1, 3, 8, 16, 8, 6, 9, 10, 13, 7

$n = 20$ and the ordered data set is:

  1   3   3   5   6   7   7   7   8   8   8   8   9   9   10   10   12   13   14   16

  $\text{Min}_x = 1$ $Q_1 = 6.5$ median $= 8$ $Q_3 = 10$ $\text{Max}_x = 16$

Test for outliers:

upper boundary	and	lower boundary
= upper quartile $+ 1.5 \times$ IQR		= lower quartile $- 1.5 \times$ IQR
$= 10 + 1.5 \times (10 - 6.5)$		$= 6.5 - 1.5 \times 3.5$
$= 15.25$		$= 1.25$

As 16 is above the upper boundary, it is an outlier.

As 1 is below the lower boundary, it is an outlier.

So, the boxplot is:

variable
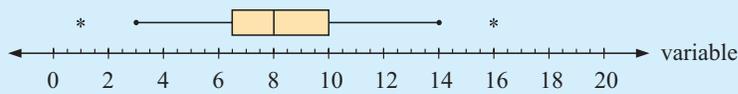
Each whisker is drawn to the last value that is not an outlier.

EXERCISE 9E

1 A boxplot has been drawn to show the distribution of marks for a particular class in a test out of 100.

score for test (%)

a What was the: **i** highest mark **ii** lowest mark scored?

b What was the median test score for this class?

c What was the range of marks scored for this test?

d What percentage of students scored 60 or more for the test?

e What was the interquartile range for this test?

f The top 25% of students scored a mark between ..... and .....

g If you scored 70 for this test, would you be in the top 50% of students in this class?

h Comment on the symmetry of the distribution of marks.

2 A set of data has lower quartile 31.5, median 37, and upper quartile 43.5 .

a Calculate the interquartile range for this data set.

b Calculate the boundaries that identify outliers.

c Which of the data 22, 13.2, 60, and 65 would be outliers?

3 Julie examines a new variety of bean. She counts the number of beans in 33 pods. Her results are:

5, 8, 10, 4, 2, 12, 6, 5, 7, 7, 5, 5, 5, 13, 9, 3, 4, 4, 7, 8, 9, 5, 5, 4, 3, 6, 6, 6, 6, 9, 8, 7, 6

a Find the median, lower quartile and upper quartile of the data set.

b Find the interquartile range of the data set.

c What are the lower and upper boundaries for outliers?

d According to **c**, are there any outliers?

e Draw a boxplot of the data set.

4 Andrew counts the number of bolts in several boxes. His tabulated data is shown below:

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

a Find the five-number summary for this data set.

b Find the **i** range **ii** IQR for the data set.

c Test for any outliers in the data set.   **d** Construct a boxplot for the data set.

F   GROUPED CONTINUOUS DATA

Sometimes data is collected in groups, and each individual value is either unknown or not important. The data is still useful and can be processed to obtain valuable information.

Below is a small part of a planning authority's survey form. It allows continuous salary data to be gathered in groups, and the groups can then be compared to see, for example, the salary range which is most common.

3. **Salary range:**

$10 000 - $19 999 ☐	$20 000 - $29 999 ☐	$30 000 - $39 999 ☐
$40 000 - $49 999 ☐	$50 000 - $59 999 ☐	$60 000+ ☐

When continuous data is grouped in **class intervals** the actual data values are not visible. Grouping is normally used for large data sets.

DISCUSSION

For data grouped in class intervals, can you explain why:

- the range cannot be found
- we cannot draw a boxplot?

Unfortunately when data has been grouped before it is presented for analysis, certain features cannot be considered. However, we can still draw a histogram for the data and describe its centre and spread.

To estimate the **mean** of the distribution we assume that all scores in each class interval are evenly spread throughout the interval and so we use the value of the **midpoint** of that interval.

Example 12   ◀)) Self Tutor

The speeds of 129 cars going past a particular point appear in the following table:

Speed ($km\,h^{-1}$)	40 - < 50	50 - < 60	60 - < 70	70 - < 80
Frequency	1	3	17	39

Speed ($km\,h^{-1}$)	80 - < 90	90 - < 100	100 - < 110
Frequency	48	17	4

a Find the modal class.   **b** Estimate the mean.

c Draw a frequency histogram for this data.

a The modal class is the most common range of speeds. This is 80 - < 90, or 80 $km\,h^{-1}$ up to 90 $km\,h^{-1}$.

b

Speed	Frequency	Midpoint of interval	Product
40 - < 50	1	45	45
50 - < 60	3	55	165
60 - < 70	17	65	1105
70 - < 80	39	75	2925
80 - < 90	48	85	4080
90 - < 100	17	95	1615
100 - < 110	4	105	420
Total	129		10 355

$$\therefore \quad \text{approximate mean} = \frac{\text{total product}}{\text{total frequency}}$$

$$\approx \frac{10\,355}{129}$$

$$\approx 80.3 \text{ km h}^{-1}$$

We assume that each unknown raw score in an interval takes the value of the interval midpoint.

c The frequency histogram for this data is:
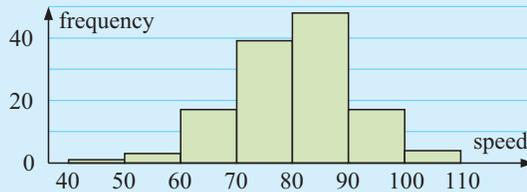
ESTIMATING THE MEDIAN

Using the same assumption as before, the median can be estimated by using the frequency histogram. Remember that frequencies are indicated by column heights.

The number of sample values n is 129   and $\frac{1}{2}(n+1) = 65$.

So,   median = 65th data value

$= 80 + \frac{5}{48} \times 10$

≈ 81

$\frac{5}{48}$ of the step is in the next interval
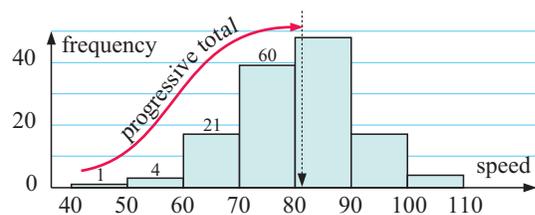
THE INTERQUARTILE RANGE

To find the **lower quartile** Q_1, we want the $\frac{1}{4}(n+1)$th   value of the ordered data set.

To find the **upper quartile** Q_3, we want the $\frac{3}{4}(n+1)$th   value.

In the speed data, $n = 129$ $\therefore$ $\frac{1}{4}(n+1) = 32.5$ and $\frac{3}{4}(n+1) = 97.5$

So, $Q_1 = 70 + \dfrac{11.5}{39} \times 10$ {21 values are < 70 and $32.5 = 21 + 11.5$}

≈ 73

and $Q_3 = 80 + \dfrac{37.5}{48} \times 10$ {60 values are < 80 and $97.5 = 60 + 37.5$}

≈ 88

The IQR $\approx 88 - 73 \approx 15$.

EXERCISE 9F

1 Consider the graph alongside:

    a Name the type of graph.

    b Is the data it represents discrete or continuous?

    c Find the modal class.

    d Construct a table for the data showing class intervals, frequencies, midpoints of intervals, and products.

    e Estimate the mean of the data.

    f Estimate:

        i the median   **ii** Q_1 **iii** Q_3.

    g Estimate the interquartile range.

Weights of dogs
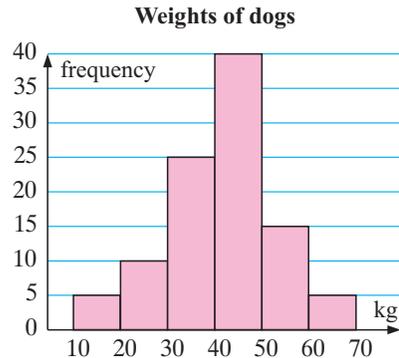
2 A survey was carried out on the age structure of the population of a country town. The results are shown alongside.

    a Can we accurately give the age of the oldest person?

    b Draw a frequency histogram of the data.

    c What is the length of each class interval?

    d What is the modal class?

    e Add further columns to help find the approximate mean.

    f Estimate: **i** the median **ii** Q_1 **iii** Q_3

    g Estimate the interquartile range.

    h Comment on the shape of the distribution.

Age	Frequency
0 - < 10	170
10 - < 20	107
20 - < 30	111
30 - < 40	121
40 - < 50	104
50 - < 60	75
60 - < 70	63
70 - < 80	32
80 - < 90	9

G   CUMULATIVE DATA

It is sometimes useful to know the number of scores that lie above or below a particular value. To do this we construct a **cumulative frequency distribution table** and a **cumulative frequency graph** to represent the data.

The **cumulative frequency** gives a *running total* of the number of data less than a particular value.

Example 13   ◀)) **Self Tutor**

The data shown are the weights of 60 male gridiron players.

Weight (w kg)	Frequency
$75 \leqslant w < 80$	3
$80 \leqslant w < 85$	11
$85 \leqslant w < 90$	12
$90 \leqslant w < 95$	14
$95 \leqslant w < 100$	13
$100 \leqslant w < 105$	4
$105 \leqslant w < 110$	2
$110 \leqslant w < 115$	1

a Construct a cumulative frequency distribution table.

b Represent the data on a cumulative frequency graph.

c Use your graph to estimate the:

   i median weight

   ii number of men weighing less than 83 kg

   iii number of men weighing more than 102 kg.

a

Weight (w kg)	Frequency	Cumulative frequency
$75 \leqslant w < 80$	3	3
$80 \leqslant w < 85$	11	14 ◀
$85 \leqslant w < 90$	12	26 ◀
$90 \leqslant w < 95$	14	40
$95 \leqslant w < 100$	13	53
$100 \leqslant w < 105$	4	57
$105 \leqslant w < 110$	2	59
$110 \leqslant w < 115$	1	60

this is $3 + 11$

this 26 means that there are 26 players who weigh less than 90 kg

b **Cumulative frequency graph of gridiron players' weights**
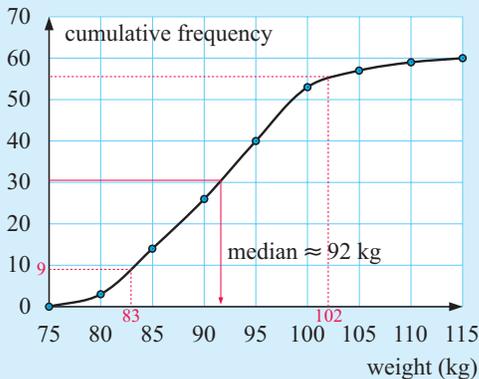
c **i** The median is the average of the 30th and 31st weights. Call it 30.5.
Reading from the graph, the median ≈ 92 kg.

ii There are 9 men who weigh less than 83 kg.

iii There are $60 - 56 = 4$ men who weigh more than 102 kg.

STATISTICS PACKAGE

EXERCISE 9G

1 The following data shows the lengths of 40 trout caught in a lake during a fishing competition. Measurements are to the nearest centimetre.

27 38 31 30 26 28 24 30 33 35 25 29 36 37
25 29 22 36 40 31 35 27 31 37 26 24 30 33
28 34 30 42 35 29 26 35 31 28 30 27

  a Construct a cumulative frequency table for salmon lengths, x cm, using the intervals
    $21 \leqslant x < 24$, $24 \leqslant x < 27$, ... and so on.
  b Draw a cumulative frequency graph for the data.
  c Use **b** to find the median length.
  d Find the median of the original data and compare your answer with **c**.
  e Find the first and third quartiles and the interquartile range for the grouped data.

2 In an examination the following scores were achieved
by a group of students:
Draw a cumulative frequency graph of the data and use
it to find:

Score	Frequency
$10 \leqslant x < 20$	1
$20 \leqslant x < 30$	3
$30 \leqslant x < 40$	6
$40 \leqslant x < 50$	15
$50 \leqslant x < 60$	14
$60 \leqslant x < 70$	28
$70 \leqslant x < 80$	18
$80 \leqslant x < 90$	11
$90 \leqslant x < 100$	4

  a the median examination mark
  b how many students scored less than 65 marks
  c how many students scored between 50 and 80 marks
  d how many students failed, given that the pass mark
    was 50
  e the credit mark, given that the top 25% of students
    were awarded credits
  f the interquartile range.

3 In a running race, the times of 80 competitors were
recorded as shown.
Draw a cumulative frequency graph of the data and use
it to find:

Times (min)	Frequency
$30 \leqslant t < 35$	7
$35 \leqslant t < 40$	13
$40 \leqslant t < 45$	18
$45 \leqslant t < 50$	25
$50 \leqslant t < 55$	12
$55 \leqslant t < 60$	5

  a the median time
  b the approximate number of runners whose time was
    not more than 38 minutes
  c the approximate time in which the fastest 30
    runners completed the course
  d the range within which the middle 50% of the data lies.

4 The following table is a summary of the distance a baseball was thrown by a number of
different students.

Distance (m)	20 - < 30	30 - < 40	40 - < 50	50 - < 60	60 - < 70	70 - < 80
Frequency	2	6	26	12	3	1

Draw a cumulative frequency graph of the data and use it to find:

  a the median distance thrown by the students
  b the number of students who threw the ball less than 55 m
  c the number of students who threw the ball between 45 and 70 m.
  d If only students who threw the ball further than 55 m were considered for further
    coaching, how many students were considered?

5 Consider the **Opening Problem** on page **194**. Using the **computer**
statistics package or otherwise, display the data and calculate its
descriptive statistics.

STATISTICS
PACKAGE

REVIEW SET 9A

1 11 of 25 measurements are below 12 cm and 11 are above 13 cm. Find the median if the other 3 measurements are 12.1 cm, 12.4 cm and 12.6 cm.

2 A microbiologist measured the diameter (in cm) of a number of bacteria colonies 12 hours after seeding. The results were as follows:

0.4, 2.1, 3.4, 3.9, 1.7, 3.7, 0.8, 3.6, 4.1, 0.9, 2.5, 3.1, 1.5, 2.6, 1.3, 3.5

  a Is this data discrete or continuous?

  b Construct an ordered stem-and-leaf plot for this data.

  c What percentage of the bacteria colonies were greater than 2 cm in diameter?

3 The data below gives the number of letters received by a household each week:

6 6 6 6 7 7 7 7 7 7 7 8 8 8 8 8 8 8 8 8 9 9 9 9 9 10 10 10 13

  a Find the range of the data set.

  b Construct a 5-number summary for the data set.

  c Construct a boxplot for the data set.

4 Six scores have mean 8. What must the seventh score be to increase the mean by 1?

5 The table below summarises the masses of 50 domestic cats chosen at random.

  a What is the length of each class interval?

  b What is the modal class?

  c Find the approximate mean.

  d Draw a frequency histogram of the data.

  e Draw a cumulative frequency graph of the data.

  f Estimate: **i** the median **ii** Q_1 **iii** Q_3.

  g Estimate the interquartile range.

Mass (kg)	Frequency
0 - < 2	5
2 - < 4	18
4 - < 6	12
6 - < 8	9
8 - < 10	5
10 - < 12	1

REVIEW SET 9B

1 The heights of the basketballers from the two squads playing in the finals were recorded in centimetres as follows:

190 188 176 191 205 187 193 177 180 175 186 183 179 181 192
185 184 172 200 183 188 186 192 179 189 194 196 180 182 181

  a Construct an ordered stem-and-leaf display for this data.

  b What percentage of the basketballers are more than 190 cm tall?

2 12 of 29 measurements are below 20 cm and 13 measurements are above 21 cm. Find the median if the other 4 measurements are 20.1 cm, 20.4 cm, 20.7 cm and 20.9 cm.

3 Consider the graph alongside.

   a Name the type of graph.

   b Is the data represented discrete or continuous?

   c Find the modal class.

   d Construct a table for the data showing class intervals, frequencies, midpoints and products.

   e Estimate the mean of the data.
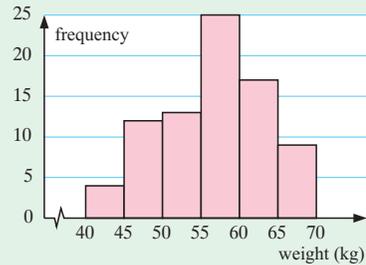
4 Jenny's golf scores for her last 20 rounds were:

   90 106 84 103 112 100 105 81 104 98

   107 95 104 108 99 101 106 102 98 101

   a Find the median, lower quartile and upper quartile of the data set.

   b Find the interquartile range of the data set and explain what it represents.

   c What are the lower and upper boundaries for outliers?

   d Are there any outliers according to **c**?   **e** Draw a boxplot of the data set.

5 The table below summarises the best times of 100 swimmers who swim 50 m.

   a What is the length of each class interval?

   b What is the modal class?

   c Draw a frequency histogram of the data.

   d Draw a cumulative frequency graph of the data.

   e Estimate: **i** the median   **ii** Q_1 **iii** Q_3.

   f Estimate the interquartile range.

Time (sec)	Frequency
25 - < 30	5
30 - < 35	17
35 - < 40	34
40 - < 45	29
45 - < 50	15

HISTORICAL NOTE   ARCHIMEDES (C. 287 B.C. - 212 B.C.)

Archimedes was born about 287 B.C. in Syracuse in Sicily. He was the son of an astronomer and studied in Alexandria in Egypt. From there he returned to Syracuse and concentrated on mathematical research. He invented a number of mechanical contraptions which made him famous but the only one of these which he recorded in writing was one which simulated the motion of the planets. There are stories that he designed convex lenses which focused on the sun's heat and caused ships to burst into flames when they came within range of the city's walls.
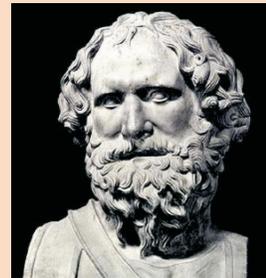
Perhaps the most famous story concerning Archimedes is his discovery of the principle of buoyancy, called Archimedes principle. It is said that Archimedes was stepping into a bath and noticed the water overflowing when he realised that a body immersed in fluid loses as much in weight as the weight of the fluid it displaces. He is said to have run naked into the street shouting "Eureka" as he was so delighted with his discovery.

He pioneered work in mechanics and invented a water screw which is still used for water irrigation in Egypt to this day. Archimedes was killed during the Roman capture of Syracuse in 212 B.C. and it is said that he was stabbed by a soldier as he was drawing a mathematical figure in the sand. In accordance with his wishes, his tomb was marked by a sphere inscribed in a cylinder. He is recognised as one of the greatest mathematicians of all time.

Chapter **10**

Probability

Contents:

The study of **probability** deals with the **chance** or **likelihood** of an event happening. For every event we can carefully assign a number which lies between 0 and 1 inclusive.

An **impossible** event which has 0% chance of happening is assigned a probability of 0.

A **certain** event which has 100% chance of happening is assigned a probability of 1.

All other events between these two extremes can be assigned a probability between 0 and 1.

The number line below shows how we could interpret different probabilities:

not likely
to happen

likely
to happen

0   0.5   1

impossible   certain

very unlikely
to happen

equal chance of
happening as not happening

very likely
to happen
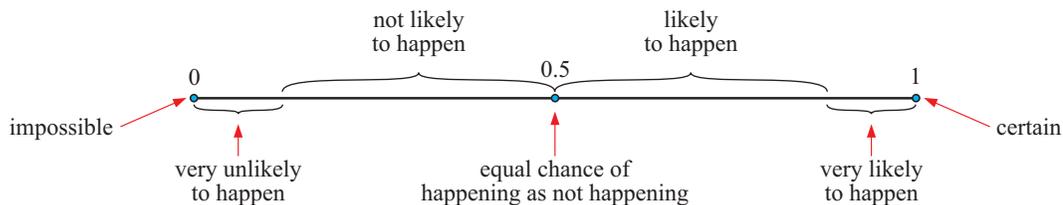
We usually assign either:

- an **experimental** probability by observing the results of an experiment,   or
- a **theoretical probability** by using arguments of symmetry.

The study of chance has important applications in physical and biological sciences, economics, politics, sport, life insurance, quality control, production planning in industry, and a host of other areas.

Probability theory can be applied to games of chance such as card and dice games to try to increase our chances of success. It may therefore appear that an understanding of probability encourages gambling.

However, in reality a better knowledge of probability theory and its applications helps us to understand why the majority of habitual gamblers "die broke".
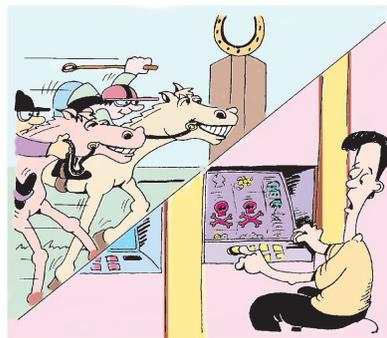
HISTORICAL NOTE

The development of modern probability theory began in 1653 when gambler Chevalier de Mere contacted mathematician **Blaise Pascal** with a problem on how to divide the stakes when a gambling game is interrupted during play. Pascal involved **Pierre de Fermat**, a lawyer and amateur mathematician, and together they solved the problem. While doing so, they laid the foundations upon which the laws of probability were formed.

Blaise Pascal   *Pierre de Fermat*

In the late 17th century, English mathematicians compiled and analysed mortality tables. These tables showed the number of people that died at different ages. From these tables they could estimate the probability that a person would be alive at a future date. This led to the establishment of the first life insurance company in 1699.

A   EXPERIMENTAL PROBABILITY

In experiments involving chance, we agree to use the following language to accurately describe what we are doing and the results we are obtaining.

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

When a small plastic cone was tossed into the air 300 times it fell on its *side* 203 times and on its *base* 97 times. We say:

- the number of trials is 300
- the outcomes are *side* and *base*
- the frequency of *side* is 203 and *base* is 97
- the relative frequency of $side = \frac{203}{300} \approx 0.677$
- the relative frequency of $base = \frac{97}{300} \approx 0.323$

side   base

In the absence of any further data, the relative frequency of each event is our best estimate of the probability of it occurring.

> The **estimated experimental probability** is the **relative frequency** of the event.

We write   Experimental P(*side*) $\approx$ 0.677,   Experimental P(*base*) $\approx$ 0.323.

EXERCISE 10A

1 A coin is tossed 100 times. It falls heads 47 times. What is the experimental probability that it falls: **a**   heads   **b**   tails?

2 A die is rolled 300 times and the results are:

Result	1	2	3	4	5	6
Frequency	52	47	50	51	49	51

What is the experimental probability of rolling:

   a   a 6   **b**   a 2   **c**   a 6 or a 2?

3 A batch of 145 paper clips was dropped onto 6 cm by 6 cm squared paper. 113 fell completely inside squares and 32 finished up on the grid lines. Find, to 2 decimal places, the estimated probability of a clip falling:

   a   inside a square   **b**   on a line.

6 cm

on

inside

6 cm

4   A pair of coins is tossed 500 times and the results are:

Result	two heads	a head and a tail	two tails
Frequency	121	251	128

What is the experimental probability of getting:

a   two heads   **b**   a head and a tail   **c**   at least two tails?

B   PROBABILITIES FROM DATA

If we are given data from a population then we can use relative frequencies to find the probabilities of various events occurring. It is clear that when more data is used, our estimates for probabilities will be more accurate.

$$\text{relative frequency} = \frac{\text{frequency}}{\text{number of trials}}$$

Example 1   ◄)) Self Tutor

The table below shows the number of cars imported into Australia from various countries in 2005 and 2006.

a   A car imported in 2005 was involved in a crash. What is the probability it came from:

　　i   Korea
　　ii   Germany?

b   Jason's car was imported in 2006. What is the chance that it came from the USA?

c   Sarah's car was imported in either 2005 or 2006. What is the probability that it was imported

　　i   in 2005   **ii**   from the UK?

Country	No. in year ended 2005	No. in year ended 2006
Japan	213 764	232 056
Korea	23 842	19 412
United Kingdom	13 977	16 304
USA	6804	10 412
France	5244	7661
Germany	11 455	18 260
Others	8777	13 162
Total	283 863	317 267

a   **i**   P(from Korea) $= \frac{23\,842}{283\,863} \approx 0.0840$

　　ii   P(from Germany) $= \frac{11\,455}{283\,863} \approx 0.0404$

b   P(from the USA) $= \frac{10\,412}{317\,267} \approx 0.0328$

c   **i**   P(from 2005) $= \frac{283\,863}{601\,130} \approx 0.472$

　　ii   P(from UK) $= \frac{13\,977 + 16\,304}{601\,130} \approx 0.0504$

EXERCISE 10B

1 Jose surveyed the length of TV commercials in seconds. Estimate, to 3 decimal places, the probability that a randomly chosen TV commercial will last:

a 20 to 39 seconds   **b** more than a minute

c between 20 and 59 seconds inclusive.

Length	Frequency
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

2 Paula keeps records of the number of phone calls she receives over a period of consecutive days.

a For how many days did the survey last?

b How many calls did Paula receive over this period?

c Estimate Paula's chance of receiving:

   i no phone calls on a particular day

   ii 5 or more phone calls on a particular day

   iii less than 3 phone calls on a particular day.
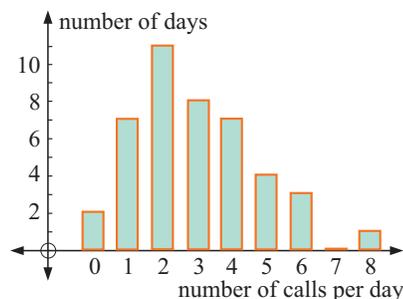
3 Pat does a lot of travelling in her car. She keeps records of how often she fills her car with petrol. The table alongside shows the frequencies of the *number of days between refills*. Estimate the likelihood that:

a there is a four day gap between refills

b there is at least a four day gap between refills.

Days between refills	Frequency
1	37
2	81
3	48
4	17
5	6
6	1

4 The table alongside gives the age distribution of prison inmates as of December 31, 2007. A prisoner was released on January 1, 2008. Find the probability that:

a the prisoner was male

b the prisoner was aged between 17 and 19

c the prisoner was 19 or under given that the prisoner was female

d the prisoner was 19 or under given that the prisoner was male

e the prisoner was female given that the prisoner was aged 60+.

Age distribution of prison inmates			
Age	Female	Male	Total
15	0	6	6
16	5	23	28
17 - 19	26	422	448
20 - 24	41	1124	1165
25 - 29	36	1001	1037
30 - 34	32	751	783
35 - 39	31	520	551
40 - 49	24	593	617
50 - 59	16	234	250
60+	5	148	153
Total	216	4822	5038

C   LIFE TABLES

WHAT ARE YOUR SURVIVAL PROSPECTS?

Life insurance companies rely on **life expectancy tables** in order to work out the premiums to charge people who insure with them. To construct these tables, they gather statistics.

The following life table shows how many people from 100 000 births can expect to survive to a given age, and the expected remaining life at a given age. More detailed tables can be obtained for your country using the internet.

LIFE TABLE

	Male			Female	
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	75.7	0	100 000	81.4
5	99 247	71.3	5	99 366	76.9
10	99 168	66.4	10	99 298	71.8
15	99 059	61.4	15	99 231	67.0
20	98 642	56.7	20	99 087	62.1
25	98 098	52.0	25	98 920	57.2
30	97 458	47.3	30	98 727	52.2
35	96 777	42.6	35	98 443	47.4
40	96 044	38.0	40	98 112	42.4
45	95 138	33.2	45	97 506	37.7
50	93 799	28.6	50	96 745	33.1
55	91 750	24.2	55	95 396	28.4
60	88 421	20.1	60	93 227	24.1
65	82 846	16.3	65	90 001	20.0
70	74 065	12.8	70	85 120	15.8
75	61 792	9.9	75	77 066	12.3
80	46 114	7.3	80	64 682	9.1
85	27 898	5.4	85	46 893	6.4
90	12 522	4.2	90	26 394	4.7
95	4006	3.5	95	10 101	3.6
100	975	3.0	100	2603	2.8
105	206	2.6	105	411	2.3

The highlighted line shows that out of 100 000 births, 99 231 males are expected to survive to the age of 15. From that age, the survivors are expected to live for another 67.0 years.

DISCUSSION

- An insurance company sells policies to people to insure them against death over a 30 year period. If the person dies during this period, the beneficiaries receive the agreed payout. Why are such policies cheaper to take out for a 20 year old than a 50 year old?

- How many of your classmates would you expect to be alive to attend a 30 year class reunion?

Example 2   ◄ϙ Self Tutor

Use the life tables to find:

a how many males out of 100 000 are expected to live to 20 years of age

b the life expectancy of males at age 20

c how many females out of 100 000 are expected to live to 45 years, and their life expectancy after that

d the probabiliy of a male surviving to the age of 20

e the probability of a 35 year old female living to the age of 70

f the probability of a 45 year old man dying before he reaches the age of 80.

---

a 98 642 are expected to reach 20 years.

b They are expected to live for a further 56.7 years.

c 97 506 are expected to reach 45. They have a life expectancy of another 37.7 years.

d P(male surviving to reach 20) $= \frac{98\,642}{100\,000} \approx 0.986$

e P(35 year old female lives to 70) $= \frac{85\,120}{98\,443} \approx 0.865$

f P(45 year old man dying before 80) $= 1 - $ P(45 year old living to 80)

$$= 1 - \frac{46\,114}{95\,138}$$
$$\approx 0.515$$

EXERCISE 10C

1 What is the expected age at death of:

   a males now 15 years   **b** females now 15 years

   c males now 50 years   **d** females now 50 years?

2 What percentage of males now aged 45 may be expected to reach the age of:

   a 50 years   **b** 65 years   **c** 80 years?

3 Repeat question **2** for the case of females.

4 A town has 1500 20 year old males and 1800 20 year old females. How many of these people do you expect to still be alive at the age of 65?

5 An insurance company offers to insure a 25 year old female against death for the next 10 years. What is the probability that the female will die within the next 10 years?

D   SAMPLE SPACES

A **sample space** is the set of all possible outcomes of an experiment.

We can display sample spaces by:
- **listing** sets of possible outcomes
- using **2-dimensional grids**
- using **tree diagrams**   (see **Chapter 20**)
- using **Venn diagrams**.

Example 3   ◄⑴ Self Tutor

List the sample space for:
  a choosing a random number between 1 and 9
  b the possible results when A plays B in two sets of tennis
  c the results when tossing a 10 cent coin, a 20 cent coin and a 5 cent coin simultaneously.

  a $S = \{2, 3, 4, 5, 6, 7, 8\}$
  b $S = \{AA, AB, BA, BB\}$ where AB means
       "A wins the first set and B wins the second set."
  c $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ where HHT means
       "a head with the 10 cent coin, a head with the 20 cent coin, and a tail with the 5 cent coin".

Example 4   ◄⑴ Self Tutor

Use a 2-dimensional grid to illustrate the sample space for tossing two coins simultaneously. List the sample space in set notation.
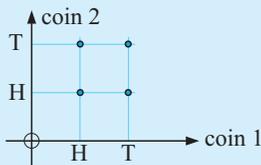
head   tail

The sample space is {HH, HT, TH, TT} where each member of the sample space is represented by one of the points on the grid.

EXERCISE 10D

1 List, in set notation, the sample space for:
  a tossing a coin
  b rolling a 6-sided die
  c the sexes of a 2-child family
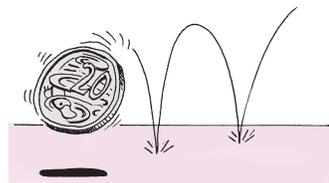
  d tossing a 20 cent and €1 coin simultaneously

  e the order in which 3 men can be lined up

  f the sexes of a 3-child family

  g tossing 4 coins simultaneously

  h the order in which 4 different rowing teams could finish a race.

2 Draw a two-dimensional grid to illustrate the sample space for:

  a rolling a die and tossing a coin simultaneously

  b rolling a pair of dice

  c twirling a square spinner marked A, B, C, D and
  a triangular spinner marked 1, 2, 3.

E   THEORETICAL PROBABILITY

Consider rolling a single die.

The sample space showing the six possible outcomes is {1, 2, 3, 4, 5, 6}.

Since the die is symmetrical we expect that each result is **equally likely** to occur. We expect each result to occur $\frac{1}{6}$ of the time and we use this fraction as the **theoretical probability** of its occurrence.

> If a sample space has n outcomes that are **equally likely** to occur when the experiment is performed once, then each outcome has a probability $\frac{1}{n}$ of occurring.

EVENTS

> An **event** is a collection of outcomes with a particular property or feature.

For example, the event of rolling a prime number with an ordinary die includes the three possible outcomes 2, 3 and 5. The probability of getting each of these results is $\frac{1}{6}$. The probability of the event occurring is $\frac{3}{6}$ as the number of members in the event is 3 and the number of members in the sample space is 6.

> If E is an event of S, a finite sample space in which the individual events are equally likely to occur, then the **theoretical probability** that E will occur is defined by
>
> $$\textbf{P(event)} = \frac{\textbf{number of outcomes in the event}}{\textbf{number of outcomes in the sample space}}$$
>
> or $\mathbf{P}(E) = \dfrac{\mathbf{n}(E)}{\mathbf{n}(S)}$ where $0 \leqslant \mathrm{P}(E) \leqslant 1$.

Example 5

🔊 **Self Tutor**

One card is randomly selected from a pack of 52 playing cards. Determine the probability that it is:

a an ace   **b** not an ace   **c** a "picture card".

$S = \{$cards from a 52 card pack$\}$ $\therefore$ $n(S) = 52$.

a Let A be the event of getting an ace. There are 4 aces in the pack, so $n(A) = 4$.
$\therefore$ $P(A) = \frac{4}{52} = \frac{1}{13}$.

b Let non-A be the event of not getting an ace. There are 48 cards that are not aces, so $n($non-$A) = 48$.
$\therefore$ $P($non-$A) = \frac{48}{52} = \frac{12}{13}$.

c Let Pi be the event of getting a "picture card". There are 3 picture cards in each suit, so $n(Pi) = 12$.
$\therefore$ $P(Pi) = \frac{12}{52} = \frac{3}{13}$.

> Picture cards are Jacks, Queens and Kings.

COMPLEMENTARY EVENTS

In **Example 5**, did you notice that $P(A) + P($non-$A) = 1$?   This should not be a surprise because one of the events *getting an ace* or *not getting an ace* **must occur**. We call these **complementary events**.

> If E is an event, then the event of E **not** occurring is written E' and called the **complementary event of E**.

$$P(E) + P(E') = 1$$
$$\text{or} \ \ P(E') = 1 - P(E)$$

EXERCISE 10E

1 A fair die is rolled. Determine the probability of getting:

  a a 3 or a 5   **b** a negative integer   **c** a 9

  d a result less than 4   **e** a non-five.

2 A poker die has faces A, K, Q, J, 10, and 9, and it is rolled once. Determine the probability of getting:

  a an ace   **b** a number   **c** an ace or a number.

3 A symmetrical octahedral die has numbers 1 to 8 marked on its faces. If it is rolled once, determine the probability of getting:

  a a 4   **b** a number less than 5

  c a number more than 8   **d** a number between 2 and 8.

4 A bag contains 4 red and 3 green buttons. One button is randomly selected from the bag. Determine the probability of getting:

   a a red   **b** a green   **c** a red or a green   **d** a red and a green.

5

A 52 card pack is well shuffled, and then one card is dealt from the top of the pack. Determine the probability that it is:

   a a Jack   **b** a non-Jack

   c a black card   **d** a diamond

   e a diamond and an ace   **f** a diamond or an ace.

6 A lottery consists of 80 tickets numbered 1 to 80. One ticket is chosen at random. Determine the probability that the ticket is:

   a a single digit number   **b** a multiple of 8

   c a multiple of 5 or 8   **d** a factor of 36.

7 Determine the probability that a person randomly selected in the street has his or her birthday in

   a May   **b** February.

F   USING 2-DIMENSIONAL GRIDS

If the sample space is not too large, a 2-dimensional grid is very useful for illustrating the possible outcomes and hence calculating probabilities. Each point on the grid represents a possible outcome, and each outcome is equally likely to occur.

Example 6   **◄)) Self Tutor**

A spinner has four equal sectors containing the numbers 1, 2, 3 and 4. It is spun and an unbiased coin is tossed simultaneously. Graph the sample space and use it to determine the probability of getting:
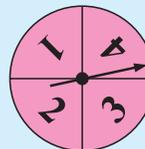
   a a head *and* a 4   **b** a head *or* a 4.
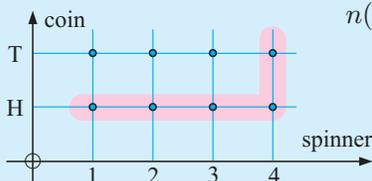
$n(S) = 8$

a P(head *and* 4) $= \frac{1}{8}$

b P(head *or* 4) $= \frac{5}{8}$ {those shaded}

EXERCISE 10F

1 Two coins are tossed simultaneously. Use a 2-dimensional grid to illustrate the sample space and hence determine the probability of getting:

    a two tails   **b** a head and a tail   **c** at least one tail.

2 A coin and a die are tossed simultaneously. Draw a 2-dimensional grid to illustrate the sample space. Determine the probability of getting:

    a a tail and a 6   **b** a tail or a 6

    c neither a 2 nor a 6   **d** neither a tail nor a 5

    e a head and an odd number   **f** a head or an odd number.

3 Draw a 2-dimensional grid to illustrate the sample space when an ordinary die is tossed and a triangular spinner labelled A, B and C is spun simultaneously. Hence, determine the probability of getting:

    a B and 5   **b** A and a prime number   **c** a non-B and a multiple of 3.

4 A pair of dice is rolled. The 36 different possible 'pair of dice' results are illustrated on the 2-dimensional grid alongside.

Use the grid to determine the probability of getting:

    a two 3s   **b** a 5 and a 6

    c a 5 or a 6   **d** at least one 6

    e exactly one 6   **f** no sixes

    g a sum of 7   **h** a sum of 7 or 11

    i a sum greater than 8

    j a sum of no more than 8.
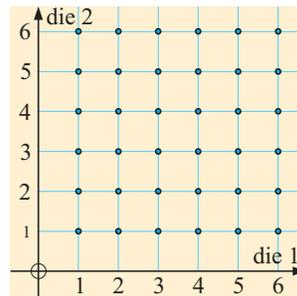
G   COMPOUND EVENTS

Consider the following problem:

Box X contains 2 blue and 2 green balls. Box Y contains 3 red and 1 white ball. A ball is randomly selected from each of the boxes. Determine the probability of getting a blue ball from X and a red ball from Y.

By illustrating the sample space on a two-dimensional grid as shown alongside, we see that 6 of the 16 possibilities are blue from X and red from Y. Since each outcome is equally likely,

P(blue from X **and** red from Y) $= \frac{6}{16} = \frac{3}{8}$.

In the investigation that follows, we will seek a quicker, easier way to find this probability.
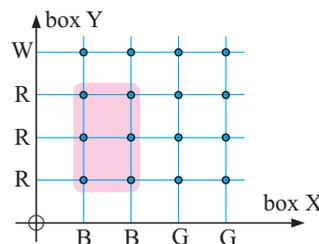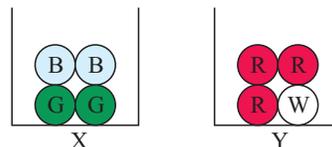
INVESTIGATION 1   PROBABILITIES OF COMPOUND EVENTS

Part 1: In this investigation we seek a rule for finding P(*A and B*)
for two events *A* and *B*.

What to do:

1 Copy and complete the following table. Each probability in the second column has
been found previously.

	Reference	P(*A and B*)	P(*A*)	P(*B*)
a	**Exercise 10F, 1b**	P(a head *and* a tail) =		
b	**Exercise 10F, 2e**	P(a head *and* an odd number) =		
c	**Exercise 10F, 3a**	P(a B *and* a 5) =		
d	Introduction above	P(blue from X *and* red from Y) =		

2 Can you see any connection between the results of the last three columns? If you
can, copy and complete   P(*A and B*) = ...

Part 2:

If a drawing pin finishes ___⊥___ we say it has finished on its *back*. If it finishes

___⟋___ we say it has finished on its *side*.

If two drawing pins are tossed simulta-
neously, the possible results are:

⊥ ⊥   ⊥ ⟋   ⟋ ⟋

two backs   back and side   two sides
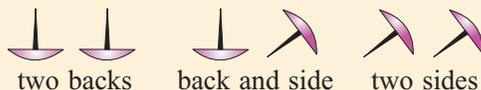
What to do:

1 Obtain two drawing pins of the same shape and size. Toss the pair 80 times and
record the outcomes in a table.

2 Obtain relative frequencies or experimental probabilities for each of the three events.

3 Pool your results with four other people and so obtain experimental probabilities
from 400 tosses. The others must have pins of the same size and shape.

4 Which gives the more reliable estimates, your results or the group's? Why?

5 From the pooled data, estimate the probabilities of getting:

    a a back from one pin   **b** a back *and* a back from two pins.

6 Is P(back *and* back) ≈ P(back) × P(back)?

From **Investigation 1**, it seems that:

> If *A* and *B* are two events where the occurrence of one of them does not affect the
> occurrence of the other, then
>
> $$P(A \textbf{ and } B) = P(A) \times P(B).$$

Before we can formulate a rule, we need to distinguish between **independent** and **dependent**
events.

INDEPENDENT EVENTS

> **Independent events** are events for which the occurrence of either one of the events **does not** affect the occurrence of the other event.

Consider again the example of the balls on page **230**. Suppose we happen to choose a blue ball from box X. This in no way affects the outcome when we choose a ball from box Y. The two events "a blue ball from X" and "a red ball from Y" are **independent events**.

In general: If A and B are **independent events** then $P(A \text{ and } B) = P(A) \times P(B)$.

This rule can be extended to any number of independent events.

For example: If A, B and C are all independent events then
$$P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C).$$

Example 7   ◁) Self Tutor

A coin and a die are tossed simultaneously. Without using a grid, determine the probability of getting a head and a 3.

The events are clearly independent.
$$\therefore \quad P(\text{head and } 3) = P(H) \times P(3)$$
$$= \tfrac{1}{2} \times \tfrac{1}{6}$$
$$= \tfrac{1}{12}$$

EXERCISE 10G

1   At a mountain village in Papua New Guinea it rains on average 6 days a week. Determine the probability that it rains on:

   a any one day   **b** two successive days   **c** three successive days.

2   A coin is tossed 3 times. Determine the probability of getting the following sequences of results:

   a head, then head, then head   **b** tail, then head, then tail.

3   A school has two photocopiers. On any one day, machine A has an 8% chance of malfunctioning and machine B has a 12% chance of malfunctioning.
Determine the probability that on any one day both machines will:

   a malfunction   **b** work effectively.

4   A couple decide that they want 4 children, none of whom will be adopted. They would really like the children to be born in the order boy, girl, boy, girl. Determine the probability that:

   a the children will be born in the desired order

   b the children will be born in some other order.

5 Two marksmen fire at a target simultaneously. John hits the target 70% of the time and Benita hits it 80% of the time. Determine the probability that:

    a they both hit the target

    b they both miss the target

    c John hits it but Benita misses

    d Benita hits it but John misses.

6 An archer always hits a circular target with each arrow shot. On average he hits the bullseye 2 out of every 5 shots. If 3 arrows are shot at the target, determine the probability that the bullseye is hit:

    a every time   **b** the first two times, but not on the third

    c on no occasion.

H   EVENTS AND VENN DIAGRAMS

In **Chapter 5** we studied **Venn Diagrams**. We saw that they consist of a rectangle which represents the universal set, and circles within it representing subsets. In probability questions, the universal set is the sample space of the experiment. The circles represent particular events. We can use Venn diagrams to solve probability questions and to establish probability laws.

This Venn diagram shows the event $E = \{1, 2\}$ when rolling a die. The sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Example 8   🔊 Self Tutor
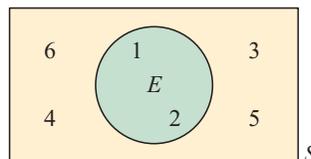
The Venn diagram alongside represents the sample space S of all children in a class. Each dot represents a student. The event E shows all those students with blue eyes. Determine the probability that a randomly selected child:

    a has blue eyes

    b does not have blue eyes.
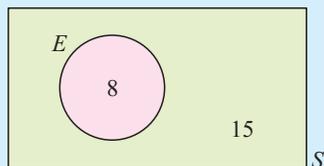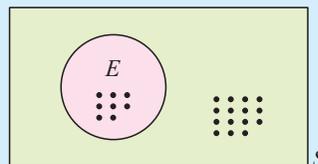
$n(S) = 23, \quad n(E) = 8$

a P(blue eyes) $= \dfrac{n(E)}{n(S)} = \dfrac{8}{23}$

b P(not blue eyes) $= \dfrac{15}{23}$

    {as 15 of the 23 are not in E}

or   P(not blue) $= 1 -$ P(blue eyes) $= 1 - \dfrac{8}{23} = \dfrac{15}{23}$ {complementary events}

UNION AND INTERSECTION

If A and B are two events in the sample space then the event A **or** B means that any member of this event is in at least one of the events A or B. The event A or B corresponds to the **union** of A and B.
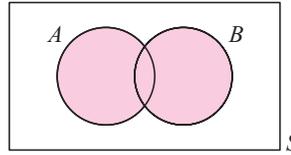
"A **and** B" means that any member of this event is in "both A and B". The event "A and B" corresponds to the **intersection** of A and B.
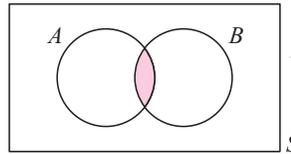
If you are in circles A **and** B you are in both circles at the same time.

Example 9   ◆ᴐ) **Self Tutor**

On separate Venn diagrams containing two non-disjoint events A and B, shade the region representing:

a A but not B **b** neither A nor B.

a

b
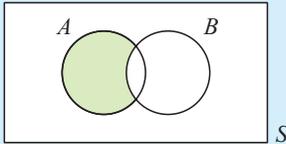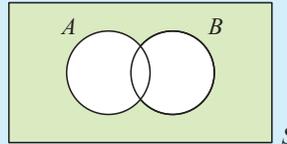
Example 10   ◆ᴐ) **Self Tutor**

The Venn diagram alongside illustrates the number of people in a sporting club who play tennis (T) and hockey (H). Determine the number of people:

a in the club   **b** who play hockey
c who play both sports   **d** who play neither sport
e who play at least one of the two sports.

a Number in the club
$= 15 + 27 + 26 + 7 = 75$

b Number who play hockey
$= 27 + 26 = 53$

c Number who play both sports $= 27$

d Number who play neither sport $= 7$

e Number who play at least one sport
$= 15 + 27 + 26 = 68$

EXERCISE 10H

1 On separate Venn diagrams for two non-disjoint events A and B, shade the region representing:

   a A **b** B **c** both A and B

   d A or B **e** B but not A **f** exactly one of A or B.

2

The Venn diagram alongside illustrates the number of students in a particular class who study Chemistry (C) and History (H). Determine the number of students:

   a in the class

   b who study both subjects

   c who study at least one of the subjects

   d who only study Chemistry.

3 In a survey at an alpine resort, people were asked whether they liked skiing (S) or snowboarding (B). Use the Venn diagram to determine the number of people:

   a who took part in the survey

   b who liked both activities

   c who liked neither activity

   d who liked exactly one of the activities.

Example 11   ◀)) Self Tutor

In a class of 30 students, 19 study Physics, 17 study Chemistry, and 15 study both of these subjects. Display this information on a Venn diagram and hence determine the probability that a randomly selected class member studies:

   a both subjects   **b** at least one of the subjects

   c Physics, but not Chemistry   **d** exactly one of the subjects.

Let P represent the event of 'studying Physics' and C represent the event of 'studying Chemistry'.

Now
$$a + b = 19 \quad \text{\{as 19 study Physics\}}$$
$$b + c = 17 \quad \text{\{as 17 study Chemistry\}}$$
$$b = 15 \quad \text{\{as 15 study both\}}$$
$$a + b + c + d = 30 \quad \text{\{as there are 30 in the class\}}$$

$$\therefore \quad b = 15, \quad a = 4, \quad c = 2, \quad d = 9.$$

a   P(P and C)

$= \frac{15}{30}$ or $\frac{1}{2}$

b   P(at least one of P and C)

$= \frac{4+15+2}{30}$

$= \frac{21}{30}$ or $\frac{7}{10}$

c   P(P but not C)

$= \frac{4}{30}$

$= \frac{2}{15}$

d   P(exactly one of P and C)

$= \frac{4+2}{30}$

$= \frac{6}{30}$ or $\frac{1}{5}$

4 In a class of 40 students, 19 play tennis, 20 play netball, and 8 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:

  a plays tennis
  b does not play netball
  c plays at least one of the two sports
  d plays one and only one of the sports
  e plays netball, but not tennis.

5 50 married men were asked whether they gave their wife flowers or chocolates for their last birthday. The results were: 31 gave chocolates, 12 gave flowers, and 5 gave both chocolates and flowers. If one of the married men was chosen at random, determine the probability that he gave his wife:

  a chocolates or flowers
  b chocolates but not flowers
  c neither chocolates nor flowers.

6 The medical records for a class of 30 children show whether they had previously had measles or mumps. The records show that 24 have had measles, 12 have had measles and mumps, and 26 have had measles or mumps. If one child from the class is selected randomly from the group, determine the probability that he or she has had:

  a mumps
  b mumps but not measles
  c neither mumps nor measles.

7

From the Venn diagram, $P(A) = \dfrac{a+b}{a+b+c+d}$.

  a Use the Venn diagram to find:
    i P(B)   **ii** P(A and B)   **iii** P(A or B)   **iv** P(A) + P(B) − P(A and B).
  b What is the connection between   P(A or B)   and   P(A) + P(B) − P(A and B)?

8 **a** The grid on the next page shows all possible results when selecting a playing card at random. Use the grid to find:
    i P(an ace *and* a spade)   **ii** P(an ace *or* a spade).
  b Use question **7 b** to check your answer to **ii**.

I EXPECTATION

In many gaming situations we want to know what results are expected or what chances there are of winning.

For example, when a fair coin is tossed, the chance of a head or a tail occurring is equally likely. If a coin is tossed 100 times we would "expect" $\frac{1}{2}$ of the results to be heads. We would thus expect 50 heads.

Similarly, if a normal six-sided die is rolled 300 times, the possible outcomes 1, 2, 3, 4, 5, and 6 are equally likely to occur with probability $\frac{1}{6}$ on each roll. We would expect $\frac{1}{6}$ of the results to be a 4, which means $\frac{1}{6} \times 300 = 50$ of them.

In general:

> If there are n trials and the probability of a single event occurring is p then the **expectation** of the occurrence of that event is $n \times p$.

Example 12 ◀)) **Self Tutor**

When an archer fires at a target there is a probability of $\frac{2}{5}$ that he hits the bullseye. In a competition he is required to fire 40 arrows. How many times would you expect him to hit the bullseye?

$p = \text{P(bullseye)} = \frac{2}{5}$ and $n = 40$

$\therefore$ the expected number of bullseyes is $np = 40 \times \frac{2}{5}$
$= 16$

EXERCISE 10I
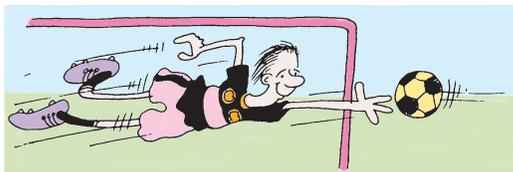
1   A football goalkeeper has probability $\frac{3}{10}$ of saving a penalty attempt. How many goals would he expect to save out of 90 penalty shots?

2

During the snow season there is a $\frac{3}{7}$ probability of snow falling on any particular day. If Dan skis for five weeks, on how many days could he expect to see snow falling?

3 If two dice are rolled simultaneously 180 times, on how many occasions would you expect to get a double?
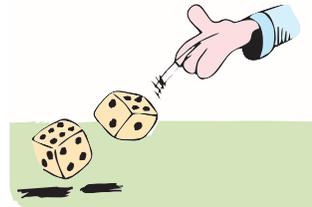
4 A hat contains three yellow discs and four green discs. A disc is drawn from the hat. If the disc is then returned to the hat and the procedure is repeated 350 times, on how many occasions would you expect a green disc to be drawn?

5 In a random survey of her electorate, politician A discovered the residents' voting intentions in relation to herself and her two opponents B and C. The results are indicated alongside:

A	B	C
165	87	48

  a Estimate the probability that a randomly chosen voter in the electorate will vote for:

    i A   **ii** B   **iii** C.

  b If there are 7500 people in the electorate, how many of these would you expect to vote for:

    i A   **ii** B   **iii** C?

Example 13   ◄》 **Self Tutor**

In a game of chance, the player spins a square spinner labelled 1, 2, 3, 4, and wins the amount of money shown in the table alongside depending on which number comes up. Determine:

Number	1	2	3	4
Winnings	$1	$2	$5	$8

  a the expected return for one spin of the spinner
  b whether you would recommend a person to play this game if it costs $5 to play one game.

---

  a Each number is equally likely, so the probability for each number is $\frac{1}{4}$

    $\therefore$ the expected return $= \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4.$

  b The expected return is $4 whereas it costs $5 to play the game, so you would not recommend that a person play the game.

---

6 A person rolls a normal six-sided die and wins the number of dollars shown on the face.

  a How much would the person expect to win for one roll of the die?
  b If it costs $4 to play the game, would you advise the person to play several games?

7 A person plays a game with a pair of coins. If a double head appears, $10 is won. If a head and a tail appear, $3 is won. If a double tail appears, $5 is lost.

   a How much would a person expect to win playing this game once?

   b If the organiser of the game is allowed to make an average of $1 per game, how much should be charged to play the game once?

8 A single coin is tossed once. If a head appears you win $2 and if a tail appears you lose $1. How much would you expect to win from playing this game three times?

INVESTIGATION 2   **ROULETTE**

Click on the icon to obtain a printable worksheet on the expected returns from playing the game of roulette.

PRINTABLE WORKSHEET

REVIEW SET 10A

1 Ninety three people arriving at the beach are asked their age. The results are shown opposite. Assuming they give correct replies, what is the probability that a randomly selected person on the beach will be aged:

Age	Frequency
0 - 9	17
10 - 19	25
20 - 29	26
30 - 39	20
40+	5

   a 30 or more   **b** between 10 and 30?

2 A netball goalshooter has probability $\frac{3}{4}$ of scoring a goal each time she shoots. If she has 52 shots at goal, how many goals would you expect her to score?

3 What is meant by saying that two events are *independent*?

4 A coin is tossed and a die is rolled simultaneously.

   a Illustrate the sample space on a grid.

   b Find the probability of getting:

      i a head and an even number   **ii** a head and a non-3

      iii a 5 or a 6   **iv** a head or an even number.

5 Three coins are tossed simultaneously.

   a Using H for a head and T for a tail, list the possible results which could occur.

   b Find the probability of getting:

      i two heads and a tail   **ii** at least one head.

6 In a golf match, Peter has a 70% chance of hitting the green when using a nine iron. Paula has a 90% chance when using the same club. If, at a particular hole, they both elect to use a nine iron to play to the green, determine the probability that:

   a both hit the green   **b** neither hits the green

   c at least one hits the green   **d** only Peter hits the green.

7 From the life tables on page **224**, determine the probability that:

    a a female will survive to the age of 60

    b a 15 year old male will survive to the age of 70

    c a 50 year old woman will die before reaching 65.

REVIEW SET 10B

1 Over a 35 day period, Lorna records the number of phone calls she receives. She then draws the graph of her data shown opposite.

    a Using this data, estimate the probability that on any day she will receive:

        i no phone calls   **ii** at least 3 phone calls

        iii between 1 and 5 calls inclusive.

    b How reliable do you believe your answers are in **a**? Give reasons for your response.

Number of calls
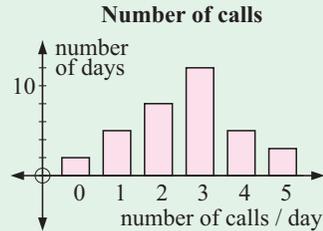
2 From past experience, a surfer has probability 0.83 of catching a wave. In one week she tries to catch 75 waves. How many do you expect her to have caught?

3 A shared garden area has 12 patches owned by 12 different people. All lines on the figure are fences. If a patch is selected at random, what is the probability that it has:
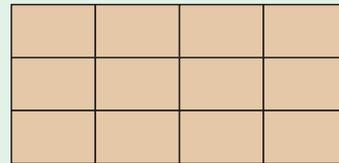
    a two shared fences   **b** three shared fences?

4 Jar X contains 3 white and 2 red marbles. Jar Y contains 6 white and 4 red marbles. A marble is selected at random from each jar. Determine the probability that:

    a both marbles are white   **b** both marbles are red

    c one marble of each colour is selected.

5 Zelda rolls a normal six-sided die. She will win twice the number of dollars as the number shown on the face.

    a How much does Zelda expect to win from one roll of the die?

    b If it costs \$8 to play the game, would you advise Zelda to play several games? Explain your answer.

6 From the life tables on page **224**, determine the probability that:

    a a male will survive to the age of 40

    b a 20 year old female will survive to the age of 80

    c a 35 year old man will die before the age of 70.

7 At a local girls' school, 65% of the students play netball, 60% play tennis, and 20% play neither sport. Display this information on a Venn diagram, and hence determine the likelihood that a randomly chosen student plays:

    a netball   **b** netball but not tennis

    c at least one of these two sports   **d** exactly one of these two sports.

Chapter

11

Financial mathematics

Contents:

OPENING PROBLEM

Ravi earns $42 000 p.a. as a sales consultant. He has $5000 savings in an account that pays 5.4% p.a. interest compounded monthly. Ravi wishes to purchase a $22 000 new car for use in his job. He can either:

- pay 15% deposit and $420 per month for 5 years, or
- take out a personal loan at 11.5% p.a. over 3 years.

After completing this chapter, you should be able to help Ravi answer the following questions:

1 How much interest would Ravi receive if he left his $5000 in the savings account for 1 year?

2 What amount of depreciation can Ravi claim as a tax deduction on his car assuming a tax rate of 30% p.a.?

3 Which car purchase option would you recommend for Ravi?

4 How should Ravi finance any deposit for his car?

5 How much value added tax (VAT) at 15% does Ravi need to pay on his car?

A   BUSINESS CALCULATIONS

Rather than work for someone else, many people choose to run their own businesses. A successful small business owner needs to have a keen understanding of percentages in order to calculate profits, discounts, mark-up, depreciation, and tax.

PERCENTAGE CHANGE USING A MULTIPLIER

Shopkeepers often need to increase or decrease the price of an item by a percentage. This is easily done using a **multiplier**.

For example:

- if a price is increased by 20%, the final price is $100\% + 20\%$ or 120% of the original price
  $\therefore$ the multiplier $= 1.2$

- if a price is decreased by 20%, the final price is $100\% - 20\%$ or 80% of the original price.
  $\therefore$ the multiplier $= 0.8$

> The multiplier provides us with a **one-step** method for increasing or decreasing quantities.

$$\text{multiplier} = \frac{\text{new value}}{\text{old value}}$$

Example 1   ◀ **Self Tutor**

What multiplier corresponds to:
  a a 40% increase   **b** a 15% decrease?

  a multiplier $= (100 + 40)\% = 140\% = 1.4$
  b multiplier $= (100 - 15)\% = 85\% = 0.85$

Example 2   ◀ **Self Tutor**

For the following multipliers, state the corresponding percentage increase or decrease:
  a 1.15   **b** 0.88

  a   1.15   **b**   0.88
     $= 1.15 \times 100\%$ $= 0.88 \times 100\%$
     $= 115\%$ $= 88\%$
     which is an increase over 100%   which is a decrease below 100%
     of 15%   of 12%

EXERCISE 11A.1

1 Find the multiplier that corresponds to:
  a a 20% increase   **b** a 20% decrease   **c** a 45% increase
  d a 45% decrease   **e** an 8% decrease   **f** a 3% increase
  g a 100% increase   **h** a 600% increase   **i** a 100% decrease

2 For the following multipliers, state the corresponding percentage increase or decrease:
  a 1.12   **b** 1.23   **c** 0.96   **d** 0.85
  e 1.45   **f** 0.67   **g** 2.4   **h** 0.3

Example 3   ◀ **Self Tutor**

  a Increase $10 500 by 8%.   **b** Decrease £120 by 12%.

  a   new amount
       $= 108\%$ of $10 500   {to increase by 8%, multiply by 108%}
       $= 1.08 \times \$10\,500$
       $= \$11\,340$

  b   new amount
       $= 88\%$ of £120   {to decrease by 12%, multiply by 88%}
       $= 0.88 \times £120$
       $= £105.60$

3 Calculate the following:

    a Increase \$50 by 5%

    b Decrease €27 by 30%

    c Decrease £780 by 20%

    d Increase \$68 by 15%

    e Increase £140 by 25%

    f Decrease €640 by 25%

    g Decrease \$36 by 2%

    h Increase €325 by 4%

    i Increase £1600 by 100%

    j Decrease \$4500 by 75%

Example 4   🔊 **Self Tutor**

A house is bought for \$156 000 and six months later is sold for \$175 500. What is the percentage increase on the investment?

Method 1:

    multiplier

$$= \frac{\text{new value}}{\text{old value}}$$

$$= \frac{175\,500}{156\,000}$$

$$= 1.125$$

$$= 112.5\%$$

∴   a 12.5% increase occurred

Method 2:

    percentage increase

$$= \frac{\text{increase}}{\text{original}} \times 100\%$$

$$= \frac{19\,500}{156\,000} \times 100\%$$

$$= 12.5\%$$

∴   a 12.5% increase occurred

> percentage change $= \dfrac{\text{change}}{\text{original}} \times 100\%$

---

4 Find the percentage change that occurs when:

    a \$80 increases to \$100

    b €9000 decreases to €7200

    c \$85 reduces to \$82

    d £96 increases to £108

    e €75 000 increases to €86 000

    f £3500 reduces to £2400.

5 A block of land is bought for €164 000 and sold later for €208 000. Calculate the percentage increase in the investment.

6 A share trader buys a parcel of shares for £3200 and sells them for £2800. Calculate the percentage decrease in the investment.

7 Paolo bought a truck costing €47 500, but a few months later he found it was no longer large enough for transporting his goods. If Paolo sold the truck for €42 000, what was his percentage loss?

MARK-UP

In order to make a profit, a retailer must add on an amount to the price at which he or she buys goods. This amount is called the **mark-up**.

For example, if the retailer buys an item for \$100 and his mark-up is 40%, he will mark the item to sell at   \$100 × 140% = \$140.   The mark-up on the item is \$40.

Example 5   ◀)) **Self Tutor**

Alfred buys shirts for €20 and marks them up by 60%.
At what price does he sell them?

Alfred sells the shirts for   €20 × 160%
                             = €20 × 1.6
                             = €32

> A mark-up increases the price of an item!

EXERCISE 11A.2

1 At what price would Alfred sell:

    a a shirt costing him €24 if he applies a 40% mark-up

    b a skirt costing him €50 if he applies a 55% mark-up

    c a jacket costing him €120 if he applies a 35% mark-up

    d a suit costing him €320 if he applies a 30% mark-up

    e a pair of trousers costing him €45 if he applies a 28% mark-up?

2 A baker puts a mark-up of 30% on the bread sold to storekeepers. If it costs the baker, on average, 90 cents per loaf, what price must the baker charge storekeepers?

3 Adriano sells fruit and vegetables. At what price should he sell:

    a apples costing him $2.30 a kg, if he marks them up by 60%

    b pineapples costing him $0.80 each, if he marks them up by 80%

    c cabbages costing him $1.20 each, if he marks them up by 70%?

4 A butcher buys chicken breast fillets for $5 per kg and adds a 65% mark-up for profit. What price per kilogram will the customer pay?

5 Kane bought a lawnmower for $450 to sell in his shop. If he marks it up by 32%, at what price does he hope to sell it?

DISCOUNT

In order to encourage a sale, a retailer may offer a **discount**. This means that the sale price is **reduced** or **discounted** by a certain amount or by a given percentage.

Example 6   ◀)) **Self Tutor**

Sally discounts her dresses by 20% at her annual sale. What is the sale price of a dress originally marked at $150?

Sale price = original price × 80%   {reduced by 20%}
           = $150 × 0.8
           = $120

> A discount reduces the price of an item!

EXERCISE 11A.3

1 For what price will Sally sell a dress if it was originally marked at:

    a $80 and it is discounted 30%   **b** $160 and it is discounted 25%

    c $240 and it is discounted 35%?

2 A CD is marked at €20 and is discounted by 15%. What is its selling price?

3 A refrigerator is marked at £850 and is discounted by 18%.
What is its selling price?

4 A bicycle marked at $560 is to be discounted by 30%.
Calculate the sale price.

5 A wok priced at RM95 is to be sold at a 45% discount.
Calculate its selling price.

Example 7   ◄)) Self Tutor

A bag of tomatoes is sold at €4.35 to make a 30% profit. Find the cost price of the tomatoes.

        selling price = 130% of cost price

     $\therefore$ €4.35 = 1.30 × cost price

     $\therefore$ $\dfrac{€4.35}{1.30}$ = cost price   So, the cost price = €3.35

---

6 A shop sells fresh fish with a 45% mark-up on the price they buy them for. How much did the fish cost the shop if they sell them for $5.50 per kg?

7 A litre of milk retails for €1.47 after being marked up by 40%. What was the cost price?

Example 8   ◄)) Self Tutor

A local electrical store offers a 15% discount on all items sold at its summer sale. If a television set is sold for $675, find its marked price.

Since there is a 15% discount,

        selling price = 85% of marked price

     $\therefore$ $675 = 0.85 × marked price

     $\therefore$ $\dfrac{\$675}{0.85}$ = marked price

So,   the marked price is $794.12

---

8 A CD was discounted by 20% and sold for ¥2360. What was the marked price?

9 A pair of bathers sold for €44 after being discounted by 45%. What was the marked price of the bathers?

TAX ON THE SALES OF GOODS

In most countries a tax is payable on the sale of goods. This tax has various names depending on the country where the transaction takes place. The two most common names are **value added tax (VAT)** and **goods and services tax (GST)**.

The standard rate of tax varies from one country to another. For example:

- in Chile it is 19%
- in Sweden it is 25%
- in Australia it is 10%
- in the UK it is 17.5%
- in Russia it is 18%
- in Singapore it is 7%.

Example 9　　　　　　　　　　　　　　　　　　　　　◀ Self Tutor

In order to make reasonable profit, an electrical store in Russia must sell a television set for 8500 rubles. If an 18% tax applies, find:

a the television's selling price

b the amount of tax.

a　Sale price $= 8500 \times 1.18$ rubles　$\{100\% + 18\% = 118\%\}$

　　　　　　　　$= 10\,030$ rubles

b　The amount of tax $= (10\,030 - 8500)$ rubles

　　　　　　　　　　　　$= 1530$ rubles

EXERCISE 11A.4

1 A washing machine has a price of £240 before tax. It is sold with a VAT of $17\frac{1}{2}\%$ which must be added to the price.

　　a What does it sell for?　　　　　**b** How much is the VAT?

2 A double bed in Canada has a price tag which says $2165 plus 6% GST.

　　a What amount is paid by a customer?　　**b** How much is the GST?

3 In Spain a case of wine sells for €350 with an IVA of 20% to be added to this price.

　　a What amount is paid by a customer?　　**b** How much is the IVA?

4 In Croatia the PDV (tax) is 22%. A boat is priced at HRK62 500.

　　a What amount is paid by a customer?　　**b** How much is the PDV?

CHAIN PERCENTAGE PROBLEMS

Sometimes we need to use a multiplier more than once within a problem.

Example 10　　　　　　　　　　　　◀ Self Tutor

A 1.25 L soft drink is bought by a deli for $0.90 (GST exclusive). The deli owner adds 70% mark-up and then 10% GST. What price does the customer pay?

Price $= \$0.90 \times 1.7 \times 1.1$

　　　$= \$1.68$

> The $0.90 is increased by 70% **and then** this amount is increased by 10%.

Example 11

◀)) **Self Tutor**

Cucumbers are initially marked up 30% above the cost price. If a 10% discount is then offered, what is the percentage profit? Assume there is no VAT or GST.

Selling price $=$ cost price $\times 130\% \times 90\%$
$\qquad\qquad = $ cost price $\times 1.3 \times 0.9$
$\qquad\qquad = $ cost price $\times 1.17$
$\qquad\qquad = 117\%$ of cost price

So, the percentage profit is 17%.

EXERCISE 11A.5

1 Katinka buys a skirt for $32 (GST exclusive). She adds 55% profit plus 10% GST before putting it on sale in her salon. What price should be put on the tag?

2 A surfboard manufactured for $300 is sold to a retail shop for a 40% profit plus 10% VAT. How much does the retail shop pay?

3 Flavoured milk is sold to a supermarket for $1.20 per carton. The supermarket adds 60% profit plus 10% GST. What price does the customer pay?

4 A fisherman sells a crayfish to a wholesaler for €35. The wholesaler sells it to a fresh fish shop for 60% profit. However, the fish shop is forced to sell the crayfish at a 15% loss. Assuming there is no tax, what price must a crayfish lover pay for this delicacy?

5 If bread is marked up 40% above the cost and then a 20% discount is offered, what is the percentage profit? Assume there is no tax.

6 If meat is marked up 60% above cost and sold in a 35% off 'market day' sale, calculate the percentage profit. Assume there is no tax.

7 A rare book is bought by an antiques dealer, and he marks its price up by 100% for sale. One of his regular customers is interested in buying the book, so the dealer offers the customer a 20% discount. What is the overall percentage profit on the sale?

B   APPRECIATION

When the value of an investment or an item such as a house increases, we say it **appreciates** in value.

You may have noticed that the prices of everyday goods and services also increase over time due to **inflation**. This is a form of appreciation.

Example 12 ◀⑨ Self Tutor

The inflation rate over the next three years is predicted to be 2% then 3% then 3.5%. If an item currently costs \$140 and the cost rises in line with the predicted inflation, what will its cost in 3 years' time be?

Cost in 3 years $= \$140 \times 1.02 \times 1.03 \times 1.035$
$$= \$152.23$$

Example 13 ◀⑨ Self Tutor

Over 4 consecutive years, the value of a house increases by 15%, increases by 9%, decreases by 4%, and increases by 18%. What is the overall percentage increase in value over this period?

Let \x be the original value.

$\therefore$ value after 1 year $= \$x \times 1.15$ {15% increase}
$\therefore$ value after 2 years $= \$x \times 1.15 \times 1.09$ {9% increase}
$\therefore$ value after 3 years $= \$x \times 1.15 \times 1.09 \times 0.96$ {4% decrease}
$\therefore$ value after 4 years $= \$x \times 1.15 \times 1.09 \times 0.96 \times 1.18$ {18% increase}
$$\approx \$x \times 1.4200$$

$\therefore$ a 42% increase in the value has occurred.

EXERCISE 11B

1 The inflation rate is predicted to be 3%, 3.5%, 4% and then 5% over the next four years. What would you expect a bicycle costing £500 today to cost in four years' time?

2 An investment of \$18 250 is left to accumulate interest over a 4 year period. During the first year the interest paid is 9.2%, and in successive years the rates paid are 8.6%, 7.5% and 5.6%. Find the value of the investment after 4 years.

3 Caitlin invests €24 000 in a fund which accumulates interest at 9.5% per annum. If the money is left in the investment for a 5 year period, what will its maturing value be?

4 An amount of money is invested at 8% for the first year, 11% for the second year, and 7% for the third year. What is the percentage increase in the value of the investment over this period?

5 A politician's wage increases by 3%, 8%, 5% and then 10% over a four year period. What is the percentage increase in the wage over this period of time?

6 A share fund reported a 9% increase in value for year 1, a 13% decrease in value for year 2, and a 4% increase in value for year 3. What was the overall percentage increase or decrease of the share fund over the 3 years?

C   COMPOUND INTEREST

Suppose you have €5000 and wish to bank it for a period of 5 years.

The bank offers you an interest rate of 7% p.a. which means 7% **per annum** or per year.

Interest is paid at the end of each year that your money is in the bank. You are paid what is known as **compound interest** which is calculated on the actual amount you have in your account at these times.

The following table shows you what you should have in your account at the end of each year. It assumes there are no bank charges or fees.

After year	Interest paid	Account balance
0		€5000.00
1	7% of €5000.00 = €350.00	€5000.00 + €350.00 = €5350.00
2	7% of €5350.00 = €374.50	€5350.00 + €374.50 = €5724.50
3	7% of €5724.50 = €400.71	€5724.50 + €400.71 = €6125.21
4	7% of €6125.21 = €428.76	€6125.21 + €428.76 = €6553.98
5	7% of €6553.98 = €458.78	€6553.98 + €458.78 = €7012.76

Your account balance is growing as the interest earned for each year is added on. You earn interest on interest, so the interest paid is increasing each year.

If your initial deposit was made on May 11, 2008, then you should expect interest payments on or about May 12 each year.

Notice the increasing amount of interest each year.

DISCUSSION   **SIMPLE INTEREST OR COMPOUND INTEREST?**

€5000 is placed in a bank where interest is paid at a rate of 7% p.a. and is calculated annually. In the long term, why would we expect the investment to grow more with compound rather than simple interest?

EXERCISE 11C.1

1 £1000 is deposited into a bank account which pays interest at 6% p.a. The money is left there for 4 years.

   a If the bank pays simple interest:
   i how much interest is earned each year
   ii what will the £1000 amount to at the end of 4 years?
   iii Draw a graph which shows the value of the investment over time.

b Suppose now that the bank pays compound interest.

  i Complete a table like the one above for this investment.

  ii Draw a graph showing the value of the investment over time on the same set of axes as the simple interest investment.

c Compare the two graphs drawn in **a** and **b**.

2 $10 000 is invested for 4 years at 8% p.a. with interest paid annually.

a What will the investment amount to at the end of the 4th year if the interest paid is:

  i simple   **ii** compound?

b What type of interest payment gives the greater final amount and by how much?

3 How much interest is earned for investments of:

a €4000 for 3 years at 9.3% p.a. compounded annually

b £20 000 for 4 years at 7.8% p.a. compounded annually

c $4750 for 5 years at 8.3% p.a. compounded annually

d 7600 pesos for 4 years at 11.4% p.a. compounded annually

e ¥400 000 for 5 years at 4.95% p.a. compounded annually

f 35 000 pesos for 6 years at 3.98% p.a. compounded annually?

THE COMPOUND INTEREST FORMULA

Consider again the deposit of €5000 into a bank account for 5 years at an interest rate of 7% p.a. compounding annually.

We can use the concept of a chain percentage increase to calculate the value of the investment after 5 years.

Each year the amount in the bank is increased by 7%.

This corresponds to a multiplier of 107% or 1.07.

So, the value after 5 years is   €5000 × 1.07 × 1.07 × 1.07 × 1.07 × 1.07
$$= €5000 \times (1.07)^5$$
$$\approx €7012.76$$

Notice that $1.07 = 1 + 0.07 = 1 + i$ where i is the annual interest rate written as a decimal number.

These observations lead to the **compound growth formula**:

$F_v = P_v(1 + i)^n$ where F_v is the **future value**

$\qquad\qquad\qquad\qquad\qquad\quad P_v$ is the **present value** or amount initially invested

$\qquad\qquad\qquad\qquad\qquad\quad i$ is the **annual interest rate** as a decimal

$\qquad\qquad\qquad\qquad\qquad\quad n$ is the **number of years of investment**.

$F_v - P_v$ is the **total interest earned** over the n year period.

Example 14   ◀) **Self Tutor**

$8000 is invested for 3 years at 8% p.a. compound interest with interest paid annually.

   a What will it amount to after this period?

   b How much interested is earned?

a $F_v = P_v(1+i)^n$ **b** Interest earned $= F_v - P_v$
$$= \$8000 \times (1.08)^3$$ $$= \$10\,077.70 - \$8000$$
$$\approx \$10\,077.70$$ $$= \$2077.70$$

FINDING THE AMOUNT TO BE INVESTED FOR A DESIRED FUTURE VALUE

We can use the compound growth formula to find how much we need to invest in order to generate a particular amount in the future.

Example 15   ◀) **Self Tutor**

How much should I invest now if I require a maturing or future value of $15\,000 in 4 years' time? The highest rate of interest I can get is 7.9% p.a. compounded annually, fixed over the period.

In this case $F_v = 15\,000,$ $n = 4,$ and $i = 7.9\% = 0.079$

    Now $F_v = P_v(1+i)^n$

$\therefore$ $15\,000 = P_v(1.079)^4$

$\therefore$ $\dfrac{15\,000}{(1.079)^4} = P_v$ {dividing both sides by $(1.079)^4$}

$\therefore$ $11\,066.38 \approx P_v$

I would need to invest $11\,066.38.

CALCULATING THE ANNUAL RATE OF INVESTMENT GROWTH

The compound growth formula can be used in non-banking problems to find the average rate at which an investment is growing.

Example 16   ◀) **Self Tutor**

Stephen Prior bought a rare New Zealand bank note for $4600. He kept it for 4 years and then sold it for $7300 at auction. What annual rate did Stephen's investment return?

$P_v = 4600,$ $n = 4,$ $F_v = 7300$

Now $P_v(1+i)^n = F_v$

$\therefore$ $4600(1+i)^4 = 7300$

$$\therefore \quad (1+i)^4 = \tfrac{7300}{4600} \qquad \text{\{dividing both sides by 4600\}}$$

$$\therefore \quad 1+i = \sqrt[4]{\tfrac{73}{46}} \qquad \text{\{fourth root of both sides\}}$$

$$\therefore \quad 1+i \approx 1.122\,38$$

$$\therefore \quad i \approx 0.122\,38$$

$$\therefore \quad i \approx 12.238\%$$

So, the annual rate of increase was about 12.2%.

EXERCISE 11C.2

1   £10 000 is invested for 4 years at 7% p.a. compound interest with interest paid annually.
   a   What will it amount to after this period?
   b   How much interest is earned?

2   €4750 is invested for 5 years at 6.8% p.a. compound interest with interest paid annually.
   a   What is the maturing or final value of the investment?
   b   How much interest will be paid into the account over this period?

3   $20 000 is placed into a bank account for 6 years. The interest is fixed at 7.5% p.a. and is compounded annually.
   a   Find the maturing value of the investment.
   b   What portion of the maturing value is interest?

4   Geetha would like $15 000 for a holiday in Fiji in 5 years' time. How much does she need to invest now at a fixed rate of 7.25% p.a. compounded annually in order to achieve her goal?

5   Abdul requires €20 000 in 4 years' time. The best interest rate he can get is 6.8% p.a. compounded annually, fixed for the period. How much does Abdul need to invest?

6   Sally has an agreement with her uncle to purchase his apartment in 4 years' time for £155 000. She figures that she can save £55 000 during that period from her weekly pay. If she can invest at a fixed rate of 8.75% p.a. compounded annually, how much does she need to invest now from current savings to ensure she has the money needed?

7   Fifi bought an antique table for €2700 and 8 years later sold it for €6350. Find the yearly rate of increase for this investment.

8   My uncle bought a section of land for $150 in November 1944. He sold it for $350 000 in November 2007. What was the annual rate of increase for his investment?

9   In 1964 a rare vase was offered for sale at $135. Fredrik bought the vase and eventually sold it for $84 000 in 2006. What was the annual rate of increase in the value of his investment?

INVESTIGATION 1   COMPOUND INTEREST

Consider an investment of $1000 with interest of 6% p.a. paid annually and compounded for x years.

Using the compound growth formula $F_v = P_v(1 + i)^n$ we get
$F_v = 1000 \times (1.06)^x$.

A graphics calculator can be used to investigate the growth of the investment. If you have difficulty using your calculator, you should consult the **graphics calculator instructions** at the start of the book.

What to do:

1 Enter the function $Y_1 = 1000 \times 1.06^X$ into your graphics calculator.

```
Plot1 Plot2 Plot3
\Y1■1000*1.06^X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

2 Set up a table that will calculate the final value of the investment for each year from Year 0 for at least 30 years.

3 View the table you have created and answer the following questions:

```
  X      Y1
0       1000
1       1060
2       1123.6
3       1191
4       1262.5
5       1338.2
6       1418.5
X=0
```

   a Find the value of the investment after:

      i 1 year   **ii** 2 years

      iii 10 years   **iv** 11 years.

   b Use your answers above to calculate the interest paid in:

      i Year 1   **ii** Year 2

     iii Year 11.

   What do you notice?

4 Change the window settings to show X values between 0 and 30 and Y values between 0 and 6000.
Graph $= 1000 \times 1.06^X$

```
Y1=1000*1.06^X
```

5 If the investment account is closed partway through a year, interest is paid for that fraction of the year. Use your graph to find the value of the investment after:

   a 20 years   **b** 25 years

   c 27.5 years   **d** 29.75 years.

```
X=27.5          Y=4964.909
```

6 We wish to know how many years the investment will take to amount to $3000. Enter the function $Y_2 = 3000$ into your calculator.

```
Plot1 Plot2 Plot3
\Y1■1000*1.06^X
\Y2■3000
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

7 Graph Y_1 and Y_2 on your calculator.
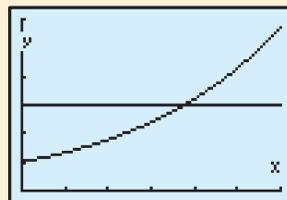
8 Find the point of intersection of the graphs. How many years will it take for the investment to amount to $3000?
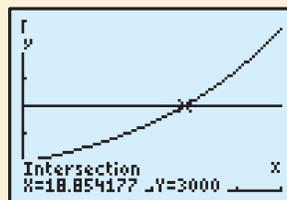
9 Use your graphics calculator and the techniques described above to investigate the following compound interest situations:

a $3000 is invested at 5% p.a. compound interest paid annually.

> **i** What is the investment worth after 5 years?
>
> **ii** What interest is paid in the 6th year?
>
> **iii** How long will the investment take to triple in value?

b Erica invests $2000 at 4% p.a. compound interest paid annually.
Ernie invests $2000 at 8% p.a. compound interest paid annually.

> **i** How long does it take Erica's money to double in value?
>
> **ii** How long does it take Ernie's money to double in value?
>
> **iii** What is each person's investment worth after $15\frac{1}{2}$ years?

10 Find out how to use your calculator's built-in finance program to investigate compound interest scenarios.

D   DEPRECIATION

Electrical equipment, furniture, vehicles and machinery all lose value over time. This may be because they become damaged or worn, or because they become *obsolete*, which means that their technology is no longer the latest and best available.

> **Depreciation** is the process by which goods lose value over time.

If a business owns goods that depreciate in value and are essential for the business to earn an income, the taxation department will usually allow the business to claim the depreciation as a **tax deduction**.

Many items are depreciated by a constant percentage for each year of their useful life. They are said to be depreciated on their **reduced balance**.

The following table shows how a video projector costing $8500 depreciates over 3 years at 30% each year. The depreciated value at the end of each year is commonly called the **book value**.

Age (years)	Depreciation	Book Value
0		$8500.00
1	30% of $8500.00 = $2550.00	$8500.00 − $2550.00 = $5950.00
2	30% of $5950.00 = $1785.00	$5950.00 − $1785.00 = $4165.00
3	30% of $4165.00 = $1249.50	$4165.00 − $1249.50 = $2915.50

The annual depreciation decreases each year as it is calculated on the reduced balance of the item.

In Year 2, $1785 can be claimed as a tax deduction.

EXERCISE 11D.1

1   a   A deep fryer was purchased by a fish and chip shop for £15 000. It depreciated by 15% each year for 3 years. Copy and complete the following table to find the value of the deep fryer at this time.

Age (years)	Depreciation	Book Value
0		£15 000
1	15% of £15 000 = £2250	
2		
3		

b   Calculate how much depreciation can be claimed as a tax deduction by the shop in:
   i   Year 1   **ii**   Year 2   **iii**   Year 3

2   What will be the book value of the following:
   a   a car costing €35 400, depreciating at 25% p.a. over 4 years
   b   a printing machine costing $895 000, depreciating at 8% p.a. over 8 years
   c   a farm machine costing £356 000, depreciating at 15% p.a. over 5 years.

DEPRECIATION FORMULA

We can also use the concept of chain percentage decreases to calculate the value of the video projector after 3 years.

Each year, the video projector is only worth $100\% - 30\% = 70\%$ of its previous value.

$$\therefore \quad \text{the value after 3 years} = \$8500 \times 0.7 \times 0.7 \times 0.7$$
$$= \$8500 \times (0.7)^3$$
$$= \$2915.50$$

When calculating depreciation, the **annual multiplier** is $(1 - i)$ where i is the annual depreciation rate expressed as a decimal.

We can use a formula to calculate the value of an asset that depreciates by a constant percentage each year.

The **depreciation formula** is:

> $F_v = P_v(1 - i)^n$ where F_v is the **future value** after n time periods
> P_v is the **original purchase price** or **present value**
> i is the **depreciation rate per period** as a decimal
> n is the **number** of periods.

Example 17   ◀⦆ **Self Tutor**

A motorcycle was purchased for €12 500 and depreciated at 15% each year.

a Find its value after five years.

b By how much did it depreciate?

a Now $F_v = P_v(1 - i)^n$ where $P_v = 12\,500$, $i = 0.15$, $n = 5$
$$\therefore \quad F_v = 12\,500 \times (1 - 0.15)^5$$
$$= 12\,500 \times (0.85)^5$$
$$\approx 5546.32$$
So, after 5 years the value is €5546.32

b The depreciation $= €12\,500 - €5546.32 = €6953.68$

EXERCISE 11D.2

1 **a** Find the future value of a truck which was purchased for €225 000 if it depreciates at 25% p.a. for 5 years.

   b By how much did it depreciate?

2 **a** If I buy a car for £32 400 and keep it for 3 years, what will its value be at the end of that period given an annual depreciation rate of 20%?

   b By how much did it depreciate?

3 The Taxation Office allows industrial vehicles to be depreciated at $7\frac{1}{2}\%$ every 6 months.

   a What will be the value in 2 years' time of vehicles currently worth €240 000?

   b By how much will they have depreciated?

> If we depreciate every 6 months for 2 years, the number of periods n is 4.

4 A $145 000 luxury car was bought new at the beginning of 2006. Its value depreciates at $22\frac{1}{2}\%$ per annum.

   a What will its value be at the end of 2009?

   b If inflation runs at 5.4% p.a. during this time, what will be the replacement cost for a similar new car then?

   c If the depreciated value is used as a deposit for the new car, what will be the balance to be paid?

INVESTIGATION 2   DEPRECIATION

Consider a $200 000 truck depreciating at a rate of 15% p.a. for x years.

Using the depreciation formula

$$F_v = P_v(1-i)^n \quad \text{we get} \quad F_v = 200\,000 \times (0.85)^x.$$

A **graphics calculator** can be used to investigate the depreciation of the truck. You may need to use the **graphics calculator instructions** at the start of the book.

What to do:

X	Y₁

X	Y1
0	200000
1	170000
2	144500
3	122825
4	104401
5	88741
6	75430

X=0

1 Enter the function $Y_1 = 200\,000 \times 0.85\text{\textasciicircum}X$ into your graphics calculator.

2 Create and view a table that calculates the value of the truck for each year from year 0 for at least 15 years.

3 Find the value of the truck in:

   a Year 0   **b** Year 1   **c** Year 2   **d** Year 6   **e** Year 7   **f** Year 8

4 Use your answers from **3** to calculate the annual depreciation in:

   a Year 1   **b** Year 2   **c** Year 7   **d** Year 8

5 Draw the graph of $Y_1 = 200\,000 \times 0.85\text{\textasciicircum}X$.

6 Use your graphics calculator to find how many years it takes for the truck to diminish in value to:

   a $50 000   **b** $10 000.   **Hint:** In **a**, graph $Y_2 = 50\,000$.

E   BORROWING

At some stage in your life you will probably need to borrow money in order to pay for goods and services.

In this section we will look at three ways of borrowing money including using a credit card, buying on terms, and taking out a personal loan.

CREDIT CARDS

Credit cards are a very popular way to borrow money because of their convenience. However, they charge a higher rate of interest than other borrowing alternatives.

Suppose we use our credit card to obtain a **cash advance**. In this case, interest will be charged from the date the advance is made. There is **no interest free period**.

Interest is calculated daily on these advances and is compounded each month. We assume interest is paid for the period inclusive of end dates. The interest is calculated using the formula

$$\text{interest} = \text{balance} \times \text{interest rate} \times \frac{\text{number of days}}{365}.$$

Example 18   ◄)) Self Tutor

Jeri obtained a $1000 cash advance on her credit card on the 8th January. She received her statement for the period ending on the 20th January, then got another cash advance on the 25th January for $500. How much interest will be charged by her next statement on the 20th February? Assume the interest is charged at 16.9% p.a. and compounds monthly.

The period 8/1 - 20/1 inclusive is 13 days, so the interest charged
$= \$1000 \times 0.169 \times \frac{13}{365} = \$6.02$

The interest is now added to the balance, so Jeri now owes $1006.02

The period 21/1 - 24/1 inclusive is 4 days, so the interest charged
$= \$1006.02 \times 0.169 \times \frac{4}{365}$
$= \$1.86$

This $1.86 is not added until the next statement.

On 25/1 the balance becomes $1006.02 + $500 = $1506.02

The period 25/1 - 20/2 inclusive is 27 days, so the interest charged
$= \$1506.02 \times 0.169 \times \frac{27}{365}$
$= \$18.83$

On 20/2 the interest is added to the balance.

So, the balance is now $= \$1506.02 + \$1.86 + \$18.83$
$= \$1526.71$

∴   the total interest charged $= \$26.71$

EXERCISE 11E.1

In the following questions, assume the initial balance of each credit card is nil.

1 Alma obtains a €5000 cash advance on her credit card on 17th November. Her statement is issued on 27th November. Calculate the interest charged on the statement if the interest rate is 15.5% p.a.

2 Bronte obtains a cash advance of $2000 on his credit card on 7th January. He repays it in full on 20th January. Calculate the interest charged on this advance when his statement for the period ending on 28th January is received. The interest rate charged is 15.8% p.a.

3 Sylvia draws advances on her credit card of £800 on July 20th and £500 on August 12th. Her statement is for the period ending on the 24th of each month and the interest rate is 16.8% p.a. How much will she owe on 24th August?

4 Steve draws cash advances on his credit card as follows:
    9th March - cash advance $1800,   18th March - cash advance $1200
His statement is for the period ending on the 12th of each month and the interest rate is 16.25% p.a. Calculate the total interest charges on these advances up to 12th April.

BUYING ON TERMS

When buying an item on terms, the purchaser pays a **deposit** at the time of the sale, and then the rest of the price at a certain rate over a certain time. The deposit may be cash or else the value of a traded item.

The **regular payments** made to pay off the loan include an interest charge.

Electrical items, cars, and whitegoods such as washing machines and refrigerators are often purchased this way.

Example 19   ◀ﻬ **Self Tutor**

A car is priced at \$23 850. The terms are 20% deposit and \$136.10 per week for 3 years. Calculate:

  a the amount borrowed   **b** the total amount of repayments
  c the total interest paid   **d** the total cost of the car.

  a   Amount borrowed = price − deposit
                = \$23 850 − 20% of \$23 850   {or 80% of \$23 850}
                = \$23 850 − \$4770
                = \$19 080

  b   Total repayments = regular payment × number of payments
                = \$136.10 × 156   {3 years × 52 weeks = 156}
                = \$21 231.60

  c   Interest paid = total repayments − amount borrowed
                = \$21 231.60 − \$19 080
                = \$2151.60

  d   Total cost of car = deposit paid + total repayments
                = \$4770 + \$21 231.60
                = \$26 001.60

> The total cost could also be calculated as price + interest.

EXERCISE 11E.2

1 A washing machine is advertised for €1050 cash or else €100 deposit followed by monthly repayments of €47.50 for 24 months.

  a What will be the total cost of the machine if it is purchased on terms?

  b How much interest will be paid?

2 A customer purchases a lounge suite costing \$5200 by paying a 10% deposit and then \$415 per month for 12 months. Calculate:

  a the total cost of the lounge suite   **b** the amount of interest charged.

3 An electrical appliance store advertises a digital television for £1800. They offer a discount of 5% if the item is paid for in cash up front, or terms of 20% deposit plus fortnightly repayments of $64 for one year. How much more would you pay if you bought the television on terms rather than paying cash?

4 Kara is moving into a flat and wants to buy a new bed. By shopping around she has found that two stores are selling the bed she wants. Their prices and terms are:

Store A: $865 cash or deposit $100 with monthly repayments of $52.95 for 18 months.

Store B: $895 cash or 10% deposit with fortnightly repayments of $36.50 for one year.

Which store should Kara purchase her new bed from? Give reasons for your answer.

5 Khoa wants to purchase a new 4WD. He is offered two deals on the same model:

Dealer A: price $41 880 or 20% deposit with weekly repayments of $275 over 3 years.

Dealer B: price $41 880 or 10% deposit with monthly repayments of $820 over 60 months.

Which dealer should Khoa choose?

PERSONAL LOANS

A common way to borrow money to purchase cars, boats, renovations, overseas holidays, education expenses or share portfolios, is to take out a personal loan. Banks, credit unions and finance companies all offer personal loans with differing terms, conditions, and interest rates.

Interest will be the biggest cost involved in repaying a personal loan.

Interest is calculated on the **reducing balance** of the loan so that the interest reduces as the loan is repaid. The interest each period is added to the balance owed and the repayment is deducted. So,

> **outstanding balance = previous balance + interest − repayment**.

Example 20   ◄⑴ Self Tutor

$2000 is borrowed at a reducing balance interest rate of 12% p.a. Monthly repayments of $400 are to be made. By completing a table, determine:

a the number of repayments to be made

b the final repayment.

An interest rate of 12% p.a. means 1% per month.

Month	Opening balance	Interest (1% per month)	Repayment	Reduction in principal	Closing balance
1	$2000.00	$20.00	$400.00	$380.00	$1620
2	$1620.00	$16.20	$400.00	$383.80	$1236.20
3	$1236.20	$12.36	$400.00	$387.64	$848.56
4	$848.56	$8.49	$400.00	$391.51	$457.05
5	$457.05	$4.57	$400.00	$395.43	$61.62
6	$ 61.62	$0.62	$62.24	$61.62	$0.00

a From the table we can see that there are five *full* monthly repayments plus a final *part* payment.

b The final repayment is $62.24.

EXERCISE 11E.3

1 Set up a table as in **Example 20** to describe a loan of $3000. The interest rate is 9% p.a. compounded monthly under reducing balance conditions, and monthly payments of $600 are to be made. How many payments are required to repay the loan?

2 Erica takes out a loan to buy a car. She borrows €20 000 for 5 years at a rate of 12% p.a. compounded monthly under reducing balance conditions. Determine the closing balance after making monthly repayments of €445 for three months.

3 A loan of £100 000 is taken out for 10 years to buy a share portfolio. Interest is charged at 12% p.a. compounded quarterly under reducing balance conditions. Determine the amount still owing after 1 year if quarterly repayments of £4350 are made.

INVESTIGATION 3   "LOAN SCHEDULES" ON SPREADSHEET

This spreadsheet will allow you to investigate how a **reducing balance loan** is paid off.

Consider Erica's loan in question **2** above. €20 000 was borrowed for 5 years at 12% p.a. with monthly repayments of €445.

What to do:

1 Click on the icon to load the **Reducing balance loans** spreadsheet.

	A	B	C	D	E
1	Reducing balance loans				
2					
3	Loan amount	20,000		SPREADSHEET	
4	Rate p.a.	12%			
5	Rate per month	=B4/12			
6	Repayment	445			
7					
8	Month	Principal	Interest	Repayment	Balance
9	0	–	–	–	=B3
10	=A9+1	=E9	=B10*B$5	=B$6	=B10+C10-D10
11	↓	↓	↓	↓	↓
12			fill down		

2 Fill row 10 down to the 60th month. Answer the following questions:

  a What is the balance after 60 months?

  b How much interest is paid in the: **i** 1st month   **ii** 60th month? Explain the difference between these answers.

3 Suppose Erica gets a pay rise, so she can now pay off more than €445 each month. When will the loan be paid off if she pays:

  a €600 per month   **b** €890 per month?

Hint: Enter the repayment in B6 and scroll down until the balance is zero. Does doubling the repayment halve the repayment time?

4 Ricky takes out a loan of £23 000 over 5 years at 13.5% p.a. Enter these details on your spreadsheet and fill down for 60 months. By trial and error, calculate the monthly repayment needed to completely pay off the loan in this time.

5 Joanne and Pavel take out a $150 000 loan to buy a house. The bank charges them 8.5% p.a. and sets their repayments at $1207.90 monthly over 25 years. Enter these details on your spreadsheet. Fill down for 300 months.

   a What is the balance after 300 months? Discuss this.

   b Calculate the total interest paid by finding the sum of column C.

   c Joanne and Pavel think they will be able to repay an extra $50 per month. If they can do this, when will their loan be paid off?
     Hint: Enter $1257.90 in B6 and scroll down.

   d They want to have their house paid off within 7 years. By trial and error, find the monthly repayment that will enable them to do this.
     Hint: Change the monthly repayment until the balance in the 84th month is zero.

LOAN REPAYMENT TABLES

When you apply for a personal loan, you will be advised of the amount you will need to repay each fortnight or each month. This amount is dependent on the loan amount, the duration of the loan, and the interest rate charged.

> The resulting monthly instalments are rounded off to the next 10 cents. For example, $485.51 becomes $485.60

Banks have **tables of repayments** for various loans. **Loan calculators** are available on bank websites. Look up the internet addresses of some major financial institutions and investigate these.

The following **table of monthly repayments** is based on **borrowing 1000 units of currency**.

Table of Monthly Repayments per 1000 units of currency							
Loan term	*Annual interest rate*						
(months)	10.0%	10.5%	11.0%	11.5%	12.0%	12.5%	13.0%
12	87.9159	88.1486	88.6151	88.8488	89.0829	89.3173	89.3173
18	60.0571	60.2876	60.5185	60.7500	60.9820	61.2146	61.4476
24	46.1449	46.3760	46.6078	46.8403	47.0735	47.3073	47.5418
30	37.8114	38.0443	38.2781	38.5127	38.7481	38.9844	39.2215
36	32.2672	32.5024	32.7387	32.9760	33.2143	33.4536	33.6940
42	28.3168	28.5547	28.7939	29.0342	29.2756	29.5183	29.7621
48	25.3626	25.6034	25.8455	26.0890	26.3338	26.5800	26.8275
54	23.0724	23.0724	23.5615	23.8083	24.0566	24.3064	24.5577
60	21.2470	21.4939	21.7424	21.9926	22.2444	22.4979	22.7531

Example 21   ◆) **Self Tutor**

Erin takes out a personal loan for $16 500 to buy a car. She negotiates a term of 4 years at 11.5% p.a. interest. Calculate the:

a monthly repayments   b total repayments   c interest charged.

a From the table, the monthly repayments on each $1000 for 4 years (48 months) at 11.5% p.a. = $26.0890

$\therefore$ the repayments on $16 500 = 26.0890×16.5 {16.5 lots of $1000}
$= \$430.4685$
$\approx \$430.50$ {to next 10 cents}

The monthly repayments are $430.50.

b Total repayments = monthly repayment $\times$ number of months
$= \$430.50 \times 48$
$= \$20\,664$

So, $20 664 is repaid in total.

c Interest = total repayments − amount borrowed
$= \$20\,664 - \$16\,500$
$= \$4164$

The total interest charged is $4164.

EXERCISE 11E.4

1 Ron takes out a personal loan for €15 000 to go overseas. He will repay it over 5 years at 11.5% p.a. Calculate the:

a monthly repayments   b total repayments   c interest charged.

2 Jai and Tatiana need £18 000 to fund house renovations. They take out a personal loan over 3 years at 13% p.a. Calculate the:

a monthly repayments   b total repayments   c interest charged.

3 Gunter needs €28 000 to buy a boat. His bank offers him a personal loan at 12.5% p.a. Calculate the total interest he will pay if he repays it over:

a 2 years   b 5 years.

Comment on your answers.
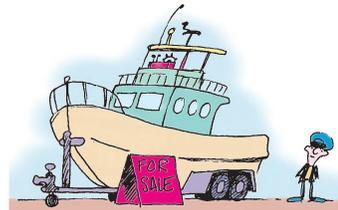
4 Trina wants to borrow $45 000 to purchase some shares. Calculate the total interest charged for the following options:

a Balance Bank   offer 11.5% over 5 years

b Cash Credit Union   offer 12.5% over 3 years.

Which option would you recommend for Trina?

PAYING OFF A MORTGAGE

LINKS
click here

Areas of interaction:
Health and social education

REVIEW SET 11A

1 **a** Decrease 140 kg by 20%.

  b Determine the selling price of an article marked at $60 and discounted $12\frac{1}{2}$%.

  c A share trader buys shares for €1100 and sells them for €1240. What percentage profit does she make?

2 I bought a house for £120 000. Over the next four years it increased in value by 9%, decreased by 3%, increased by 7%, and finally increased by 12%. What is the value of the house after these four years?

3 Sally has $12 000 to invest for 10 years. She is offered an investment which pays 5% p.a. compounded annually.

  a What will the investment amount to after this period?

  b How much interest will she earn?

4 A new caravan costing £15 000 depreciates at a rate of 16% p.a. What will be its value in 5 years?

5 A €2000 loan has an interest rate of 18% p.a. and is reducible monthly. If €600 is repaid each month, draw up a loan schedule to find:

  a the number of payments necessary   **b** the size of the last payment.

6 Terri wants to buy a car priced at €24 990. She does not have the full amount saved and is therefore offered the option of paying a 40% deposit and €75.50 per week over 5 years. If she takes up this option, determine:

  a the deposit she pays

  b the total amount she pays for the vehicle

  c the amount of interest she pays.

7 Mizumi takes out a personal loan for ¥600 000 to go on a holiday. Her bank offers her a rate of 11% p.a. over 3 years on a reducing balance basis. Calculate her:

  a monthly repayments using the table of monthly repayments on page **263**.

  b total repayments   **c** interest charged.

REVIEW SET 11B

1 **a** Increase €3725 by $12\frac{1}{2}$%.

  b An item of clothing is marked at $49 but sold in a 15% discount sale. Find the sale price.

c The value of a rare stamp increases by 70% one year, but the next year it decreases in value by 35%. What is its overall percentage change in value?

2 A lounge suite was bought for a basic cost of $1780. It was marked up for a 35% profit, and then 10% GST was added on. Determine the final selling price of the lounge suite.

3 Business equipment purchased for £325 000 depreciated at $17\frac{1}{2}\%$ p.a. Calculate the value of the equipment when the company closed down five years after the purchase.

4 A bank offers an investment package to its customers of 7.4% p.a. with interest paid annually, provided no money is withdrawn during the investment period. What would be the maturing value of an investment of $130 000 for 5 years?

5 Mizumi obtained the ¥600 000 necessary for her holiday by getting a cash advance on her credit card. Interest is charged on this advance at 16.5% p.a.. Her parents paid off the loan 20 days later. How much interest will Mizumi need to pay if her card began with a nil balance?

6 A softball bat was discounted 20% and sold for $120 including 15% VAT.
   a How much VAT was paid by the customer?
   b What was the original marked price including VAT?

7 A company car is purchased for €46 500. Depreciation is calculated at 17.5% of its value each year. Find:
   a the value of the car after four years
   b the amount by which the car has depreciated after four years.

8 A 45 000 rupee car is purchased using a deposit of 15 000 rupees. The balance is repaid over 5 years at $10\frac{1}{2}\%$ per annum on a reducing balance basis.

Calculate:
   a the amount borrowed
   b the size of each monthly repayment
   c the total to be repaid
   d the amount of interest paid.

Table of Monthly Repayments per 1000 rupee

Loan term	Annual interest rate		
(months)	10.0%	10.5%	11.0%
12	87.9159	88.1486	88.6151
18	60.0571	60.2876	60.5185
24	46.1449	46.3760	46.6078
30	37.8114	38.0443	38.2781
36	32.2672	32.5024	32.7387
42	28.3168	28.5547	28.7939
48	25.3626	25.6034	25.8455
54	23.0724	23.0724	23.5615
60	21.2470	21.4939	21.7424

Chapter 12

Trigonometry

Contents:

OPENING PROBLEM

Nindi's office building has a satellite dish on the roof.   It is angled at $41°$ to horizontal. Nindi is concerned that a new building across the road may obstruct the path of the satellite connection. The new building is 26 m higher than the satellite dish, and is 30 m from it. Can you answer Nindi's concern?
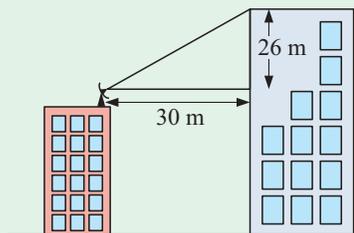
Problems like the **Opening Problem** can be solved by either a **scale diagram** or by right angled triangle **trigonometry**.

> **Trigonometry** is the study of the relationship between lengths and angles of geometrical figures.

Trigonometry uses other branches of mathematics such as algebra, arithmetic and geometry to find unknown lengths and angles of triangles.

A   USING SCALE DIAGRAMS

To represent a physical situation we can often draw a **scale diagram**. All lengths on the diagram are *in proportion* to those in reality according to the diagram's **scale**. The angles in a scale diagram will equal the corresponding angles in reality.

Scale diagrams can be used to find the lengths of sides and angles of geometrical figures.

Example 1   ◀⑦ Self Tutor

From the top of an embankment, Fiona measured the angle between the horizontal ground and the top of a tower to be $47°$. The embankment is 35.4 m from the centre of the base of the tower. How high is the tower?
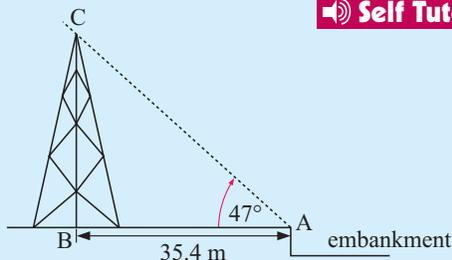
First we choose a suitable *scale*, in this case
$1 \text{ cm} \equiv 10 \text{ m}$ or $1 \text{ mm} \equiv 1 \text{ m}$.

We draw a horizontal line segment [BA] which is 35.4 mm. At the left end we draw a vertical line using a set square.

We use a protractor to draw a $47°$ angle at A. The point where the two lines meet is C, which is the top of the tower.

We measure BC in mm and use the scale to convert back to metres. $BC \approx 38$ mm.
∴   the tower is approximately 38 metres high.
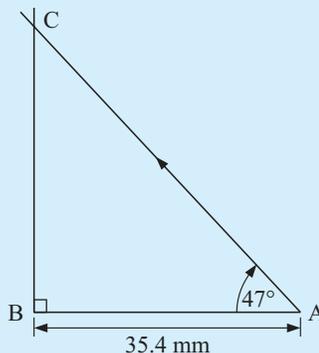
DISCUSSION

What factors might cause errors when using scale diagrams?

Comment on the accuracy of answers obtained using scale diagrams.

EXERCISE 12A

1 Convert this rough sketch into an accurate scale diagram using a scale of 1 cm ≡ 1 m.

Use the scale diagram to find as accurately as you can the length of:

    a [QR]   **b** [PR].
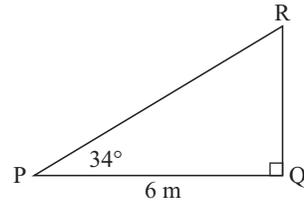
2
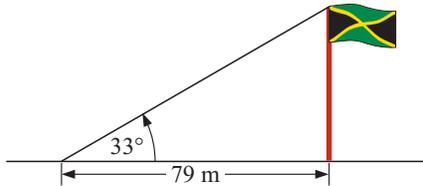
Use a scale diagram to find the height of the flagpole.

3 The triangular garden XYZ has XY = 12 m, YZ = 10.2 m, and XZ = 8.4 m. Use a compass and ruler to draw an accurate scale diagram of the garden and hence find the measures of the garden's angles.

4 A mountain rises in the distance from a horizontal plane. A surveyor makes two angle measurements from the plane to the top of the mountain. The results are shown in the diagram alongside. Use a scale diagram to find the height of the mountain.
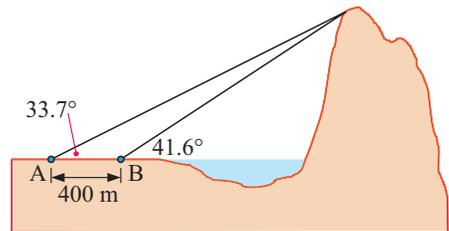
5 Try to solve the **Opening Problem** on page **268** using a scale diagram. Do you think a scale diagram is sufficient to answer Nindi's concerns in this case?

B   LABELLING TRIANGLES

Trigonometry enables us to find lengths and angles to greater accuracy than is possible using scale diagrams.

Loosely translated, *trigonometry* means *triangle measure*.

In this section we will consider only **right angled triangles**.

The **hypotenuse** is the longest side of a right angled triangle, and is opposite the right angle.
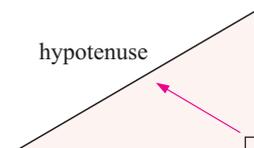
For a given angle θ in a triangle, the **opposite** side is opposite the angle θ.

The third side is next to the angle θ and so is called the **adjacent** side.

For example:
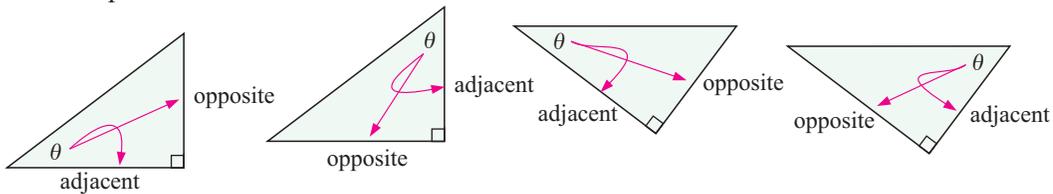
Example 2	◀) Self Tutor

In the diagram alongside, find the:

a hypotenuse

b side opposite angle A

c side adjacent to angle A

d side opposite angle C

e side adjacent to angle C.
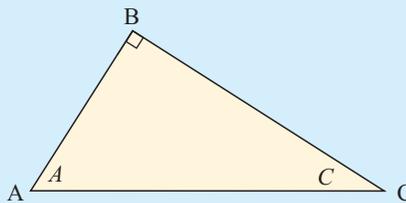
a The hypotenuse is [AC].   **b** [BC] is the side opposite angle A.

c [AB] is the side adjacent to angle A.   **d** [AB] is the side opposite angle C.

e [BC] is the side adjacent to angle C.

EXERCISE 12B

1 In the diagrams below, name the:

    i hypotenuse   **ii** side opposite angle θ

    iii side adjacent to angle θ

a **b** **c**
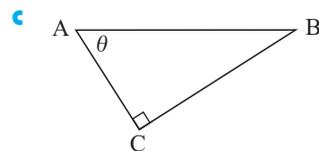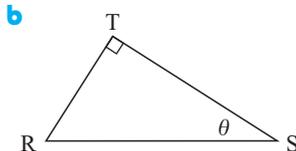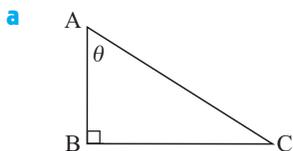
2 The right angled triangle alongside has hypotenuse of length a units and other sides of length b units and c units. θ and ϕ are the two acute angles. Find the length of the side:

    a opposite θ **b** opposite ϕ

    c adjacent to θ **d** adjacent to ϕ
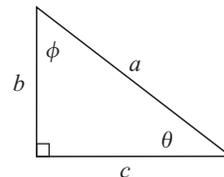
C	THE TRIGONOMETRIC RATIOS

Consider the right angled triangle opposite. We use:

  HYP to indicate the length of the hypotenuse

  OPP to indicate the length of the side opposite θ

  ADJ to indicate the length of the side adjacent to θ.
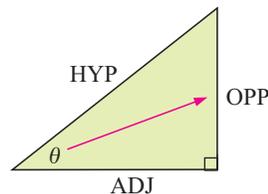
We use these abbreviations to define the three basic **trigonometric ratios**:

> The **sine** of angle θ is $\sin \theta = \dfrac{\text{OPP}}{\text{HYP}}$.
>
> The **cosine** of angle θ is $\cos \theta = \dfrac{\text{ADJ}}{\text{HYP}}$.
>
> The **tangent** of angle θ is $\tan \theta = \dfrac{\text{OPP}}{\text{ADJ}}$.

These ratios are the tools we use for finding the lengths of unknown sides and angles of right angled triangles.

FINDING TRIGONOMETRIC RATIOS

Example 3 ◀)) **Self Tutor**

For the triangle shown, find:

a $\sin \theta$ **b** $\cos \phi$ **c** $\tan \theta$
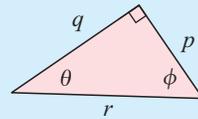
a $\sin \theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{p}{r}$ **b** $\cos \phi = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{p}{r}$ **c** $\tan \theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{p}{q}$

EXERCISE 12C.1

1 For each of the following triangles find:

   i $\sin \theta$ **ii** $\cos \theta$ **iii** $\tan \theta$ **iv** $\sin \phi$ **v** $\cos \phi$ **vi** $\tan \phi$

a

b

c

d

e

f
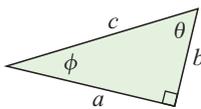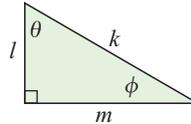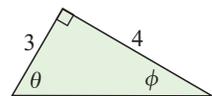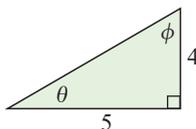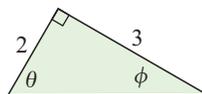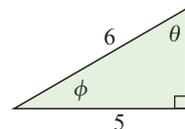
FINDING SIDE LENGTHS

In a right angled triangle, if we are given another angle and a side length we can find:

- the third angle using the 'angle sum of a triangle is 180°'
- the other side lengths using trigonometry.

The method:

Step 1: Redraw the figure and mark on it HYP, OPP, and ADJ relative to the given angle.

Step 2: For the given angle choose the correct trigonometric ratio which can be used to set up an equation.

Step 3: Set up the equation.

Step 4: Solve to find the unknown.

When we solve equations involving trigonometry, we evaluate the trigonometric ratios for angles using our calculator.

Example 4 ◀) Self Tutor

Find, to 2 decimal places, the unknown length in the following triangles:

a

58°   10 cm

x cm

b

8 m   x m

39°

a

58°   10 cm
ADJ   HYP

x cm
OPP
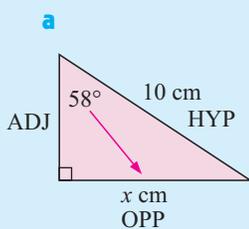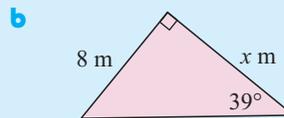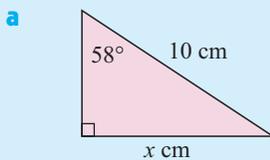
$$\sin 58^o = \frac{x}{10} \qquad \left\{\sin \theta = \frac{\text{OPP}}{\text{HYP}}\right\}$$

$$\therefore \quad \sin 58^o \times 10 = x \qquad \{\text{multiplying both sides by } 10\}$$

$$\therefore \quad x \approx 8.48 \qquad \{\boxed{\text{sin}}\ 58\ \boxed{)}\ \boxed{\times}\ 10\ \boxed{\text{ENTER}}\}$$

The length of the side is about 8.48 cm.

b

OPP   ADJ
8 m   x m
      39°
     HYP
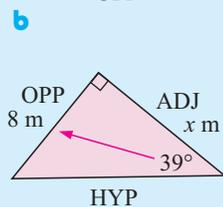
$$\tan 39^o = \frac{8}{x} \qquad \left\{\tan \theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$$

$$\therefore \quad x \times \tan 39^o = 8 \qquad \{\text{multiplying both sides by } x\}$$

$$\therefore \quad x = \frac{8}{\tan 39^o} \qquad \{\text{dividing both sides by } \tan 39^o\}$$

$$\therefore \quad x \approx 9.88 \qquad \{8\ \boxed{\div}\ \boxed{\text{tan}}\ 39\ \boxed{)}\ \boxed{\text{ENTER}}\}$$

The length of the side is about 9.88 m.

EXERCISE 12C.2

1 Set up a trigonometric equation connecting the angle and the sides given:

a

x

65°   a

b

x

32°

b

c

56°

c

x

d

d

37°

x

e

49°

e   x

f

f

x

73°

g

h

i
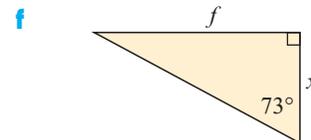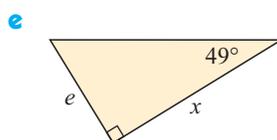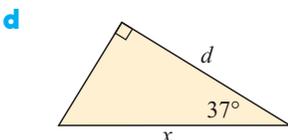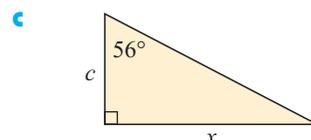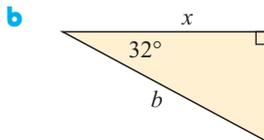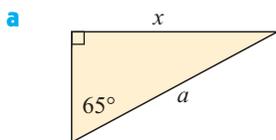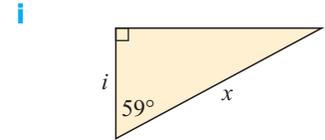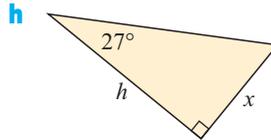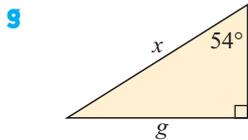
2 Find, to 2 decimal places, the unknown length in:

a

b

c

d

e

f

g

h

i

j

k

l
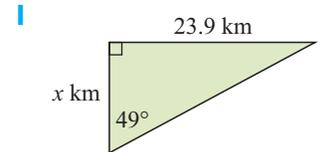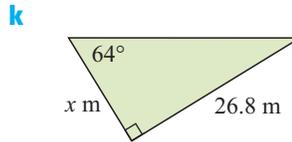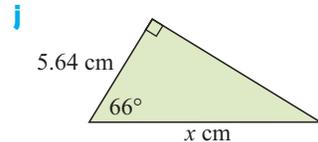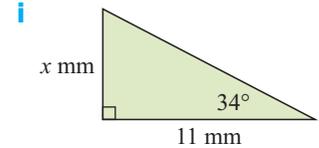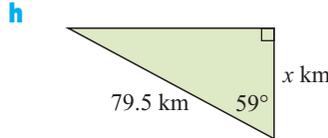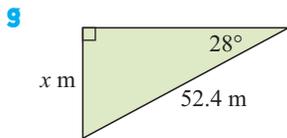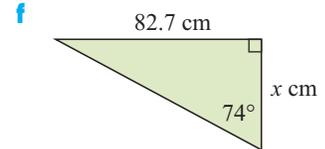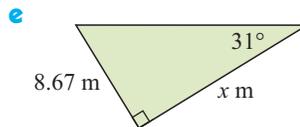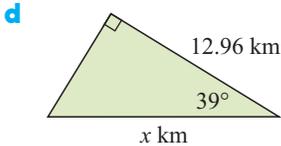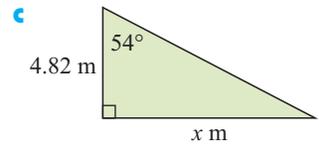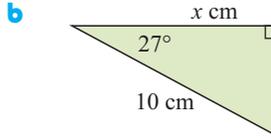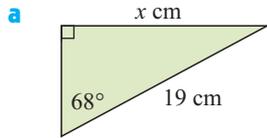
3 Find, to one decimal place, *all* unknown angles and sides of:

a

b

c
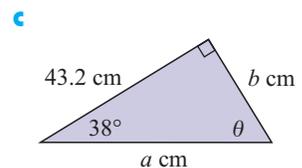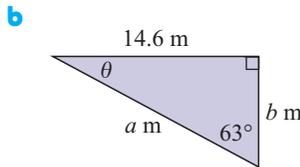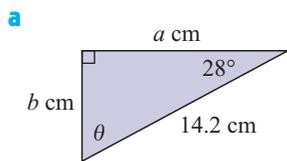
FINDING ANGLES

In the right angled triangle shown, $\cos \theta = \frac{3}{5}$.

So, to find θ we need an angle with a cosine of $\frac{3}{5}$.

If $\cos^{-1}(\ldots)$ reads   "the angle with a cosine of ......",
we can write $\theta = \cos^{-1}\left(\frac{3}{5}\right)$.
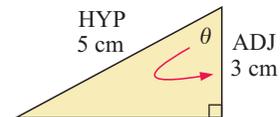
We could also say that "θ is the *inverse cosine* of $\frac{3}{5}$ ".

In a similar way we can define the inverse sine and inverse tangent of an angle. These inverses can also be found on a calculator. For graphics calculator instructions, see page **14**.

To find θ in this case, press: 2nd cos 3 ÷ 5) ENTER . The answer is $\theta \approx 53.1^{o}$.

With an ordinary scientific calculator you may need to press INV , 2nd F   or SHIFT
and then press cos (3 ÷ 5) = .

Example 5   ◀) **Self Tutor**

Find, to one decimal place, the measure of the angle marked θ in:

a

5 cm

θ

8 cm

b

3.17 m

5.88 m

θ

---

a

HYP

OPP
5 cm

θ

8 cm
ADJ

$\tan \theta = \frac{5}{8}$ $\left\{ \tan \theta = \dfrac{\text{OPP}}{\text{ADJ}} \right\}$

$\therefore \quad \theta = \tan^{-1} \left(\frac{5}{8} \right)$
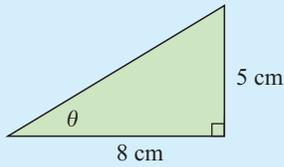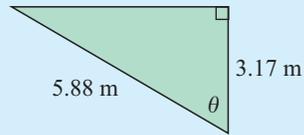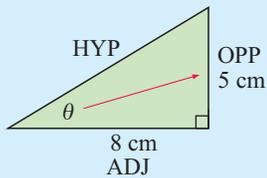
$\therefore \quad \theta \approx 32.0^o$ $\{$ 2nd tan 5 ÷ 8) ENTER $\}$

So, the angle measure is about 32.0^o .

b

3.17 m
ADJ

5.88 m

HYP θ
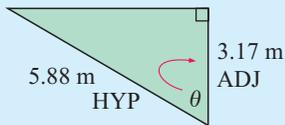
$\cos \theta = \frac{3.17}{5.88}$ $\left\{ \text{as } \cos \theta = \dfrac{\text{ADJ}}{\text{HYP}} \right\}$

$\therefore \quad \theta = \cos^{-1} \left(\frac{3.17}{5.88} \right)$

$\therefore \quad \theta \approx 57.4^o$ $\{$ 2nd cos 3.17 ÷ 5.88) ENTER $\}$

So, the angle measure is about 57.4^o.

EXERCISE 12C.3

1 Find, to one decimal place, the measure of the angle marked θ in:

a

5 cm

4 cm

θ

b

θ

6 cm

4 cm

c

4.8 cm

3.7 cm

θ

d

3.2 m

θ

5.2 m

e

θ

3.1 km

2.1 km

f

3.1 m

4.2 m

θ

g

5.1 km

θ

4.1 km

h

4.4 m

6.1 m

θ

i

θ

7.5 mm

12.5 mm
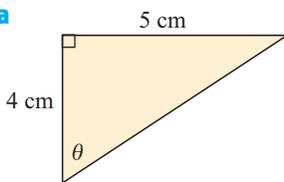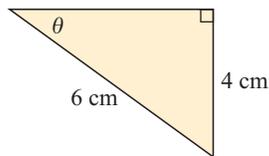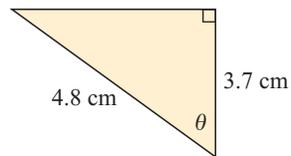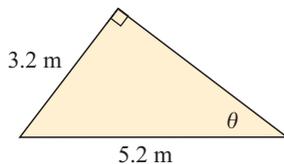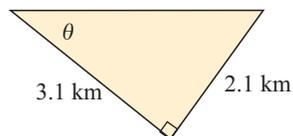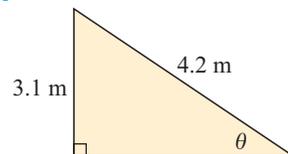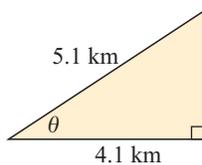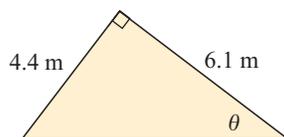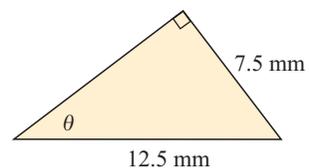
2 Find, to 1 decimal place, all unknown sides and angles in the following:

a

x cm

θ

5 cm

8 cm

ϕ

b

α

x m

3.5 m

β

6.1 m

c

9.45 km

x km

a

b

12.62 km
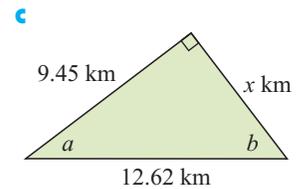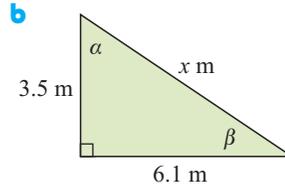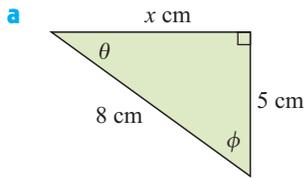
3 Check your answers for x in question **2** using Pythagoras' theorem.

4 Find θ in the following using trigonometry. What conclusions can you draw?

a

8.12 m

6.45 m

θ

b

17.9 mm

θ

15.6 mm

c

11.7 km

θ

11.7 km
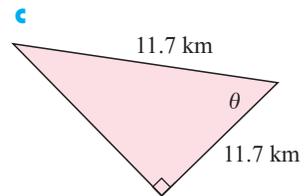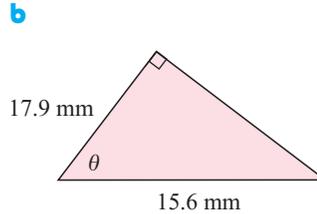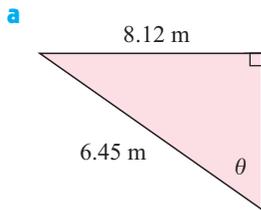
D   TRIGONOMETRIC PROBLEM SOLVING

The trigonometric ratios can be used to solve a wide variety of problems involving right angled triangles. When solving these problems it is important to follow the steps below:

Step 1:	**Read** the question carefully.
Step 2:	Draw a **diagram**, not necessarily to scale, with the given information clearly marked.
Step 3:	If necessary, **label** the vertices of triangles in the figure.
Step 4:	State clearly any assumptions you make which will enable you to use **right angled triangles** or properties of other geometric figures.
Step 5:	Choose an appropriate **trigonometric ratio** and substitute into a trigonometric equation connecting the quantities. On some occasions more than one equation may be needed.
Step 6:	**Solve** the equation(s) to find the unknown.
Step 7:	Answer the question in a **sentence**.

ANGLES OF ELEVATION AND DEPRESSION

Suppose you are standing at the top of a cliff. As you look out you can see an aeroplane in the distance, high overhead. The angle between horizontal and your line of sight to the aeroplane is called the **angle of elevation**.

Looking down, there is a boat out to sea. The angle between the horizontal and your line of sight to the boat is called the **angle of depression**.

angle of elevation

angle of depression
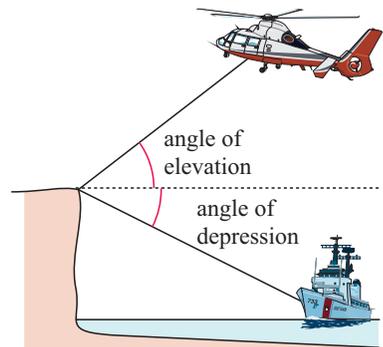
If the angle of elevation from A to B is θ, then the angle of depression from B to A is also θ.

For example:

When using trigonometry to solve problems we often use:

- true bearings
- angles of elevation and depression
- the properties of isosceles and equilateral triangles
- the properties of circles and tangents.
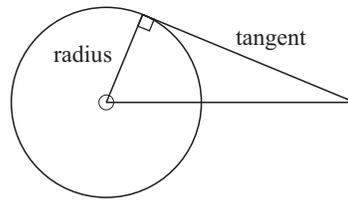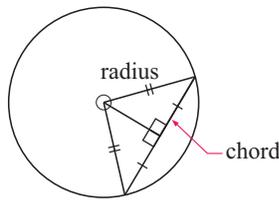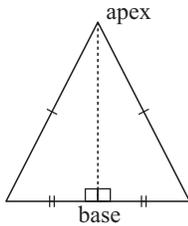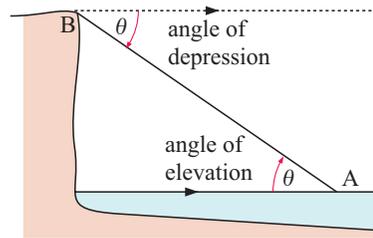
Example 6   ◀)) Self Tutor
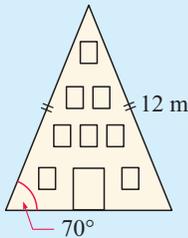
An A-frame house has the shape of an isosceles triangle with base angles of $70°$. The oblique walls of the building are 12 m long.

How wide is the building at ground level?
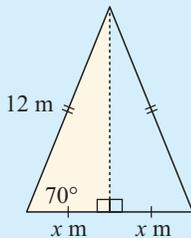
$$\cos 70° = \frac{x}{12} \qquad \{ \cos \theta = \frac{\text{ADJ}}{\text{HYP}} \}$$

$$\therefore \quad 12 \times \cos 70° = x$$

$$\therefore \quad x \approx 4.1042$$

$$\therefore \quad 2x \approx 8.21$$

At ground level, the building is about 8.21 m wide.

Example 7   ◀)) Self Tutor

A staircase 3.6 m in length is needed to access a platform which is 2.9 m above ground level. Find:

a the angle the staircase makes with the ground

b the distance from the foot of the staircase to the post supporting the platform.
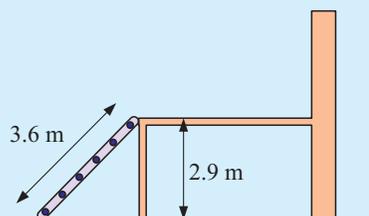
a
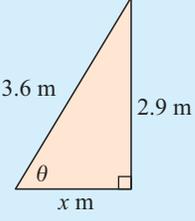
$$\sin \theta = \frac{\text{OPP}}{\text{HYP}} = \frac{2.9}{3.6}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{2.9}{3.6}\right)$$

$$\therefore \quad \theta \approx 53.7^o$$

So, the staircase makes an angle of about 53.7^o with the ground.

b

$$\cos \theta = \frac{\text{ADJ}}{\text{HYP}}$$

$$\therefore \quad \cos 53.7^o = \frac{x}{3.6}$$

$$\therefore \quad 3.6 \times \cos 53.7^o = x$$

$$\therefore \quad 2.13 \approx x$$

So, the foot of the staircase is about 2.13 m from the post.

EXERCISE 12D

In this exercise, round your answers correct to 3 significant figures.
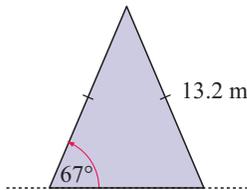
1   An A-frame house has the shape of an isosceles triangle with base angles 67°. The oblique walls are 13.2 m long. Find:

  a how wide the building is at ground level

  b the height of the apex above the ground.

2 For the triangular roof truss illustrated, find:

  a the length of a rafter if the beam is 13.8 m and the pitch is 20°

  b the pitch of the roof if the rafter is 8.85 m long and the beam is 13.2 m long.
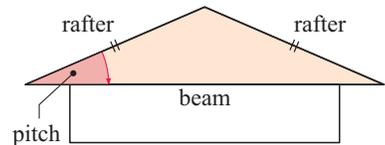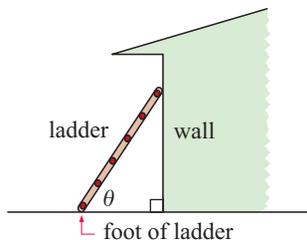
3   Consider a ladder leaning against a vertical wall. Find:

  a how far up the wall a ladder of length 4.2 m reaches if its angle θ with the ground is 70°

  b the length of a ladder if it makes an angle of 67° with the ground and its feet are 0.9 m from the wall

  c the angle between the ground and a 4.8 m long ladder that reaches 4.1 m up the wall.

4 The lamp in a lighthouse is 64 m above sea level. The angle of depression from the light to a fishing boat is 11°. How far horizontally is the boat from the lighthouse?

5 Find the height of a vertical cliff if the angle of elevation from a point 318 m from the base of the cliff is 23°.

6 A ramp for wheel-chair access is illustrated. Find:

  a the length needed to rise 0.66 m at an angle of 8°

  b its angle if the ramp rises 1.1 m over a length of 10.8 m.
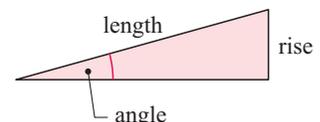
7 A cyclist in France rides up a long incline with an average rise of 6°. If he rides for 6.2 km, how far has he climbed vertically?

8
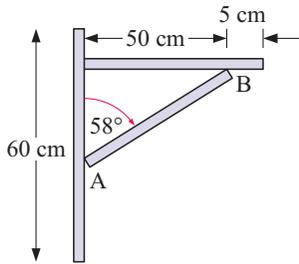
A bracket is made from flat steel to the dimensions shown.

  a Find the length of the oblique strut [AB].

  b Find the length of metal needed to make one bracket.

  c If the metal costs $5.60 per metre, find the cost of the metal needed to make 200 brackets.

9 From horizontal ground, the angle of elevation from point B to the top of a tower is 32°. Given that B is 586 m from A at the base of the tower, find the tower's height.
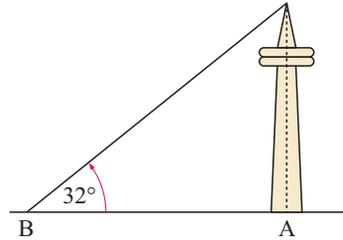
10
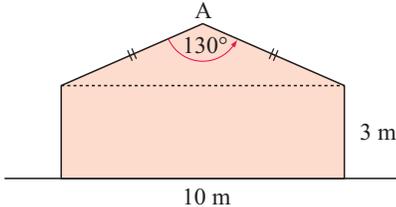
Illustrated is the end wall of a garage. Find:

  a how high A is above ground level

  b the area of this wall.

11 A housing block is 42 m by 18 m. Find:

  a θ correct to 4 significant figures

  b the length of [AC] using trigonometry

  c the length of [AC] using Pythagoras' theorem.
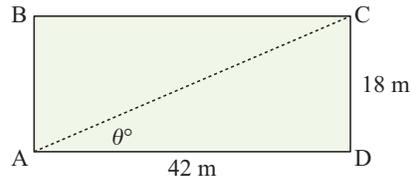
12 A circle has radius 8 cm and a chord [AB] of length 13 cm. If the circle's centre is O, find the measure of $A\widehat{O}B$.

13 The angle between the tangent from point P to a circle and the line from P to the centre of the circle is 23°. Determine the length of the line from P to the centre of the circle if the radius is 4 cm.

14 A tangent from a point X to a circle of radius 5 cm is 12 cm long. Find:

  a the distance of X from the centre of the circle

  b the size of the angle between the tangent and the line joining X to the centre of the circle.

15 A rhombus has sides of length 12 cm. The angle between two adjacent sides is 68°. Find the length of the shorter diagonal of the rhombus.

16 A flag pole casts a shadow of length 12.8 m when the sun is at an elevation of 68°. How tall is the flag pole?

17 Chord [AB] is drawn in a circle with centre O.
Find:

  a the length of the radius if chord [AB] is 8 cm long and makes an angle of 87° at the centre O

  b the length of the chord if the radius is 10 cm and angle $\widehat{AOB}$ measures 108°.
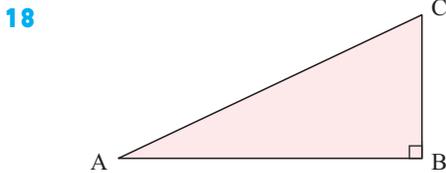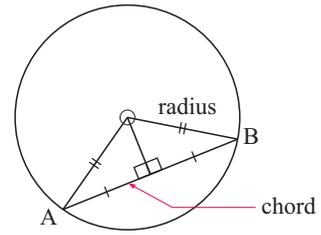
18 The hypotenuse [AC] of triangle ABC is three times longer than side [BC]. Find the measure of $\widehat{BCA}$.

19 An aeroplane takes off at an angle of 18°. Its average speed in the first 20 seconds of flight is 240 km h^{-1}. What is the altitude of the plane at the end of this time?

20 An observer notices that an aeroplane flies directly overhead. Two minutes later the aeroplane is at an angle of elevation of 25°. If the speed of the aeroplane is 350 m s^{-1}, find the altitude of the aeroplane.

21 Answer the **Opening Problem** on page **268**.

E   BEARINGS

COMPASS BEARINGS
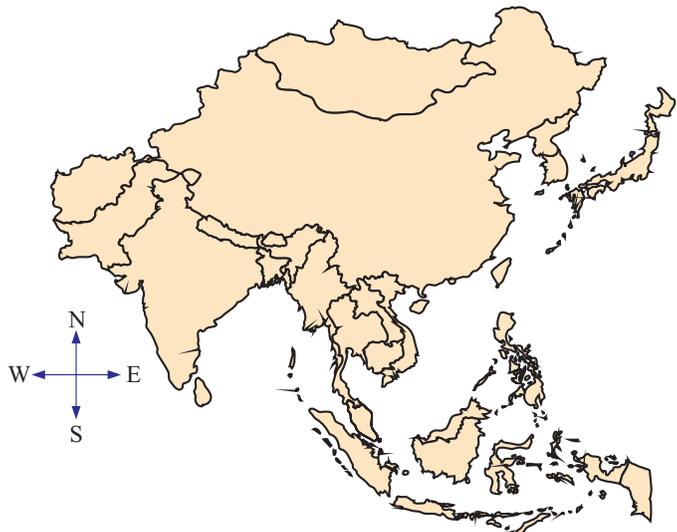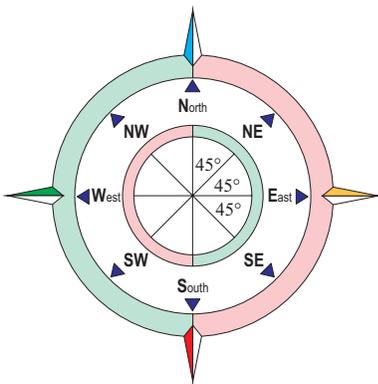
TRUE BEARINGS

We can measure a direction by comparing it with the **true north direction**. We call this a **true bearing**. Measurements are always taken in the **clockwise** direction.

Imagine you are standing at point A, facing north. You turn **clockwise** through an angle until you face B. The **bearing of B from A** is the angle through which you have turned.

So, the bearing of B from A is the clockwise measure of the angle between [AB] and the 'north' line through A.

In the diagram alongside, the bearing of B from A is 120° from true north. We write this as 120^oT or 120°.

To find the true **bearing of A from B**, we place ourselves at point B and face north. We then measure the clockwise angle through which we have to turn so that we face A. The true bearing of A from B is 300°.
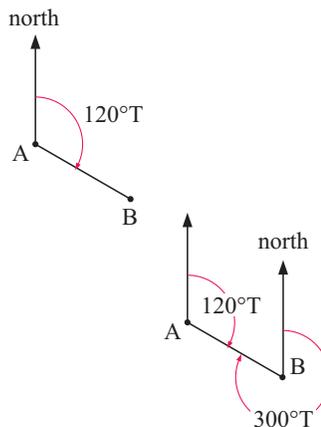
Note:
- A true bearing is always written using three digits. For example, we write 070° rather than 70°.
- The bearings of A from B and B from A always differ by 180°.
  You should be able to explain this using angle pair properties for parallel lines.

True north lines are parallel, so we can use angle pair properties to find unknown angles in bearing problems.

EXERCISE 12E

1 Draw diagrams to represent bearings from O of:

    a 055° **b** 140° **c** 330° **d** 255°

2 Find the bearing of Q from P if the bearing of P from Q is:

    a 124° **b** 068° **c** 244° **d** 321°

3 A, B and C are checkpoints in an orienteering course. For each of the following, find the bearing of:

    i B from A   **ii** C from B   **iii** B from C
    iv C from A   **v** A from B   **vi** A from C.

a

b
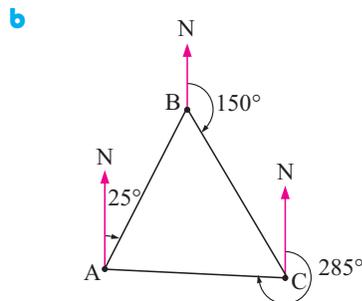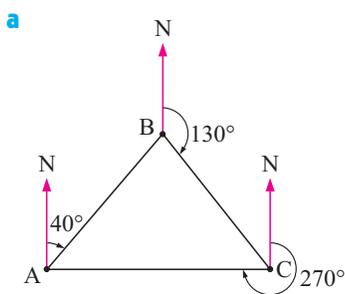
Example 8	◀)) Self Tutor

A cyclist rides 21.3 km due west and then 13.8 km due north. Find, to the nearest degree, the bearing of the finishing point from the starting point.
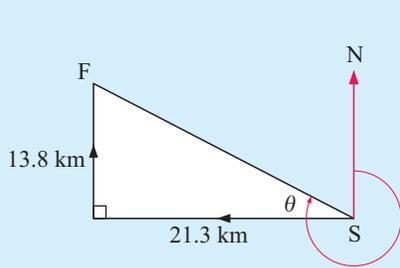
$$\tan \theta = \frac{13.8}{21.3}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{13.8}{21.3}\right) \approx 32.9^o$$

$$\therefore \quad \theta \approx 33^o$$

and so $270^o + \theta \approx 303^o$

So, the bearing of F from S is 303°.

4   A hiker walks 900 m east and then 500 m south. Find the bearing of his finishing position from his starting point.

5   Two runners meet at an intersection, then leave it at the same time. Runner P runs at 12 km h^{-1} due north, while runner Q runs at 14 km h^{-1} due east. Find the distance and bearing of runner Q from runner P after 30 minutes.

6   A helicopter pilot flies in the direction 147° and lands when she is 12 km south of her starting point. How far did she fly?

7   A ship sails for 180 km on the bearing 058°. How far is the ship north of its starting point?

8   An aeroplane travels on the bearing 315° until it is 650 km west of its starting point. How far has it travelled on this bearing?

Example 9   ◀ **Self Tutor**

Carmen departs from point A and walks in the direction 055° for 8.91 km to X. She then changes direction and walks for 7.29 km on the course 145° to point B.

a   Draw a fully labelled sketch of the situation.
b   Find the distance of B from A, correct to 4 significant figures.
c   Find the bearing of B from A.
d   In what direction must Carmen travel to return from B to A?

a

b $\widehat{AXB} = 145^o - 55^o = 90^o$
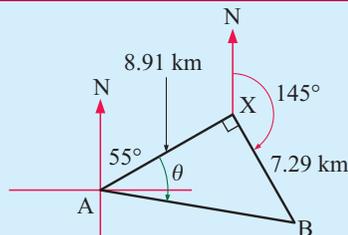
$$\therefore \quad AB^2 = 8.91^2 + 7.29^2$$
$$\{\text{Pythagoras}\}$$
$$\therefore \quad AB = \sqrt{8.91^2 + 7.29^2}$$
$$\therefore \quad AB \approx 11.51 \text{ km}$$

c $\tan \theta = \dfrac{7.29}{8.91}$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{7.29}{8.91}\right)$$

$$\therefore \quad \theta \approx 39.3^o$$

$\therefore$ the bearing of B from A is
$55^o + 39.3^o = 094.3^o$

d $94.3^o + 180^o = 274.3^o$

$\therefore$ the bearing of A from B is
274.3^o.

Carmen must travel in this direction.

9 A cyclist departs point R and rides on a straight road for 2.3 km in the direction 197°. She then changes direction and rides for 1.8 km in the direction 107° to point S.

   a Draw a fully labelled sketch of this situation.

   b Find the distance between R and S.

   c Find the bearing from R to S.

10 A fishing trawler sails from a port in the direction 083° for 14 km. Another boat radios a message that they have found large schools of fish, so the trawler captain changes course and heads in the direction 173° for 20 km.

   a Draw a fully labelled sketch of this situation.

   b Find the distance of the trawler from the port.

   c Find the bearing of the trawler from the port.

   d Find the direction the trawler needs to travel in order to return to the port.
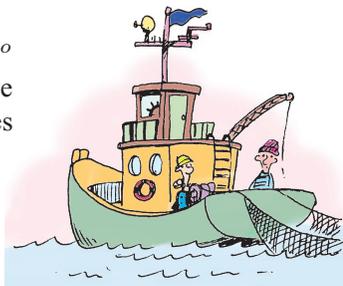
F   3-DIMENSIONAL PROBLEM SOLVING

Right angled triangles occur frequently in 3-dimensional figures. We can therefore use trigonometry to find unknown angles and lengths.

THE ANGLE BETWEEN TWO LINES

To find the angle between two lines, we consider a triangle in which two sides are these lines.

Example 10   ◄ Self Tutor

A cube has sides of length 12 cm. Find the angle between the diagonal [AB] and one of the edges at B.
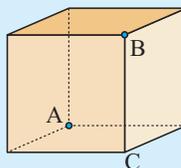
The angle between [AB] and *any* of the edges at B is the same.

∴ the required angle is $A\widehat{B}C$.
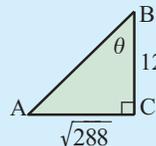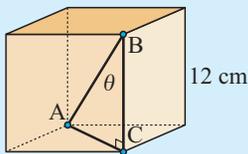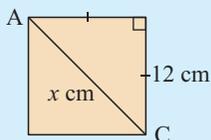
By Pythagoras:

$$x^2 = 12^2 + 12^2$$
$$\therefore \quad x^2 = 288$$
$$\therefore \quad x = \sqrt{288} \quad \{x > 0\}$$

$$\tan \theta = \frac{\sqrt{288}}{12}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{288}}{12}\right)$$

$$\therefore \quad \theta \approx 54.7^o$$

∴ the required angle is about 54.7^o.

THE ANGLE BETWEEN A LINE AND A PLANE

To find the angle between a line and plane, we consider the **projection** or shadow cast onto the plane by a light shining perpendicularly above the line. The required angle is the angle between the line and its shadow.

Click on the icon to obtain a demonstration which shows how to locate the angle between a line and a plane.

DEMO
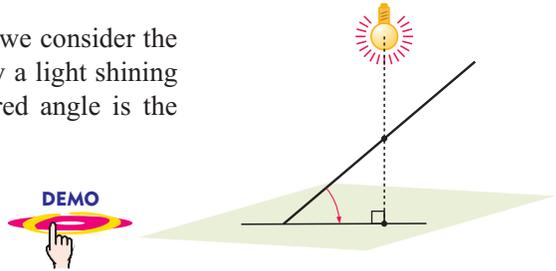
EXERCISE 12F

1 The figure alongside is a cube with sides of length 10 cm. Find:

  a EG   **b** $\stackrel{\wedge}{GAE}$

2
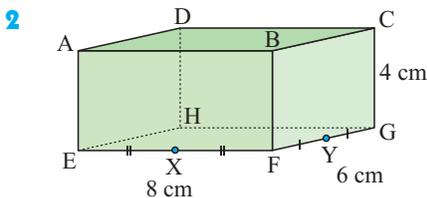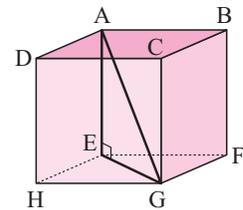
     The figure alongside is a rectangular prism. X and Y are the midpoints of the edges [EF] and [FG] respectively. Find:

  a HX   **b** the size of $\stackrel{\wedge}{DXH}$

  c HY   **d** the size of $\stackrel{\wedge}{DYH}$

3 In the triangular prism alongside, find:

  a DF

  b the size of $\stackrel{\wedge}{AFD}$.

4
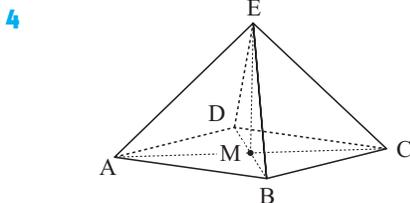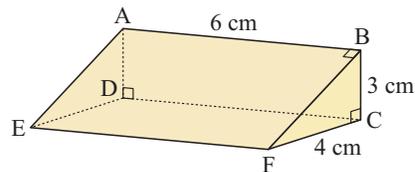
     Every edge of the illustrated pyramid has length 20 m. Find:

  a the length of diagonal [AC]

  b the angle that [EC] makes with [AC].

5 Find the shadow or projection of each of the following in the base plane if a light is shone from directly above the figure:

  a [UP]   **b** [WP]   **c** [VP]   **d** [XP]
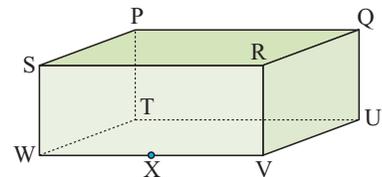
6 Find the projection on the base plane CDEF of:

  a [BD]   **b** [AE]   **c** [AF]   **d** [AX]
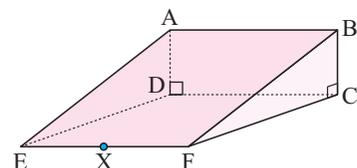
7 In the square-based pyramid shown, find the projection on the base plane ABCD of:

   a [PA]   **b** [PN]
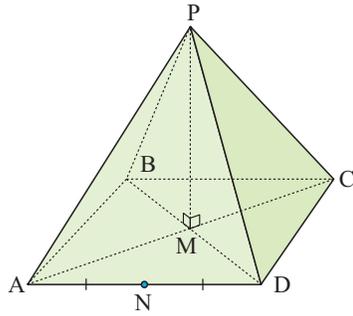
Example 11

🔊 **Self Tutor**

The given figure shows a square-based pyramid with apex directly above the centre of its base.

The base is 10 m by 10 m and its slant edges are 14 m long. Find:

   a the length of [MC]

   b the angle that [NC] makes with the base ABCD.

a
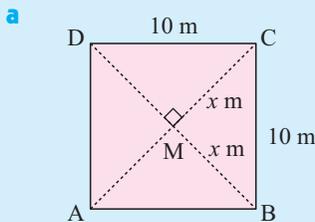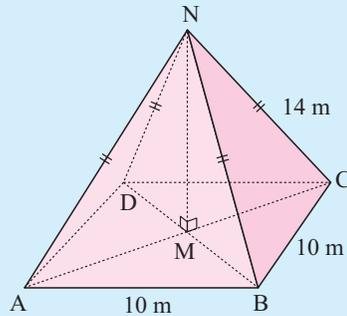
$\triangle$MBC is right angled and isosceles.

Let MC = MB = x m

$\therefore$ $x^2 + x^2 = 10^2$ {Pythagoras}

$\therefore$ $2x^2 = 100$

$\therefore$ $x^2 = 50$

$\therefore$ $x = \sqrt{50}$ {as $x > 0$}

So, [MC] is about 7.07 m long.

b

The shadow of [NC] cast onto plane ABCD is [MC], so the required angle is $\widehat{\text{NCM}}$, marked θ.
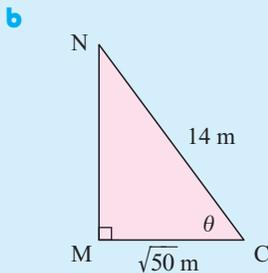
Now $\cos\theta = \dfrac{\sqrt{50}}{14}$

$\therefore$ $\theta = \cos^{-1}\left(\dfrac{\sqrt{50}}{14}\right) \approx 59.66^o$

So, [NC] makes an angle of about 59.7^o with ABCD.

---

8 Find the size of the angle between the following lines and the base plane EFGH:

   a [DG]   **b** [BH]

9
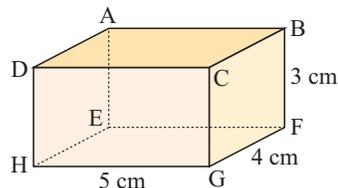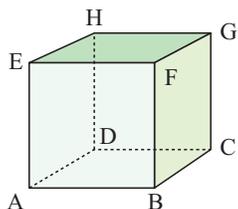
The diagram alongside shows a cube with side length 2 cm. Find the size of the angle between base ABCD and:

    a [AF]   **b** [AG].

10 Find the size of the angle between base PQRS and:

    a [PU]   **b** [PM].

11
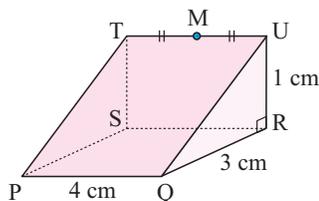
The given figure is an equilateral triangular prism with *all* edges of length 10 cm. Find the size of the angle between base ABCD and [ME].
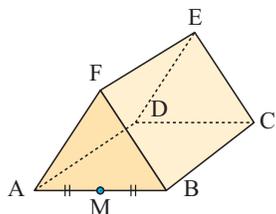
12 In the pyramid alongside, the apex E is directly above the middle of the rectangular base ABCD. Find the size of the angle that [EC] makes with the base ABCD.
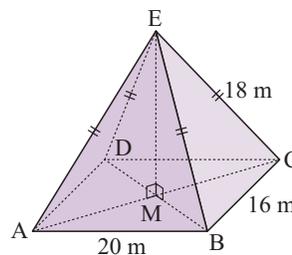
REVIEW SET 12A

1 **a** Find the exact value of x.

    b Find as a fraction:

        i $\cos\theta$ **ii** $\sin\theta$ **iii** $\tan\theta$
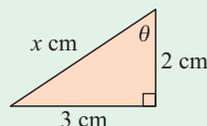
2 Find the value of a in:

    a   **b**   **c**
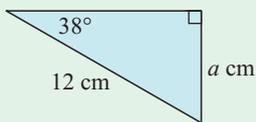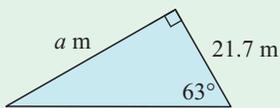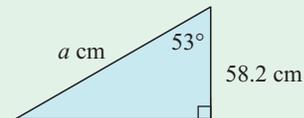
3 Find the value of θ in:
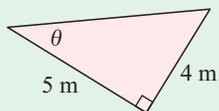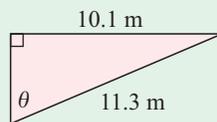
    a   **b**   **c**
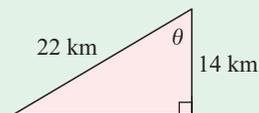
4 Using a compass, draw a circle of radius 2.6 cm. Carefully draw any chord of the circle such that the chord has length 4 cm.

   a Using the diagram only, find the shortest distance from the chord to the circle's centre.

   b Use trigonometry to find a more accurate answer to **a**.

5 From the top of a 56.2 m high building, the angle of depression to the base of a shed is $58°$. How far is it from the base of the building to the base of the shed?
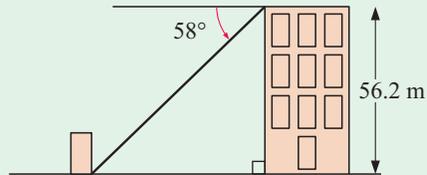
6 A boat sails 50 km on the bearing $054°$. How far is the ship north of its starting point?

7 A 30 m high tower is supported by 4 ropes. The ropes are tied 5 m from the top of the tower to four points on the ground. Each rope makes an angle of $60°$ with the ground.

   a Assuming the ropes do not sag, find the length of each one.

   b If 3% extra rope is needed to allow for sagging and tying, how much rope is needed to tie down the tower?

8
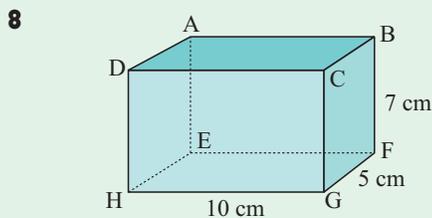
Find the angle that:

   a [BG] makes with [FG]

   b [AG] makes with the base plane EFGH.

9 Find:

   a the size of angle BFC

   b the length of [DF]

   c the angle [AF] makes with the base plane DCFE.
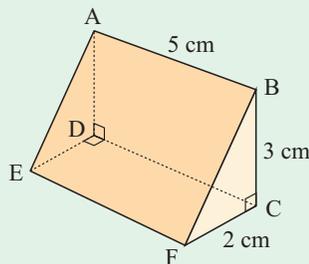
REVIEW SET 12B

1   **a** Find the exact value of y.

    b Find as a fraction:

      i $\cos\theta$ **ii** $\sin\theta$ **iii** $\tan\theta$

2 Find the value of b in:

   a   **b**   **c**

3 Find the value of ϕ in:

a

4.2 m

ϕ

5.2 m

b

317 cm

ϕ

424 cm

c

ϕ

8 m

11 m
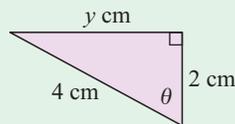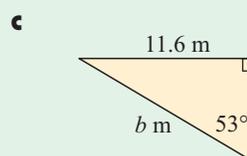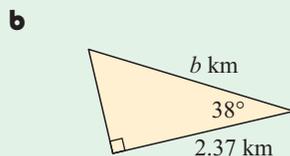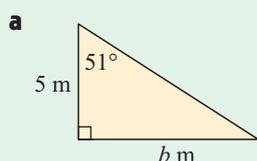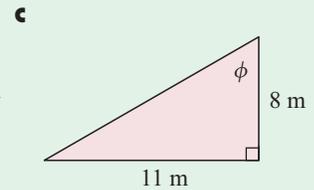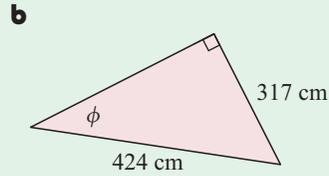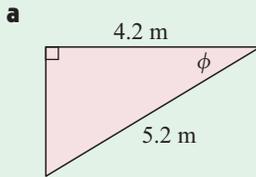
4 A motorcyclist travels 87 km west and then 63 km north.

    a How far is she from her starting point?

    b What is her bearing from her starting point?

    c What is her bearing to her starting point?

5   **a** Use a compass to construct a scale diagram of a triangular field with sides 100 m, 100 m and 60 m. Use a scale of 1 cm $\equiv$ 20 m.

    b Use a protractor to measure the smallest angle of the triangle to the nearest degree.

    c Use trigonometry to find a more accurate answer to **b**, correct to 1 decimal place.

6 The angle of elevation to the top of a vertical cliff is 13.8^o. This measurement was taken 1.234 km from the base of the cliff. How high is the cliff, to the nearest metre?

7 A ship leaves port A and travels for 60 km in the direction 038°. It then sails 40 km in the direction 128° to an island port B.

    a How far is B from A?

    b If the ship wishes to sail back directly to A from B, in what direction must it sail?

8

E

H

F

G

D

C

A   M   B
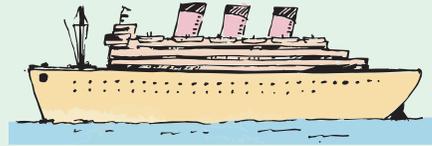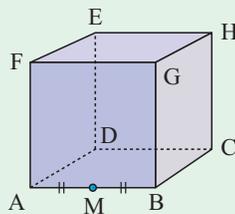
The illustrated figure is a cube with sides 6 m. Find:

    a the lengths of [BD] and [DG]

    b the angle [DG] makes with plane ABCD

    c the angle [DG] makes with [DM].

9 M is the midpoint of [AB] on the triangular prism.

Find the angle that [FM] makes with base plane ABCD.

E

6 m

F

D

4 m

A

M

C

3 m

B
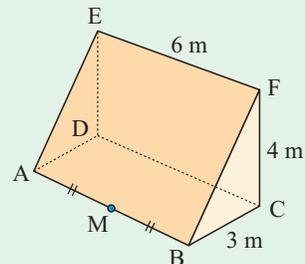
HISTORICAL NOTE   EVARISTE GALOIS 1811 - 1832

Evariste Galois was born at Bourg-la-Reine, France, in 1811. He had no formal teaching until he entered the College Royal de Louis-de Grand in 1823. These were turbulent times immediately following the French revolution. Galois found lessons frustrating and boring when teachers insisted that he show details of working in solutions which to him appeared trivial. He only showed interest in Mathematics and one teacher wrote: "The mathematical madness dominates the boy. I think his parents had better allow him to take only mathematics, for he is wasting his time here, and all he does is to torment his teachers and get into trouble."

At 16, he took an entrance examination to the Ecole Polytechnique, a famous school for mathematics and science. His failure stunned his teachers and fellow students who believed that he possessed a mathematical genius of the highest order.

At 17, under teacher Louis-Paul-Emile Richard, Galois made great discoveries in the theory of equations and the conditions under which they could be solved. He wrote a major article with a great number of new ideas and discoveries. Cauchy, the leading French mathematician of the time, promised to present the article to the Academy of Sciences, but he forgot, and later lost the manuscript.

Before he turned 18, Galois again failed to impress the examiners for entry to the Polytechnique because he did all his calculations in his head and merely wrote down the answers - all of which happened to be correct. During the oral examination, one of the examiners argued with Galois over a certain answer. The examiner was wrong and obstinate and Galois, losing his temper, threw a blackboard eraser at him.

At 19, Galois was eventually admitted to university where he worked in solitude on his own ideas. He wrote three more original papers on the theory of equations and submitted them to the Academy of Science in competition for the coveted Grand Prize in Mathematics. The Secretary received the papers, but died before examining them, and they were lost.

Galois grew more embittered and his hatred for society grew. He turned to politics and radicalism and was twice arrested for his republican views and activities. He was imprisoned for six months on a trumped up charge and during this time concentrated on developing his mathematical theories.

On his release and still not 21, he again ran foul of his political enemies and died as a result of a duel. The night before the duel, he frantically dashed off his mathematical "last will and testament" - 60 pages of work on the theory of equations and abstract algebra. This one night's work has kept mathematicians occupied ever since.

Chapter 13

Formulae

Contents:

A **formula** is an equation which connects two or more variables

For example, the simple interest formula $I = Crn$ connects the variables interest (I), principal (C), rate of interest (r), and time or duration (n) of the loan.

In this case we have a formula for the interest I in terms of the other variables, so we say that I is the **subject** of the formula.

The plural of formula is **formulae** or **formulas**.

For example, $I = Crn$ and $A_n = A_0 \left(1 + \dfrac{r}{100}\right)^n$ are *formulae* used in financial mathematics.

OPENING PROBLEM

A closed cylinder has radius r and height h.

Can you:

- show that the outer surface area of the cylinder can be found using the formula $A = 2\pi rh + 2\pi r^2$

- substitute into this formula to find the outer surface area of a cylinder of radius 5 cm and height 20 cm

- make the h the subject of the formula

- adjust the formula for a cylindrical container with an open top?
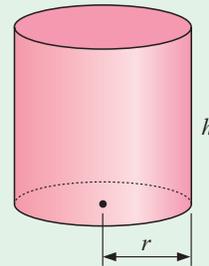
A   SUBSTITUTING INTO FORMULAE

A formula is normally written with one variable by itself on the left hand side of the equation. We call this variable the **subject** of the formula. The other variables are contained in an expression on the right hand side.

To find the value of any of the variables we need to know the values of all the other variables in the formula. To do this we:

- write down the formula and state the values of known variables
- substitute the known variables into the formula
- solve the equation for the unknown variable.

Example 1
◀) **Self Tutor**

If $v = u + gt$,   find:

a v when $u = 15$, $g = 9.8$ and $t = 10$
b t when $v = 45$, $u = 20$ and $g = 9.8$.

a $v = u + gt$ where $u = 15$, $g = 9.8$ and $t = 10$
∴ $v = 15 + 9.8 \times 10$
∴ $v = 15 + 98$
∴ $v = 113$

b $v = u + gt$ where $v = 45$, $u = 20$ and $g = 9.8$
∴ $45 = 20 + 9.8t$
∴ $25 = 9.8t$
∴ $\frac{25}{9.8} = t$
∴ $t \approx 2.55$ {to 3 significant figures}

Example 2
◀) **Self Tutor**

The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.   Find:

a the volume of a sphere of radius 9.3 cm
b the radius of a sphere which could be cast from 40 cm^3 of lead.

a $V = \frac{4}{3}\pi r^3$ where $r = 9.3$
∴ $V = \frac{4}{3}\pi \times (9.3)^3$
∴ $V \approx 3369.28$ { $(\ 4 \div 3\ ) \times \pi \times 9.3 \wedge 3$ ENTER }
∴   the volume is about 3370 cm^3.

b $V = \frac{4}{3}\pi r^3$ where $V = 40$
∴ $40 = \frac{4}{3}\pi r^3$
∴ $120 = 4\pi r^3$ {multiplying both sides by 3}
∴ $\frac{120}{4\pi} = r^3$
∴ $r^3 = \frac{30}{\pi}$
∴ $r = \sqrt[3]{\frac{30}{\pi}}$
∴ $r \approx 2.12$ { $(\ 30 \div \pi\ ) \wedge (\ 1 \div 3\ )$ ENTER }
∴   the radius is about 2.12 cm.

EXERCISE 13A

Where necessary, give answers correct to 3 significant figures.

1 When Fredrik skis down a hill, his average speed is given by the formula $s = \dfrac{d}{t}$ where d is the distance travelled and t is the time taken. Find:

   a Fredrik's average speed in $m\,s^{-1}$ if he travels 520 m in 65 seconds

   b the time taken for him to travel 835 m at an average speed of $9.51\ m\,s^{-1}$.

2 $I = Crn$ is the simple interest formula which connects the interest I, the principal C, the rate of interest r expressed as a decimal, and the period n of an investment. Find:

   a the simple interest earned by an investment of \$10 000 at 8% p.a. for 5 years

   b the rate of interest given that an investment of €8000 over $3\frac{1}{2}$ years earned €1848 in simple interest.

3 When a stone is dropped down a well, the distance fallen is given by $D = 4.9t^2$ metres where t is the time taken in seconds.

   a How far will the stone fall in:

   i 1 second   **ii** 2 seconds   **iii** 3 seconds?

   b Find the approximate depth of the well if it takes 4.3 seconds to hit the bottom.

   c If a well is 100 m deep, how long will it take for a stone to hit the bottom?

4
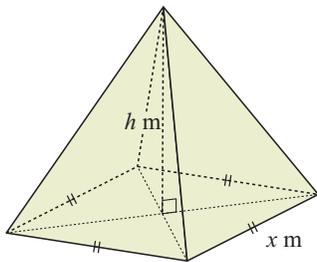
A pyramid has a square base with sides x m and height h m. Its volume is given by the formula $V = \frac{1}{3}x^2h$. Find:

   a the volume of the pyramid if its base length is 20 m and its height is 12 m

   b the height of a square-based pyramid with volume 2487 m³ and base length 23.8 m

   c the base length measurement of a square-based pyramid with volume 22700 m³ and height 32.8 m.

5 Fulvio's formula for finding the price of his pizzas is $P = F + \dfrac{r^2}{k}$ euros where r is the radius of the dish, k is dependent on the topping, and F is a constant for profit. The value of P is then rounded to the nearest 20 cents. Find:

   a Fulvio's prices for pizzas with radii: **i** 10 cm   **ii** 15 cm   **iii** 20 cm
   given $F = 3.8$ and $k = 40$

   b Fulvio's price for a pizza with radius 17.5 cm and $k = 38$ if he has increased his profit F to 4.2 euros.

6 Heron's formula for finding the area of a triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \dfrac{a+b+c}{2}$ and a, b and c are the lengths of the triangle's three sides. Use Heron's formula to find:

  a the area of a triangular garden plot with sides 6 m, 7 m and 8 m

  b the area of an isosceles triangle with sides $3x$, $3x$ and $2x$ units.

7 A circular pond has area given by $A = \pi r^2$ where r is its radius. Find:

  a the radius of the pond if its area is 10 m^2

  b the length of fencing needed to surround the pond at a distance of 1 m from the pond's edge.

8 The formula $D \approx 4\sqrt{h}$ kilometres allows us to calculate the approximate distance to the horizon when we are h metres above our surroundings. Find:

  a the approximate distance to the horizon for a person at the top of a cliff 100 metres above sea level

  b the height of a tower if a person standing at the top can see a distance of 44 km to the horizon.

B   REARRANGING FORMULAE

In the formula $M = 3x + k$, we say that M is the **subject** of the formula because M is expressed in terms of the other variables.

The formula can be **rearranged** to make x the subject. The new formula is $x = \dfrac{M-k}{3}$, which is **equivalent** to the original formula.

We **rearrange formulae** using the same methods which we use to solve equations. We perform **inverse operations** to isolate the variable we want to make the subject.

Example 3   ◄) **Self Tutor**

Make y the subject of $3x + 5y = 11$.

$$3x + 5y = 11$$
$$\therefore \quad 3x + 5y - 3x = 11 - 3x \qquad \{\text{subtracting } 3x \text{ from both sides}\}$$
$$\therefore \qquad 5y = 11 - 3x \qquad \{\text{simplifying}\}$$
$$\therefore \qquad \frac{5y}{5} = \frac{11 - 3x}{5} \qquad \{\text{dividing both sides by 5}\}$$
$$y = \frac{11 - 3x}{5}$$

Example 4   ◀)) **Self Tutor**

Make a the subject of $N = ab + c$.

$$N = ab + c$$

$$\therefore \quad ab + c = N$$

$$\therefore \quad ab + c - c = N - c \qquad \{\text{subtracting } c \text{ from both sides}\}$$

$$\therefore \quad ab = N - c \qquad \{\text{simplifying}\}$$

$$\therefore \quad \frac{ab}{b} = \frac{N - c}{b} \qquad \{\text{dividing both sides by } b\}$$

$$\therefore \quad a = \frac{N - c}{b}$$

> Sometimes it helps to start by interchanging the sides.

Example 5   ◀)) **Self Tutor**

Make g the subject of $n = \dfrac{a}{g}$.

$$n = \frac{a}{g}$$

$$\therefore \quad n \times g = \frac{a}{g} \times g \qquad \{\text{multiplying both sides by } g\}$$

$$\therefore \quad ng = a \qquad \{\text{simplifying}\}$$

$$\therefore \quad \frac{ng}{n} = \frac{a}{n} \qquad \{\text{dividing both sides by } n\}$$

$$\therefore \quad g = \frac{a}{n}$$

EXERCISE 13B

1 Make y the subject of:

   a $3x + 4y = 13$
   b $5x + 7y = 18$
   c $3x - 2y = 6$

   d $4x + 9y = 17$
   e $5x - 2y = 7$
   f $6x - 5y = -10$

   g $7x - 12y = 8$
   h $9x + 3y = x + 4$
   i $a + 4y = b$

2 Make a the subject of:

   a $b + a = d$
   b $ab = c$
   c $2m + a = g$

   d $a - b = c$
   e $a + 2b = e$
   f $P = 3a + m$

   g $Q = d - 4a$
   h $ax + b = n$
   i $ax + by = c$

   j $2b + na = c$
   k $K = c - 3a$
   l $ax + b = 2b$

   m $ax + b = ay$
   n $4a + b = 2a - c$
   o $b - a = c - 2a$

3 Make the variable in brackets the subject of the formula:

   a $K = \dfrac{g}{a}$ (g)
   b $K = \dfrac{g}{a}$ (a)
   c $A = \dfrac{bh}{c}$ (h)

   d $A = \dfrac{bh}{c}$ (b)
   e $A = \dfrac{bh}{c}$ (c)
   f $V = \dfrac{lb}{d}$ (b)

g $V = \dfrac{lb}{d}$ (d) **h** $H = \dfrac{a}{2b}$ (b) **i** $D = \dfrac{a+b}{c}$ (c)

ALGEBRAIC FRACTIONS

When two algebraic fractions are equal we can remove the fractions by **cross multiplying** the denominators:

$$\text{If} \quad \frac{a}{b} = \frac{c}{d} \quad \text{then} \quad ad = bc.$$

Proof: If $\dfrac{a}{b} = \dfrac{c}{d}$ then $\dfrac{a}{b} \times bd = \dfrac{c}{d} \times bd$

$\therefore \quad ad = bc$

For example, we know that $\frac{6}{3} = \frac{8}{4}$ and that $6 \times 4 = 3 \times 8$.

Example 6   ◀) **Self Tutor**

Make c the subject of the formula $\dfrac{b}{a} = \dfrac{2}{c}$.

$\dfrac{b}{a} = \dfrac{2}{c}$

$\therefore \quad bc = 2a$ {multiplying both sides by ac}

$\therefore \quad c = \dfrac{2a}{b}$ {dividing both sides by b}

---

4   Make the variable in brackets the subject of the formula:

a $\dfrac{3}{a} = \dfrac{c}{4}$ (a) **b** $\dfrac{a}{b} = \dfrac{c}{d}$ (c) **c** $\dfrac{a}{b} = \dfrac{c}{d}$ (b)

d $\dfrac{a}{b} = \dfrac{c}{d}$ (d) **e** $\dfrac{b}{z} = \dfrac{z}{a}$ (b) **f** $\dfrac{b}{z} = \dfrac{z}{a}$ (z)

g $\dfrac{b}{z} = \dfrac{z}{a}$ (a) **h** $\dfrac{b}{2} = \dfrac{a}{b^2}$ (b) **i** $\dfrac{a}{\sqrt{b}} = \dfrac{2}{n}$ (b)

C   CONSTRUCTING FORMULAE

Formulae are often constructed by the generalisation of numerical cases.

For example,
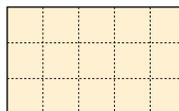
given the rectangle   the area $A = 3 \times 5$ m^2

In general,

l m

given the rectangle   the area $A = l \times w$ m^2
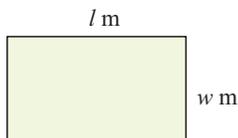
w m   i.e., $A = lw$ m^2

Example 7

🔊 **Self Tutor**

A plumber charges a £60 call-out fee and £50 per hour thereafter.

a Find the plumber's fee £F for a job which takes:

    i 4 hours   **ii** x hours

b If the call-out fee is now £C, find a formula for the plumber's fee for a job which takes x hours.

> In **a i** we do not simplify the amount. We do this so we can see how the formula is put together.

a   **i** The total fee is £60 plus £50 for each of 4 hours

      ∴ $F = 60 + 50 \times 4$ pounds

  ii $F = 60 + 50x$ pounds {replacing 4 by x}

b $F = C + 50x$ pounds {replacing 60 by C}

EXERCISE 13C.1

1 An electrician charges a $70 call-out fee and $60 per hour thereafter. Find the electrician's total fee F for a service lasting:

  a 3 hours   **b** h hours.

2 Find a formula for the amount A in a savings account given monthly deposits of:

  a $250 over 6 months   **b** $250 over m months   **c** $D over m months.

3 Write a formula for the total cost €C of hiring a bobcat for shifting earth given a fixed call-out fee of:

  a €100 plus €80 per hour for 5 hours of work

  b €100 plus €80 per hour for t hours of work

  c €100 plus €y per hour for t hours of work

  d €F plus €y per hour for t hours of work.

4 Find a formula for the total cost C of a taxi trip given a flagfall charge of:

  a $4 plus $1.20 per km for 8 km   **b** $4 plus $1.20 per km for k km

  c $4 plus $$d$ per km for k km   **d** $$F$ plus $$d$ per km for k km.

5 Find a formula for the amount of money ¥M in a bank account if initially the balance was:

  a ¥800 000 and ¥20 000 was withdrawn each week for 3 weeks

  b ¥800 000 and ¥x was withdrawn each week for a weeks

  c ¥A and ¥x was withdrawn each week for a weeks.

6 Claude digs wells for a living. He charges a £200 call-out fee and a variable fee of £$125 \times d^{1.3}$ where d is the depth dug in metres. All wells have the same diameter.

  a Find a formula for the total cost £C in terms of d.

  b How much would it cost to dig a 20 m deep well?

GEOMETRIC FORMULAE

Formulae can be constructed for geometrical situations using existing formulae.

Example 8

◀ঠ **Self Tutor**

The illustrated door consists of a semi-circle and a rectangle. Find a formula for the area of the door in terms of the width w and height h of the rectangular part.

The area of the rectangle $=$ height $\times$ width
$$= h \times w$$
$$= hw$$

The radius of the semi-circle is $\dfrac{w}{2}$

$\therefore$ the area of the semi-circle $= \frac{1}{2} \times$ (area of full circle)
$$= \tfrac{1}{2} \times \pi r^2$$
$$= \tfrac{1}{2} \times \pi \times \left(\frac{w}{2}\right)^2$$
$$= \tfrac{1}{2} \times \pi \times \frac{w^2}{4}$$
$$= \tfrac{1}{8}\pi w^2$$

$\therefore$ the total area is $A = hw + \frac{1}{8}\pi w^2$

EXERCISE 13C.2

1 Find a formula for the areas of the following shapes:

a

b

c

d

e

f

2   A hollow cylinder is illustrated.

  a Explain why the area of tin-plate needed to make the cylinder is given by the formula $A = 2\pi rh$.

  b How can this formula be adjusted to find the total area of tin-plate needed to make an open bin of radius r and height h?

  c Find the area of tin-plate used to make an open bin with base radius 15 cm and height 40 cm.

  d If a manufacturer uses 4000 cm² of tin-plate to make a bin with base radius 20 cm, how high will the bin be?

  e Make h the subject of the formula $V = 2\pi rh + \pi r^2$.   Use this rearrangement to check your answer to **d**.

D   FORMULAE BY INDUCTION

Induction is a method of finding a formula for a general situation by examining simple cases and looking for a pattern.

For example, the set of even numbers is {2, 4, 6, 8, 10, 12.....}.

We observe that: the 1st term is 2×1

the 2nd term is 2×2

the 3rd term is 2×3 , and so on.

We see from the pattern that the 13th term will be 2×13.

So, we generalise by saying that "the nth even number is $2n$".

Notice how the 2 in $2n$ indicates that the terms increase by 2 each time n is increased by 1.

Example 9   ◀)) **Self Tutor**

  Find the nth term of the number set   {2, 5, 8, 11, 14, ...}.

The sequence increases by 3 each time.

This suggests the formula should contain $3n$.

The expression $3n$ generates the set   {3, 6, 9, 12, 15, ...}   whereas our sequence is always one less than these values.

So, the nth term for the given number set is $3n - 1$.

Example 10

◀) **Self Tutor**

Examine the matchstick pattern: △ , △▽ , △▽△ , ...

How many matches are needed to make:

a the first diagram   **b** the second diagram   **c** the third diagram

d the 4th diagram   **e** the nth diagram

a 3   **b** 5   **c** 7

d The fourth diagram is △▽△▽ which contains 9 matches.

e So far, the sequence is $\{3, 5, 7, 9, ...\}$. We are adding 2 matches each time, so the formula must involve $2n$.

The expression $2n$ would generate the set $\{2, 4, 6, 8, ...\}$ which is always 1 less than our values here.

$\therefore$ there are $2n + 1$ matches in the nth diagram.

Check: if $n = 1$, $2(1) + 1 = 3$ ✓

            if $n = 2$, $2(2) + 1 = 5$ ✓

            if $n = 3$, $2(3) + 1 = 7$ ✓

EXERCISE 13D

1 Find a formula for the nth odd number, which is the nth term of the number set $\{1, 3, 5, 7, 9, ...\}$.

2 Find the nth term of:

   a 7, 9, 11, 13, 15, ...   **b** 4, 7, 10, 13, 16, ...   **c** 5, 9, 13, 17, 21, ...

   d 3, 7, 11, 15, ...   **e** 7, 14, 21, 28, 35, ...   **f** 2, 9, 16, 23, 30, ...

   g 2, 4, 8, 16, 32, ...   **h** 1, 2, 4, 8, 16, 32, ...   **i** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, ...

   j $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}$, ...

3 Examine the matchstick pattern: ▢ , ▢▢ , ▢▢▢ , ...

How many matchsticks make up the:

   a 1st, 2nd, 3rd, 4th and 5th diagrams

   b 10th diagram

   c nth diagram?

4 Repeat **3** for the patterns:

   a ◻ , ◻◻ , △▽△▽ , △▽△▽△ , ...

   b ⬡ , ⬡⬡ , ⬡⬡⬡ , ⬡⬡⬡⬡ , ...

   c ⯰ , ⯰ , ⯰ , ⯰ , ...

5 The sum of the first n positive odd numbers is $1 + 3 + 5 + 7 + 9 + \ldots\ldots +$ ▮

    a What should be written in place of ▮ ?

    b Find: **i** $1 + 3$

               ii $1 + 3 + 5$

               iii $1 + 3 + 5 + 7$

               iv $1 + 3 + 5 + 7 + 9$

    c If S_n is the sum of the first n positive odd numbers then $S_1 = 1$, $S_2 = 4$, and $S_3 = 9$. Find a formula for S_n.

6 The sum of the first n powers of 2 is $1 + 2 + 4 + 8 + 16 + \ldots\ldots +$ ▮

                        or $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots\ldots +$ ▮ .

    a What should be written in place of ▮ ?

    b Find: **i** $1 + 2$

               ii $1 + 2 + 4$

               iii $1 + 2 + 4 + 8$

               iv $1 + 2 + 4 + 8 + 16$

    c Find a formula for the sum of the first n terms of the number set $\{1, 2, 4, 8, 16, \ldots\ldots\}$.

7 Consider $S_n = 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + 5 \times 2^4 + \ldots\ldots + u_n$. u_n is the nth term of the sequence and S_n is the sum of its first n terms.

    a Write down u_n in terms of n.

    b Complete this pattern down to S_6 :

$$S_1 = 2 \times 2 = 1 \times 2^2$$
$$S_2 = 2 \times 2 + 3 \times 2^2 = 16 = 2 \times 2^3$$
$$S_3 = 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 = 48 = 3 \times 2^4$$
$$\vdots$$

    c Predict a formula for S_n in terms of n.

    d Find the actual value of $2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + 5 \times 2^4 + \ldots\ldots + 21 \times 2^{20}$.

INDUCTION DANGERS

LINKS
click here

Areas of interaction:
Human ingenuity/Approaches to learning

REVIEW SET 13A

1 Given that $A = c - dt$, find:

    a A when $c = 60$, $d = 4$ and $t = -8$

    b t when $A = 45$, $c = 21$ and $d = 5$.

2 The surface area of a sphere can be calculated using $A = 4\pi r^2$ where r is its radius. Find:

    a the surface area of a sphere of radius 7.3 cm

    b the radius of a sphere which has an area of 20 m²

    c the outer surface area of the small silo illustrated, which is sitting on a concrete base.

3 Make y the subject of:

    a $3x + 4y = 10$ **b** $5x - 3y = 8$ **c** $\dfrac{3}{y} = \dfrac{a}{b}$ **d** $\dfrac{a}{b} = \dfrac{b}{\sqrt{y}}$

4 A gardener charges a fixed amount for travel and then an amount per hour for the time he works. Find a formula for the total charge C dollars if he works for n hours and:

    a the fixed amount is $40 and the amount per hour is $30

    b the fixed amount is $F and the amount per hour is $a.

5 This diagram consists of a rectangle and a right angled triangle. Find a formula for calculating its:

    a area A **b** perimeter P.

6 **a** What is the nth multiple of 3?

    b Find the nth term of

        i 1, 4, 7, 10, 13, ...   **ii** $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, ...

7 Examine the matchstick pattern: ⌐|⌐| , ⌐|⌐|⌐| , ⌐|⌐|⌐|⌐| , ...

    How many matchsticks make up the:

    a 1st, 2nd, 3rd, and 4th diagrams   **b** 7th diagram

    c nth diagram?

8 The sum of the first n positive even numbers is $2 + 4 + 6 + 8 + ... + \blacksquare$

    a What should be written in place of $\blacksquare$?

    b Find: **i** $2 + 4$

             ii $2 + 4 + 6$

             iii $2 + 4 + 6 + 8$

             iv $2 + 4 + 6 + 8 + 10$

    c Hence write a formula for the sum of:

        i the first n positive even numbers   **ii** the first n integers.

REVIEW SET 13B

1 Given that $M = a + \dfrac{b^2}{c}$, find:

    a M when $a = 5$, $b = 8$ and $c = -2$

    b b when $M = 210$, $a = 70$ and $c = 14$.

2 Make b the subject of:

    a $2a + 3b = c$ **b** $a - \dfrac{4}{b} = c$ **c** $\dfrac{3}{b} = \dfrac{b}{e}$ **d** $\dfrac{a}{c} = \dfrac{4}{b}$

3   **a** What is the nth multiple of 7?

    b Find the nth term of:

        i 2, 9, 16, 23, 30, ...   **ii** 100, 93, 86, 79, ...

4   **a** A cylinder is capped with a cone as shown. Find a formula for the total volume of the solid in terms of d. You may assume the formulae $V_{\text{cylinder}} = \pi r^2 h$ and $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$.

    b Find the total volume if $d = 3$ m.

    c If the total volume is 20 m³, find d.

5 For painting a house, Hendrik and Gemma charge a fixed amount of €A for setting up their equipment, plus €V per hour of time spent painting. Find the total cost €M if they spend:

    a 50 hours   **b** h hours painting.

6

This figure shows the cross-section of a gutter. It consists of a rectangular centre with two congruent triangles at its ends.

    a How long is [AB]?

    b How long is [AC]?

    c Find a formula for the area of the figure A in terms of a, b and h.

7 Examine the matchstick pattern:

How many matchsticks make up the nth diagram?

8 Suppose $S_1 = \dfrac{1}{1 \times 2}$, $S_2 = \dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3}$, $S_3 = \dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4}$, ...

and so on.

    a Find the value of S_1, S_2, S_3 and S_4.

    b Predict a formula for S_n.

Chapter **14**

Comparing numerical data

Contents:

OPENING PROBLEM   SCREW CAPS ON BOTTLES

At the local soft drink factory there are two machines which place screw caps on the plastic bottles. The quality control officer Hugh suspects that machine A causes fewer faulty screw caps than machine B. Each day 1000 bottles are randomly selected from each machine and their caps are tested for faults. Over a period of 60 working days, the number of faulty caps were recorded each day. The results were:

Machine A

| 4 7 5 6 6 | 5 2 6 7 6 | 6 5 5 8 6 | 6 5 7 4 8 | 9 6 7 5 6 | 12 5 6 6 5 |
| 6 8 6 5 9 | 8 13 6 6 5 | 6 8 7 5 6 | 6 4 6 5 5 | 7 5 6 4 6 | 5 6 6 5 6 |

Machine B

| 5 9 9 7 3 | 9 8 10 8 6 | 7 7 9 8 5 | 9 9 4 11 9 | 6 9 7 9 8 | 9 5 9 9 7 |
| 9 7 9 8 8 | 9 4 10 8 6 | 9 7 3 8 7 | 5 9 8 7 6 | 8 4 9 10 9 | 7 5 8 6 9 |

Things to think about:

- Can you clearly state the problem that the quality controller wants to solve?
- How has Hugh tried to make a fair comparison?
- How could Hugh make sure that his selection is at random?
- What is the best way of organising the data?
- How can we display the organised data?
- Are there any abnormally high or low results and how should they be treated?
- How can Hugh best indicate the most typical number of faulty caps?
- How can Hugh indicate the spread of the data for each machine?
- Can a satisfactory conclusion be made?

A   GRAPHICAL COMPARISON

Two distributions can be compared by using:

- side-by-side column graphs
- back-to-back bar graphs

- back-to-back stemplots

	12	6
4	11	2 3 7
7 1	10	6 7 8 8
9 8 7 6 6 5 3 3 2 0	9	0 3 4 5 5 7 9
7 5 3 2 1	8	2 2 4 5 8
2 0	7	0 5 6
	6	1

For Hugh's data in the **Opening problem**, the organised data is:

Machine A

Number faulty	Frequency
2	1
3	0
4	4
5	16
6	24
7	6
8	5
9	2
10	0
11	0
12	1
13	1

Machine B

Number faulty	Frequency
3	2
4	3
5	5
6	5
7	10
8	11
9	20
10	3
11	1

DEMO

Click on the icon to find how to make back-to-back charts using a spreadsheet.

Possible comparative graphs for this data are:

Side-by-side column graph of screw cap data

Back-to-back bar graph of screw cap data

Example 1   🔊 Self Tutor

A Sales and Service company for photocopiers sells about equal numbers of two brands, A and B.

The number of service calls per day was recorded over many days and the results were used to construct the graph alongside.

What can be deduced from this graph?

The distribution for A is almost symmetric whereas that for B is positively skewed towards higher numbers of call outs.

Overall there are more call outs for brand A than for brand B, so brand B is more reliable.

EXERCISE 14A

1 In each of the following graphs, two data sets of the same size are compared. What can be deduced from each graph?

a

b

c

d

e

f

2 Pedro sells small ceramic items on an internet auction site. He lives in the USA and sells within his country and overseas. He uses the standard postal service for local deliveries and a private freight company for international ones. Pedro records the number of items that are broken each month over a 70 month period.

  a Comment on the shape of each distribution.

  b Which delivery service is Pedro happier with? Explain your answer.

B   PARALLEL BOXPLOTS

The graphs in **Section A** are fine for basic comparisons provided the number of sample values is about the same for each distribution. However, to make meaningful statements about the data being compared, **parallel** or **side-by-side boxplots** are more useful.

Parallel boxplots enable us to make a visual comparison of the distribution of the data, and also the descriptive statistics such as median, range, and interquartile range.

Parallel boxplots can be horizontal or vertical.

The following parallel boxplot is horizontal. It shows a sample of travel times when Juan catches the bus or train to work.

DISCUSSION

What information can be obtained by comparing the boxplots for Juan's "travel to work" data?

Example 2   ◀) Self Tutor

Jeffrey can travel to school either by car or by bus. He has collected data giving the travel times using both of these modes of transport. He wants to know which mode of transport will get him to school quicker and which is more reliable.

> *Car travel times (min):*   17, 21, 14, 9, 29, 23, 24, 10, 14, 39, 15, 18, 26, 18, 20
>
> *Bus travel times (min):*   19, 14, 12, 12, 26, 16, 17, 14, 14, 13, 16, 17, 24, 13, 12

Prepare parallel boxplots for the data sets and use them to compare the two methods of transport for speed and reliability.

The 5-number summaries are:

For car travel:	For bus travel:
min $= 9$	min $= 12$
$Q_1 = 14$	$Q_1 = 13$
median $= 18$	median $= 14$
$Q_3 = 24$	$Q_3 = 17$
max $= 39$	max $= 26$

> Outliers can sometimes be removed if we are certain an error has been made. We then recalculate the statistics. If you are not certain, do not remove them.

In the data sets we identify some outliers: 39 minutes by car, and 24 and 26 minutes by bus. They are represented as asterisks on the boxplot, and are not included in the whiskers.

The median bus travel time was 14 minutes whereas the median car travel time was 18 minutes. So, bus travel is generally *quicker*.

Comparing spread: range for car $= 39 - 9$ range for bus $= 26 - 12$
$$= 30$$ $$= 14$$
$$\text{IQR} = Q_3 - Q_1$$ $$\text{IQR} = Q_3 - Q_1$$
$$= 24 - 14$$ $$= 17 - 13$$
$$= 10$$ $$= 4$$
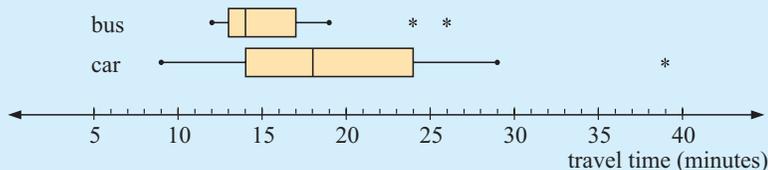
By comparing these spread measures, the bus travel times are less 'spread out' than the car travel times. They are more predictable or reliable.

EXERCISE 14B

1 Write brief comparative reports about the distributions shown in these parallel boxplots:

a

b

c

d

e

f
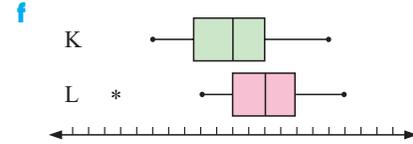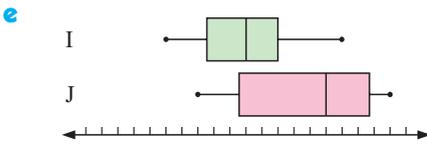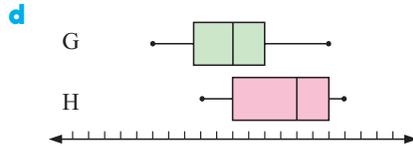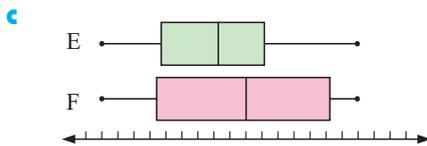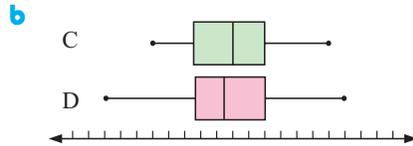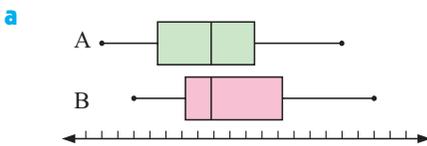
2 The following boxplots compare the time students in years 10 and 12 spend on homework over a one week period.
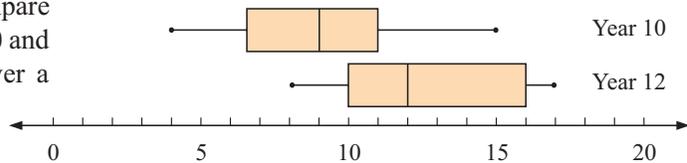
  a Find the 5-number summaries for both the year 10 and year 12 students.

  b Determine the **i** range **ii** interquartile range   for each group.

3 Barramundi are a species of fish caught in the tropical waters of northern Australia and Indonesia. Alongside is a parallel boxplot of the lengths of barramundi caught in Indonesian and Australian waters.

  a For each region, find the:

   i greatest length   **ii** shortest length

   iii range   **iv** interquartile range

  b If the legal length for a barramundi to be kept is 50 cm, what percentage of fish caught were of legal length in

   i Indonesia   **ii** Australia?
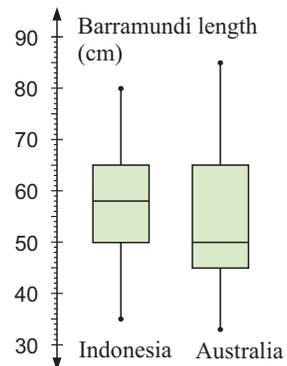
c Describe the fish length distributions for: **i** Indonesia   **ii** Australia.

d Copy and complete:

   i The fish caught in ...... generally have greater length.

   ii The fish caught in ...... have greater length variability.

4 The batting averages for the English and West Indian teams for the 2001 test series in India were as follows:

England   109.8, 48.6, 47.0, 33.2, 32.2, 29.8, 24.8, 20.0, 10.8, 10.0, 6.0, 3.4, 1.0

West Indies   83.83, 56.33, 50.67, 28.83, 27.00, 26.00, 21.00, 20.00, 17.67, 11.33, 10.00, 6.00, 4.00, 4.00, 1.00, 0.00

a Record the 5-number summary for each country and test for outliers.

b Construct parallel boxplots for the data.

c Compare and comment on the centres and spread of the data sets.

d Should any outliers be discarded and the data reanalysed?

> Recall that:
> lower boundary $= Q_1 - 1.5 \times IQR$
> upper boundary $= Q_3 + 1.5 \times IQR$

5 Samples of lobster were caught in two adjacent bays on the coast of California. The following data shows the average weights in pounds for the two bays over a 20 day catching period.

Bay 1 :   2.6, 2.5, 2.7, 2.4, 2.9, 2.7, 2.6, 2.7, 2.8, 2.5
            2.7, 2.6, 2.8, 2.6, 2.5, 2.8, 2.5, 2.4, 2.7, 2.3

Bay 2 :   2.7, 3.0, 2.6, 2.9, 2.7, 2.8, 2.9, 2.6, 2.7, 2.7
            2.9, 3.1, 2.6, 2.7, 2.7, 2.8, 3.2, 2.7, 2.8, 2.8

a Find the five-number summary for each of the data sets. Test for outliers and construct parallel boxplots.

b Compare and comment on the distributions of the data.

6 Maria and Sophie play in the same softball team. They are fierce but friendly rivals when it comes to scoring the most runs. During a season the numbers of runs scored by each batter were:

Maria: 3 3 1 6 2 0 3 4 1 4 2 3 0 3 2 4 3 4
         4 3 3 4 2 4 3 2 3 3 0 5 3 5 3 2 4 3

Sophie: 2 0 7 2 4 8 1 3 4 2 3 0 5 3 5 2 3 1
         0 1 4 3 4 0 3 3 0 2 5 1 1 2 2 5 1 4

STATISTICS PACKAGE

a Is the data discrete or continuous?

b Enter the data into a graphics calculator or statistics package.

c Produce a vertical column graph for each data set.

d Are there any outliers? Should they be deleted before we start to analyse the data?

e Describe the shape of each distribution.

g Compare the spreads of each distribution.

h Obtain side-by-side boxplots.

i If you are using the statistics package, print out the graphs, boxplots, and relevant statistics.

j What conclusions can be drawn from the data?

7 Frank makes batteries. He claims that his brand lasts *twice as long* as those manufactured by a competitor. Thirty of each brand of battery were randomly selected and tested. The results to the nearest hour were:

| *Frank's* | 135 139 141 145 150 163 135 128 145 143 140 149 136 146 129 |
| *batteries* | 139 142 132 146 146 135 137 144 146 145 128 151 136 154 152 |

| *Competitor's* | 87 89 76 88 86 90 88 91 84 94 90 97 92 93 87 |
| *batteries* | 89 87 92 94 87 88 93 83 89 88 90 85 95 87 96 |

a Enter the data into a graphics calculator or statistics package.

b Are there any outliers? Should they be deleted before we start the data analysis?

c Compare the measures of centre and spread.

d Obtain a side-by-side boxplot for the data sets.

e Use **d** to describe the shape of each distribution.

f What conclusion can be made from this data?

g Frank doubted the validity of analysing this data. What do you think his argument was based on?

8 The graphs and statistics below are for Hugh's screw caps in the **Opening problem**. Use them to help answer some of these questions:

a What is the problem that Hugh wants to solve?

b What are the advantages and disadvantages of the two forms of graphical representation?

c Are there any outliers and should they be deleted?

d What do the measures of the middle of the distribution tell you?

e What do the measures of spread (range and IQR) tell you?

f Is the interquartile range of much use in this case?

g Can a satisfactory conclusion be made? If so, what is it?
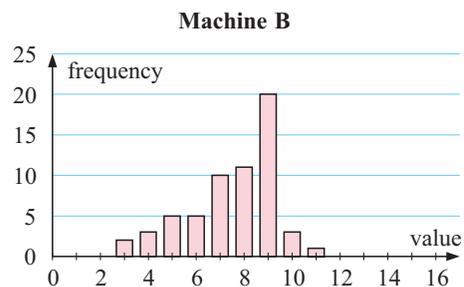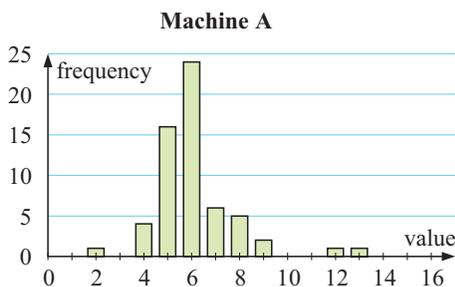
Machine A

Machine B

Machine A					
n	60	min	2	range	11
mean	6.1167	Q1	5	IQR	1.5
mode	6	median	6		
		Q3	6.5		
		max	13		

Machine B					
n	60	min	3	range	8
mean	7.5333	Q1	6.5	IQR	2.5
mode	9	median	8		
		Q3	9		
		max	11		
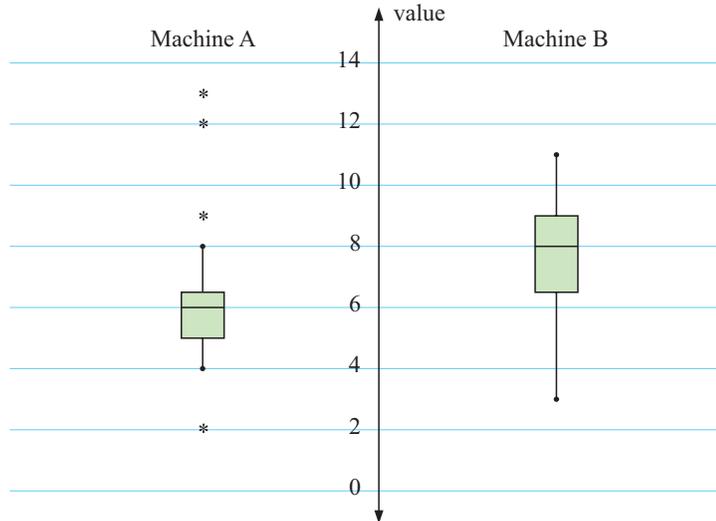
C   A STATISTICAL PROJECT

We are now equipped with the mathematics necessary to carry out our own statistical project. We will carry out our **statistical enquiry** or **investigation** using the following steps:

Step 1:   Examine the problem and pose the correct questions.

Step 2:   Collect sufficient data at random.

Step 3:   Organise the data in table form.

Step 4:   Display the data.

Step 5:   Analyse the data and make a conclusion in the form of a conjecture.

Step 6:   Write a full report.

INVESTIGATION   REACTION TIME

Many males believe that they *have a faster reaction time* than females of the same age. Is this true?

We could attempt to answer this question for high school students by collecting and analysing data. We need to make sure that the samples are sufficiently large to be properly representative.

Equipment needed: A metre ruler, pen, paper.

The experiment:

In this experiment your partner holds the ruler vertically from the top. Position your hand so that your thumb and forefinger are just below the lower end. These fingers should start 5 cm apart, but ready to catch the ruler when your partner lets it fall.

Your partner lets the ruler drop without warning and you catch it as quickly as you can. Your reaction time is related to the length of the ruler.

What to do:

1   Making the timing device:
We can place small markers on the ruler corresponding to reaction times of 0.06, 0.08, 0.10, 0.12, ... up to 0.40 seconds.

These markers are placed from the bottom end of the ruler. The formula $s = 490t^2$ is used to calculate the distances s in cm.

For example: when $t = 0.06$, $s = 490 \times (0.06)^2 \approx 1.764$ cm ≈ 18 mm

when $t = 0.08$, $s = 490 \times (0.08)^2 \approx 3.136$ cm ≈ 31 mm.

2   Obtaining the data:
For each person, measure the reaction time over 10 drops. Find the average reaction time for each person. A random sample of at least 25 males and 25 females should be used.

3   Analysing the data:

- State the problem which you are attempting to solve by statistical methods.
- Organise the data for males and females using a back-to-back stemplot and a side-by-side column graph.
- Find and record the five-number summary for both sets of data and test for outliers.
- Construct parallel boxplots for the data sets.
- Compare the distributions of the data for both males and females. Comment, giving the relevant statistics, on the centre, spread, and shape of the distributions.

4   Conjecture and report:
Make a conjecture from your findings, and write a brief report.

REVIEW SET 14A

1   Write brief comparative reports about the distributions illustrated in these parallel boxplots:
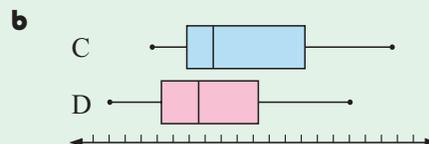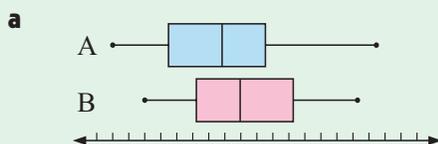
2 The given parallel boxplots represent the 100 metre sprint times for the members of two athletics squads.
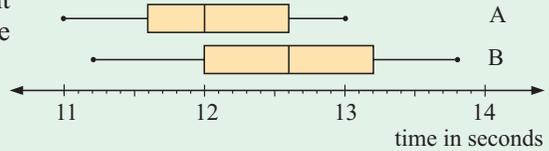
a Determine the 5-number summaries for both squads.

b Determine the **i** range **ii** interquartile range   for each group.

c Copy and complete:

  i The members of squad ...... generally ran faster times.

  ii The times in squad ...... were more varied.

3 Zac and Imran are football goalkeepers. A number of their goal kicks were measured (in metres) to determine their lengths. The results were:

Zac : 46 43 49 41 43 43 44 47 45 44 43 37 46 38 44
       46 45 42 43 44 42 42 46 40 41 41 40 38 44 42

Imran : 49 45 43 33 45 50 42 46 46 47 45 42 45 44 48
        35 45 38 54 45 40 51 46 49 46 44 44 45 46 47

a Represent the data with parallel boxplots.

b Who generally kicks the ball further? How did you come to this conclusion?

c Which goalkeeper has greater variability in the lengths of their kicks?

4 A researcher believes that boys are not as good at mathematics as girls. To examine this claim, 32 Year 10 boys and 32 Year 10 girls were selected at random. They were given a test containing 20 questions worth one mark each. The results were:

Boys : 18 10 11 13 12 14   7 15 14 13 16 12 16 17 15 13
        18 14 12 15 17 11 17 16 15   9 17 12 15 14 13 14

Girls : 8 14 16 19 12 14 15 13 10 16 15 15 14 18 17 17
        11 14 14 15 17 15 18 16 14   9 13 12 15 13 11 13

a State clearly the problem the researcher is trying to solve.

b Plot a side-by-side column graph for this data.

c Find 5-number summaries for each data set.

d Draw parallel boxplots to compare the statistics.

e For each data set, state the values of:

  i the median   **ii** the mean   **iii** the IQR.

f What conclusions can be drawn from the analysis?

REVIEW SET 14B

1 Write brief comparative reports about the distributions shown in these parallel boxplots:

a

b
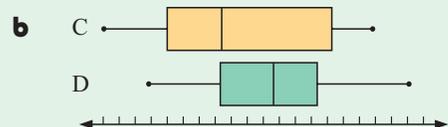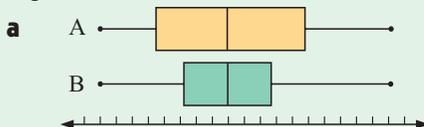
2 The parallel boxplot compares times taken by an athletics squad to run 1000 m at the start of the athletic season and at the end of it. The times are shown in seconds.

a For each set of data, find:

i the fastest time

ii the slowest time

iii the median time

iv the range

v the IQR.
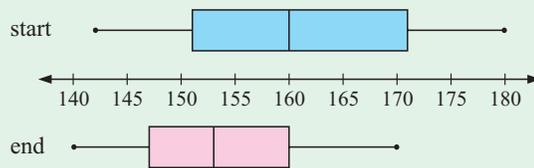
b What percentage of athletes took 160 seconds or more to run the 1000 m:

i at the start of the season   **ii** at the end of the season?

c Comment on any improvement during the season.

d At which recording time was the variability greatest?

3 A speed reading tutor wants to test the effectiveness of a new program on his group of readers. He records their ability before the program begins and then one month after it starts. The results in seconds are:

Before: 20.4 20.8 21.3 21.5 21.7 22.6 25.1 25.8 26.8 28.1
28.6 28.6 29.3 29.8 31.7 33.7 34.6 37.6 39.0 39.1

After: 20.4 20.5 21.0 21.5 21.8 22.2 22.4 25.8 26.9 28.3
28.4 29.6 30.2 31.2 31.2 32.7 33.5 34.6 35.4 42.7

a For these data sets find: **i** the mean **ii** the median **iii** the IQR.

b Form a conjecture about the effectiveness of the new program.

4 Two taxi drivers, Peter and John, are friendly rivals. Each claims that he is the more successful driver. They agree to randomly select 25 days on which they work and record their fare totals for the day. The data collected to the nearest dollar were:

Peter: 194   99 188 208   95 168 205 196 233 183 155 190 147
116 132 153 205 191 182 118 140 270   93 154 223

John: 260 152 127 163 180 161 110 153 139 110 147 162 223
139 142 161   97 116 129 215 241 160 159 174 158

a State clearly the problem which needs to be solved.

b Draw back-to-back stemplots for this data.

c Find measures of the middle of each data set.

d Find measures of the spread of each data set.

e Which driver do you conjecture is more successful? Give reasons for your answer.

Chapter 15

Transformation geometry

TRANSFORMATIONS

A change in the size, shape, orientation or position of an object is called a **transformation**.

Reflections, rotations, translations and enlargements are all examples of transformations. We can describe these transformations mathematically using **transformation geometry**.

In **transformation geometry** figures are changed (or transformed) in size, shape, orientation or position according to certain rules.

The original figure is called the **object** and the new figure is called the **image**.

One transformation of interest is known as the *rubber stretching transformation*. Imagine drawing a figure on a piece of rubber such as a balloon and then stretching the rubber to distort the shape. We could create almost any other shape we could wish.

For example, we can see how a circle can be transformed into a square:

The following example illustrates how a salmon can be transformed into a schnapper. {Ref: On Growth and Form. *D'Arcy W Thompson*}
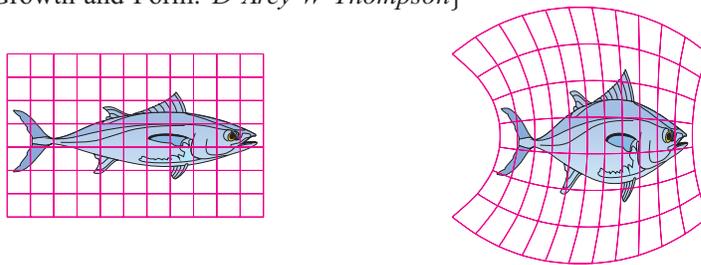
The transformations that we will consider in this chapter are:

- **translations**, where every point moves a fixed distance in a given direction
- **rotations** where we turn objects about a point
- **reflections** or mirror images
- **enlargements** and **reductions**, where objects are transformed into larger or smaller objects of the same shape.

For example:

a **translation**

> When this shape is repeatedly translated it forms a **tessellation**. It can completely cover an area without gaps.
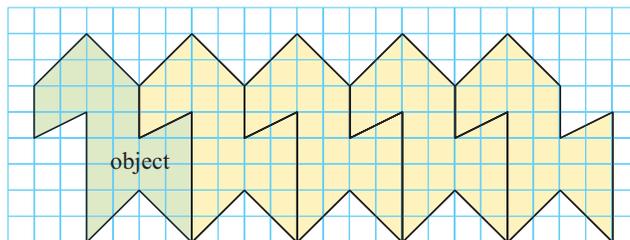
slide the original 4 units to the right to find each new image

a **rotation** about O

object

120°

centre of
rotation

O

image

240°

image
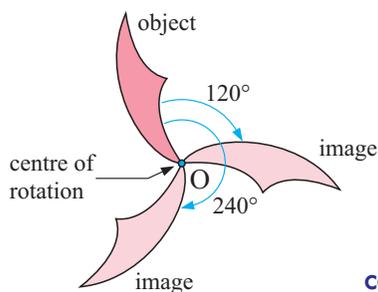
**COMPUTER
DEMO**

a **reflection**

object

image

mirror line

an **enlargement**

centre of
enlargement

object

image
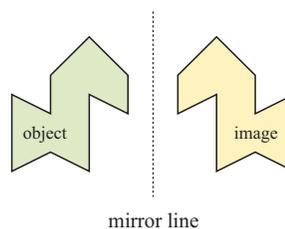
**COMPUTER
DEMO**

> The study of enlargements can be applied to **similar figures** and has applications in **trigonometry**.
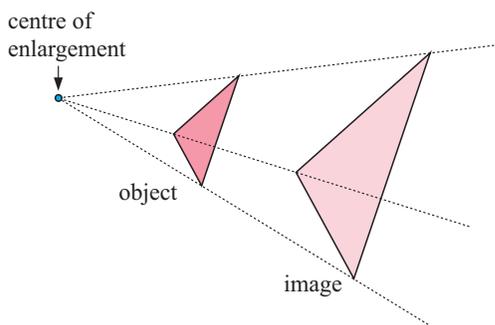
OPENING PROBLEM

Dean needs to accurately find the height of a tree in his garden. Local Council regulations prohibit the growth of garden trees beyond 20 m. They argue that in storm conditions, falling branches and uprooted large trees are dangerous to houses and lives.

1.2 m

1.72 m   14.68 m
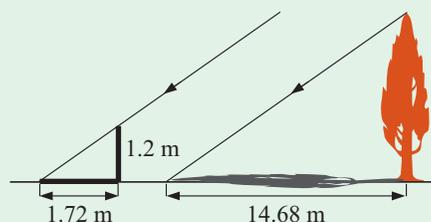
As Dean cannot climb the tree he decides to use shadows to help solve the problem. On a windless sunny day he stands a 1.2 m stick vertically on the ground. The length of the stick's shadow is 1.72 m and at the same time the tree's shadow measures 14.68 m.

Can you:

- produce an accurate scale drawing of the situation
- use the scale drawing to find the tree's height and the angle of elevation of the sun (the angle that the sun's rays make with the horizontal ground)
- recognise the transformation occurring in this problem
- use ratios to find the height of the tree?

A   TRANSLATIONS

A **translation** is a transformation in which every point of a figure moves a fixed distance in a given direction.
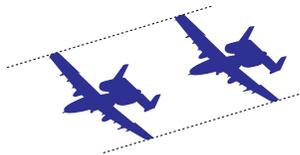
Consider this translation:
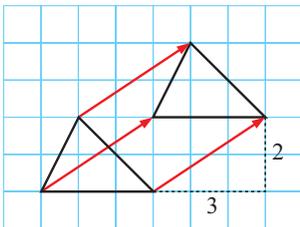
In this case, every point on the object moves 3 units to the right and 2 units up to form the image.

We can *specify* or give details of the transformation using:

- the **translation vector** $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ where the top number indicates the x-step and the bottom number indicates the y-step

- a **directed line segment** which clearly shows the direction and distance.

Using Pythagoras, we see that every point moves a distance of $\sqrt{13}$ units.

Example 1   ◀) Self Tutor

Translate the given figure by the translation vector $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$.
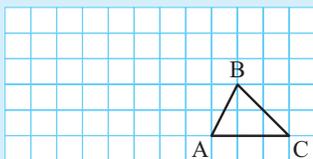
The translation $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ means 5 units to the left and 2 units up.

2 up

5 left
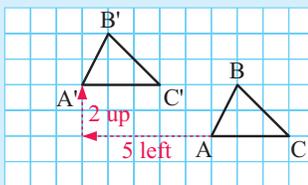
A′ should be read as "the new point A".

EXERCISE 15A

1 The object A has been translated to give an image B in each diagram. Specify the translation in each case using a translation vector.

a

b

c

d   **e**   **f**
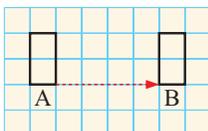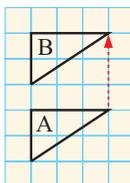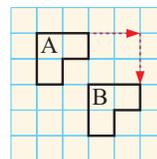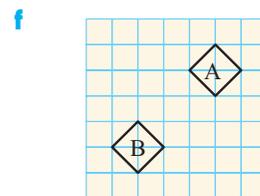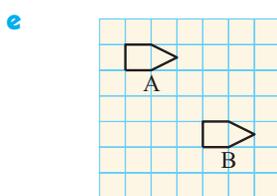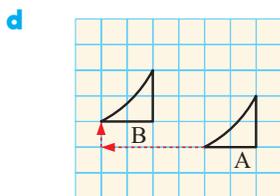
2 The figure A has been translated to B, then B
has been translated to C.

    a Give the translation vector from A to B.

    b Give the translation vector from B to C.

    c What translation vector would move A to C?
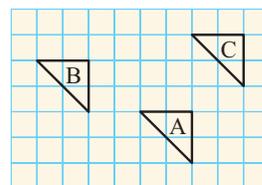
3 Copy the figures onto grid paper and translate using the vector or directed line segment
given.

a under $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** under $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **c** under $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

d under   **e** under   **f** under
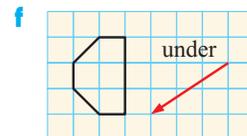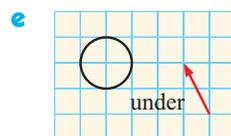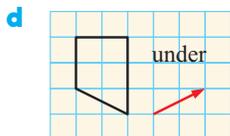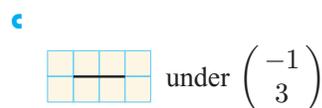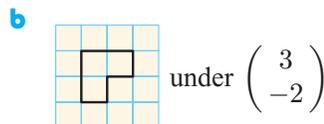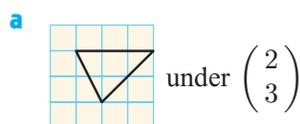
4 Which of the
following
represent
translations?
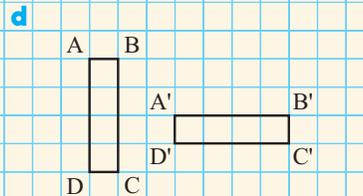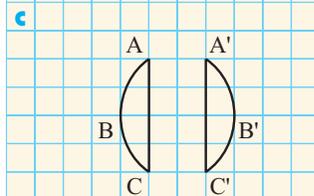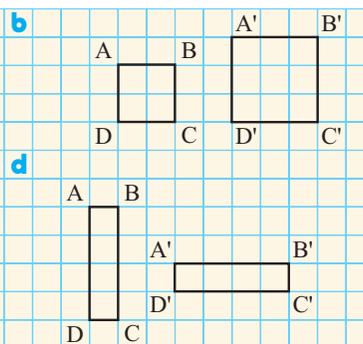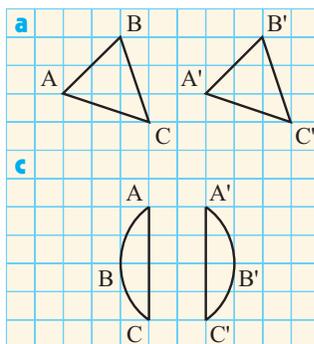
5   **a** Write down the coordinates of A, B, C and D.

    b Each point is translated 5 units to the right and 2
units up. What are the coordinates of the image
points A′, B′, C′ and D′?
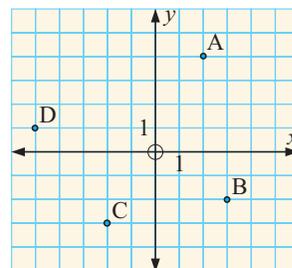
6 Draw triangle ABC where A is (−1, 3), B is (4, 1)
and C is (0, −2).

    a Translate the figure with translation vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

    b State the coordinates of the image vertices A′, B′ and C′.

    c Find the slope of [AA′], [BB′] and [CC′].

    d By what distance has every point moved?

B   ROTATIONS

When a wheel moves about its axle, we say that the wheel *rotates*.

The centre point on the axle is called the **centre of rotation**.

The angle through which the wheel turns is called the **angle of rotation**.

Other examples of rotation are the movement of the hands of a clock, and opening and closing a door.
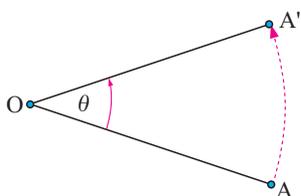
Rotations are transformations in which one point called O is fixed and all other points are rotated through the same angle θ about O.

O is called the **centre of the rotation** and θ is known as **the angle of rotation**.

Notice that $OA = OA'$ and AA' is an arc of a circle with centre O.

To completely describe a rotation we need to know:

- the **centre** of the rotation
- the **direction** of the rotation (clockwise or anticlockwise)
- the **angle** of the rotation.

ACTIVITY   ROTATIONS USING PLASTIC SHEETS OR ROTAGRAMS

Rotations of figures can be nicely demonstrated on an overhead projector.

Suppose we wish to rotate a figure such as a triangle about a point O.

centre of rotation

figure to be rotated

pin

tracing paper or rigid clear plastic sheet

image
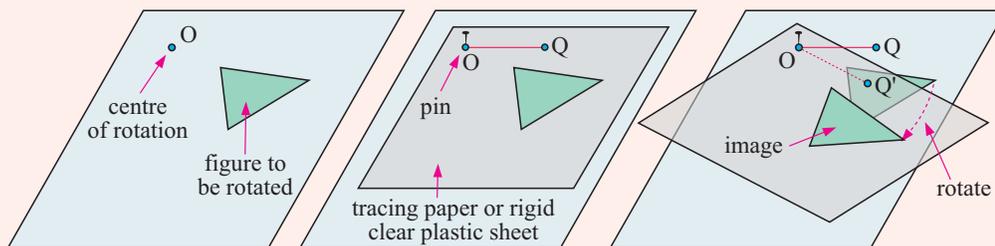
rotate

Steps:

- On an overhead transparency draw the figure to be rotated and indicate the centre of rotation O. Draw a line segment [OQ] from the centre of rotation.
- Place another transparency or piece of tracing paper over the figure to be rotated. Join the sheets together with a split pin at the centre of the rotation.

VIDEO CLIP

- Trace the object figure and the line segment onto the top sheet.
- Rotate the top sheet in the desired direction through the desired angle. This angle should be the angle formed between [OQ] and its image [OQ′].
- Use pins to mark the positions of the vertices of the image through the top sheet onto the bottom sheet. Draw the image on the bottom sheet.

What to do:

1 Using the method described, draw these diagrams on plastic sheet and rotate them *clockwise* through the angle given. Make sure that the centre of rotation O is also on the sheet.

    a Rotate 90° about O.   **b** Rotate 45° about O.   **c** Rotate 60° about O.
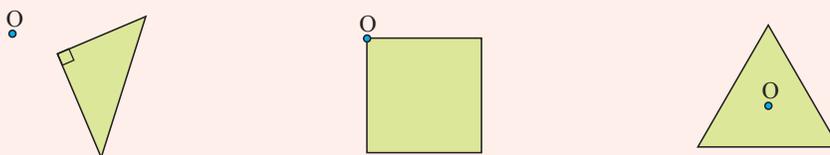
2   **a** Name any points on your sheet that do not move during a rotation.

    b What can you say about the size and shape of the image formed under a rotation?

    c Comment on your results in **1c**.

From the **Activity** above you should have found that for any rotation, the point O does not move. We call this an **invariant point**.

Example 2   ◀) **Self Tutor**

Rotate the given figures about O through the angle indicated:

  a   **b**   **c**

     180° clockwise 90° anticlockwise 90° clockwise

  a   **b**   **c**
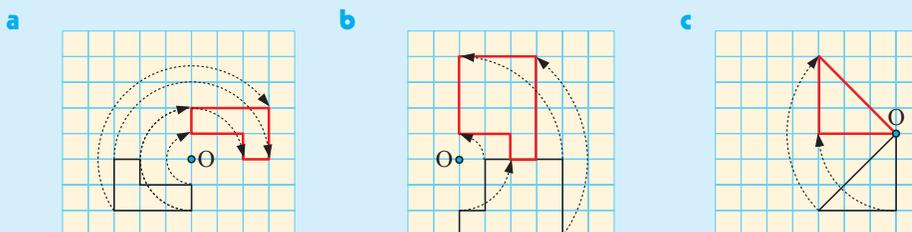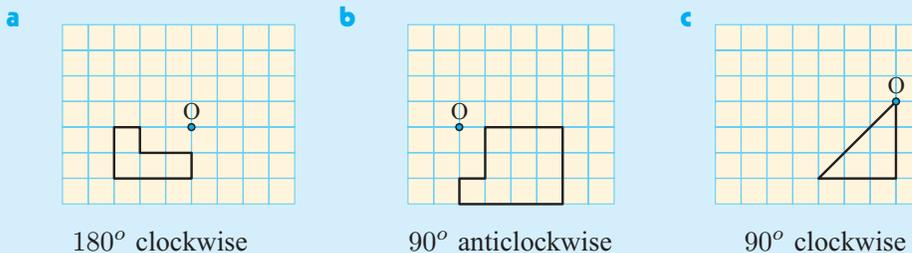
EXERCISE 15B

1 Rotate the figures through the angle given about O:

a

180°

b

180°

c

90° anticlockwise

d

90° clockwise

e

90° clockwise

f

90° anticlockwise
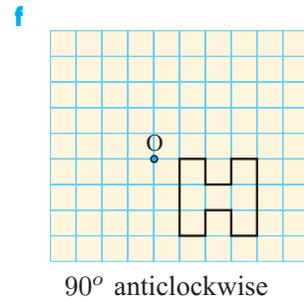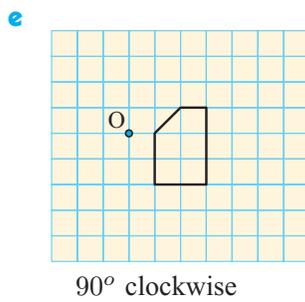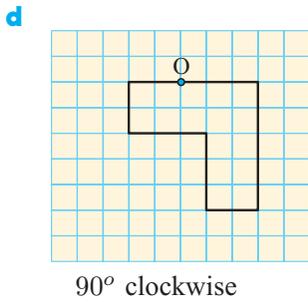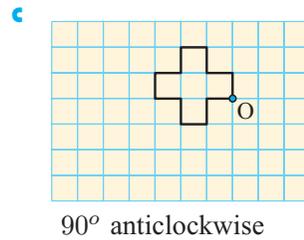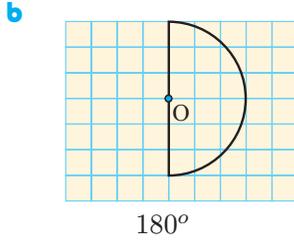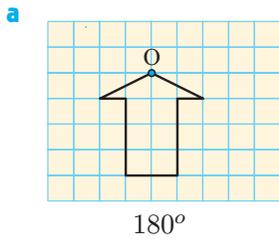
2 Which of the following letters of the alphabet either remain the same letter or become a different letter under a rotation of some kind? Do not include full rotations of 360°.

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

3 Which of the following transformations represent rotations?

a

b

c

d

4
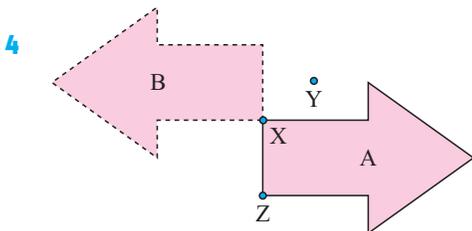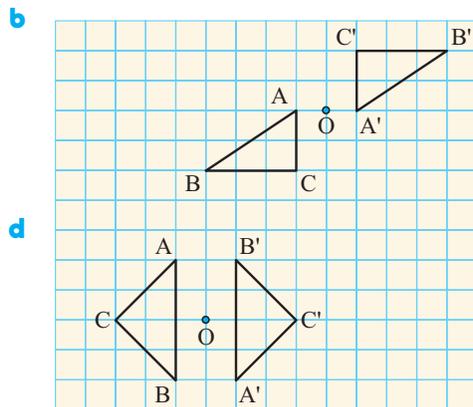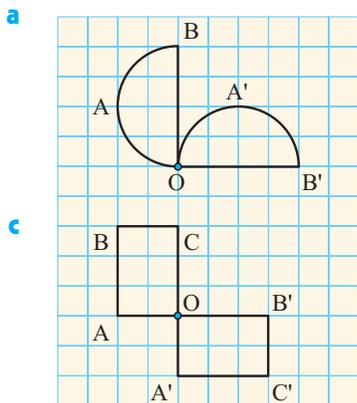
Figure B is the image of A after a rotation.

a Through how many degrees has A been rotated?

b Which point (X, Y or Z) was the centre of rotation?

5 A figure is rotated 75° clockwise about a point O. Describe a rotation which will return the figure to its original position.

Example 3   ◀ᴺ **Self Tutor**

Find the coordinates of the image of (3, 1) under a rotation about O through:

  a 90° anticlockwise   **b** 180° **c** 90° clockwise
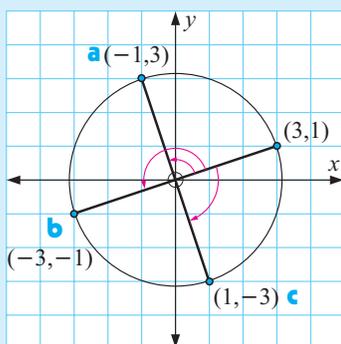
  a (3, 1)   becomes $(-1, 3)$

  b (3, 1)   becomes $(-3, -1)$

  c (3, 1)   becomes $(1, -3)$

---

6 Find the coordinates of the images of the following points under a 90° anticlockwise rotation about O:

  a (4, 1)   **b** $(5, -2)$ **c** $(0, -5)$ **d** $(-1, -5)$ **e** $(-3, 4)$

7 Find the coordinates of the images of the following points under a 180° rotation about O:

  a $(-3, 5)$ **b** (3, 0)   **c** (1, 4)   **d** $(-3, -2)$ **e** $(4, -2)$

8 Find the coordinates of the images of the following points under a 90° clockwise rotation about O:

  a $(0, -2)$ **b** (4, 2)   **c** $(-1, -5)$ **d** $(4, -2)$ **e** $(-3, 2)$

9 O is the midpoint of [AC] and AB $\neq$ BC.
Redraw the figure and rotate it through 180° about O.

  a Name the resulting figure which is made up of the original and its image.

  b Use the resulting figure to list its geometrical properties. Give reasons for your answer.
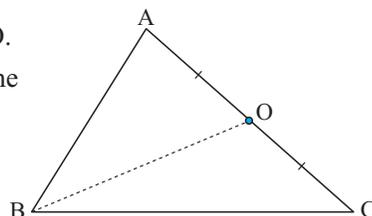
10
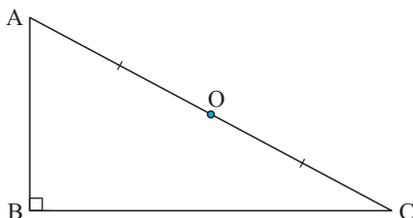
Repeat question **9** for the figure alongside.

C   REFLECTIONS

We encounter reflections every day as we look in a bathroom mirror, peer into a pond of water, or glance at the traffic in a car rear-view mirror.
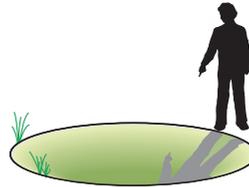
INVESTIGATION 1   REFLECTIONS

You will need: A mirror, paper, pencil, ruler.

What to do:

1   Make two copies of the figures shown below:

a   **b**   **c**
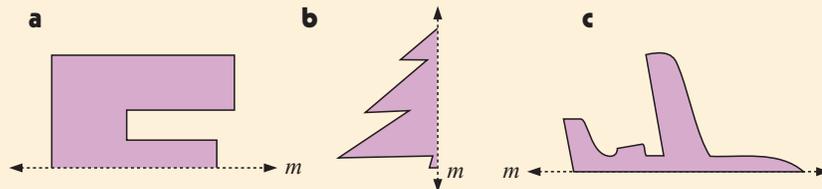
2   Put the mirror along the mirror line m on one copy. What do you notice in the mirror?

3   Draw the reflection as accurately as you can on the second copy.

4   Cut out the second copy with its reflection and fold it along the mirror line. You should find that the two parts of the figure can be folded exactly onto one another along the mirror line.

INVESTIGATION 2   PROPERTIES OF REFLECTION

The diagram shows a triangle and its reflection in a mirror line.

B B'

A   A'
    object   image

C   C'
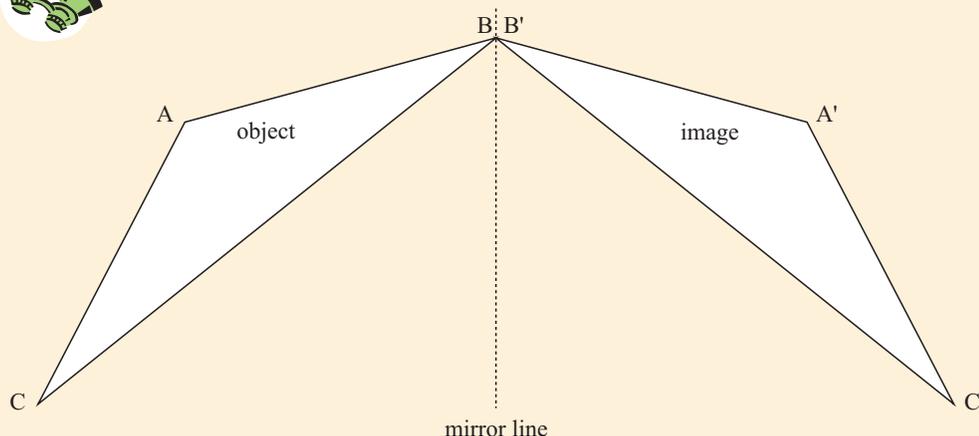
mirror line

What to do:

1 Copy the diagram.

2 Join point A to its image point A′ on the reflection, and join C to C′.

3 Measure the distance from A to the mirror line and the distance from A′ to the mirror line. What do you notice?

4 Measure the angles that the line [AA′] makes with the mirror line.

5 Repeat **3** and **4** using the point C and its image point C′.

6 Measure the lengths of sides of triangle ABC and the lengths of sides of triangle A′B′C′. What do you notice?

7 Measure the angles of triangle ABC and the angles of triangle A′B′C′. What do you notice?

8 What do you notice about the image of a point that is on the mirror line?

From the **Investigation**, you should have found that for any reflection:

- the image is as far behind the mirror line as the object is in front of it
- the line joining any image point to the corresponding point on the object is at right angles to the mirror line
- all lengths and angles are the same size in the image as they were in the object
- points on the mirror line do not move.

The mirror line is the **perpendicular bisector** of every point on the object and its corresponding point on the image.
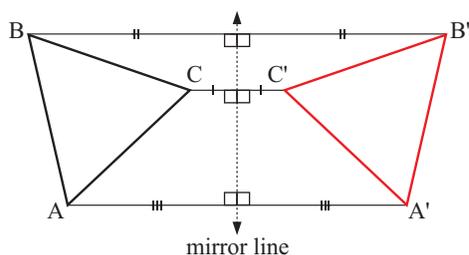
We can use these facts to help us draw reflections. It is easy to work on grid paper because we can count squares and see right angles.

Example 4   ◀) Self Tutor

Reflect the following figures in the given mirror lines:
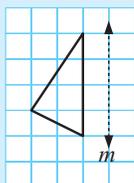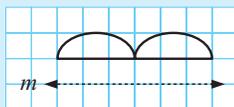
a   **b**   **c**
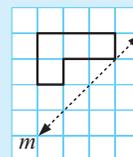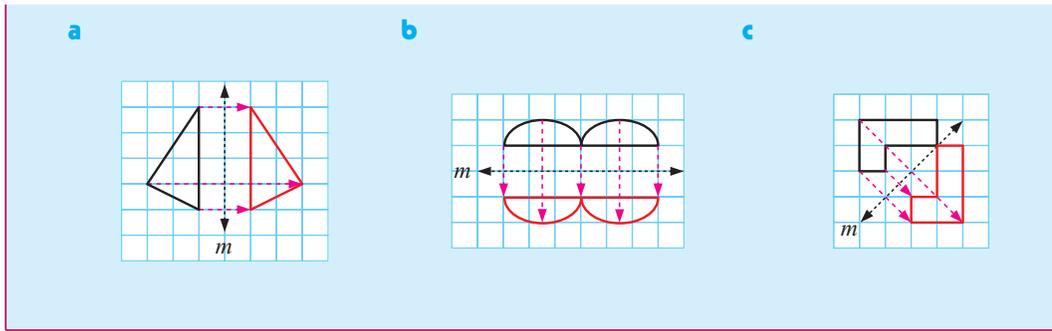
EXERCISE 15C.1

1 Copy the following figures onto grid paper and reflect them in the given mirror lines:
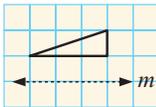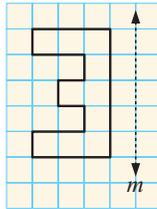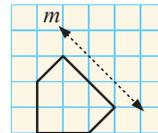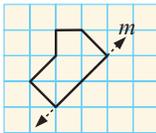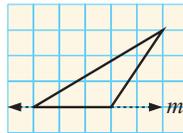
2 Which of the following transformations represent reflections?
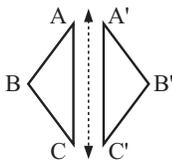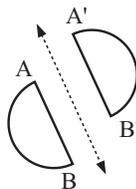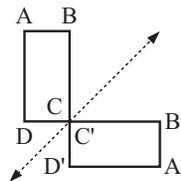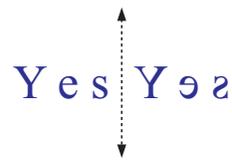
3 **a** Copy the word and reflect it in the line given:

    i   no   **ii**   think

  b Print your name and reflect it in a similar way. This is called mirror writing. Hold it up to a mirror and you should be able to read it properly.

  c 'TOOT' is one example of a word which looks the same in mirror writing. Can you think of some more? Check your answers in a mirror.

4   **a**   By plotting points on grid paper, complete the table alongside showing the images of the given points under a reflection in the x-axis.

	P	P'
i	(5, 2)	
ii	(−3, 4)	
iii	(−2, −5)	
iv	(3, −7)	
v	(4, 0)	
vi	(0, 3)	

b   Use your table from **a** to complete the following statement:

Under a reflection in the x-axis, (a, b) maps onto (...., ...).

5   **a**   Copy the table from question **4** and this time reflect the original points in the y-axis.

b   Use your table to complete the following statement:

Under a reflection in the y-axis, (a, b) maps onto (...., ...).

6   The line $y = x$ can be used as a mirror line.

a   Copy the table from question **4** and this time reflect the given points in the line $y = x$.

b   Copy and complete: For a reflection in the line $y = x$, point (a, b) becomes (....., ...).
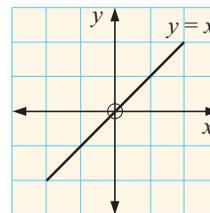
7   A roadway must go from A to the pipeline and then to B by the shortest route.
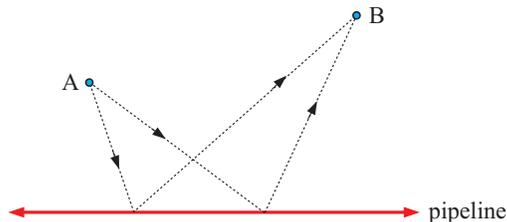
Alec Smart's solution is as follows:

Step 1:   Use the pipeline as a mirror line to reflect B.

Step 2:   Join [AB′], and let [AB′] meet the pipeline at X.

Step 3:   Join [BX].

Step 4:   AX + XB provides us with the shortest route.
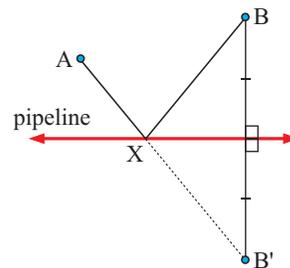
a   What can be said about triangle BXB′? Why?

b   Can you explain why Alec's solution is correct?

Hint: Let Y be any point on the pipeline other than X. Join [AY] and [YB′].

LINE SYMMETRY

A figure has an **axis** or **line of symmetry** if it can be reflected in that line so that each half of the figure is reflected onto the other half of the figure.

For example, an isosceles triangle has one axis of symmetry. The line drawn from its apex to the midpoint of its base is the axis of symmetry.
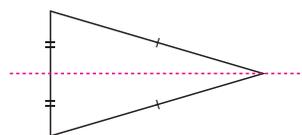
A square has 4 axes of symmetry.
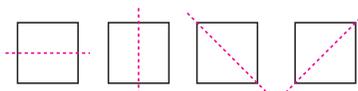
A figure has an axis of symmetry if it can be folded onto itself along that line.

Example 5   ◄)) **Self Tutor**

For the following figures, draw all lines of symmetry:
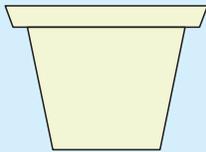
a   b   c
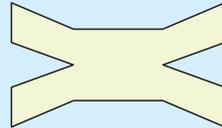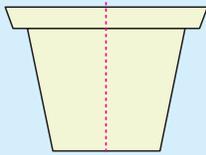
a   b   c
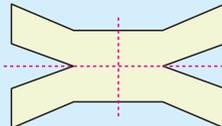
1 line of symmetry   no lines of symmetry   2 lines of symmetry

EXERCISE 15C.2

1 Copy the following figures and if possible draw their lines of symmetry:
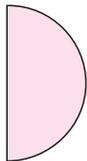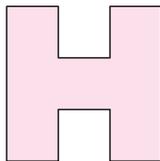
a   b   c
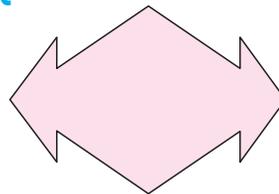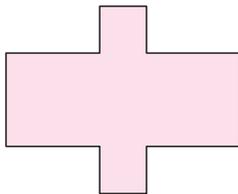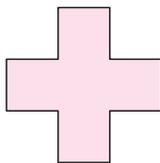
d   e   f
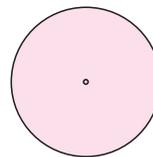
2 Draw the following figures and if possible draw their axes of symmetry. Cutting and folding may help. Record the number of axes of symmetry for each.

a a square	**b** an equilateral triangle	**c** a parallelogram
d a rectangle	**e** a rhombus	**f** a regular pentagon
g a circle	**h** a kite	**i** an isosceles triangle

D   ENLARGEMENTS AND REDUCTIONS

We are all familiar with enlargements in the form of photographs, zoom tools in computer software, or looking through a microscope. Plans and maps are examples of reductions.

The size of the image is enlarged or reduced, but the proportions are the same as the original. Most photocopiers can make images either smaller or larger than the original.
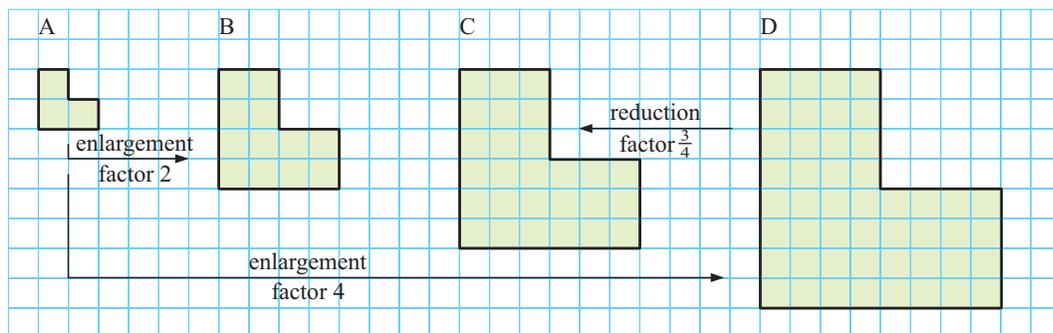
Look carefully at the diagrams above, with the smallest shape A as our starting shape. Shape B has each length twice the size it was in shape A. We say shape B is an **enlargement** of shape A with a **scale factor** of 2.

> A scale factor of 2 **does not** mean that the shape formed is 2 times larger than the original. In fact, it is 2×2, or 4 times larger. Count the squares inside A and B to confirm this.

Shape C is an enlargement of shape A with a scale factor of 3.

Shape D is an enlargement of shape A with a scale factor of 4.

You might also notice that shape D is an enlargement of shape B with a scale factor of 2.

Now, suppose that shape D is our starting shape.

Shape B is a **reduction** of shape D with scale factor $\frac{1}{2}$ since each length gets halved. Similarly, shape A is a reduction of shape D with scale factor $\frac{1}{4}$.

Shape C is a reduction of shape D with scale factor $\frac{3}{4}$. This means that for every 3 units of length in C, there are 4 units of length in D.

> For any **enlargement** the scale factor is **greater** than 1.
>
> For any **reduction** the scale factor is **less** than 1.

EXERCISE 15D.1

1 In the following diagrams, A has been enlarged to A′. Find the scale factor.

a

b

c
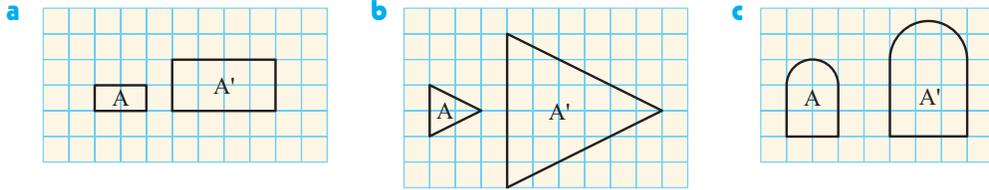
2 In the following diagrams, B′ is a reduction of B. Find the scale factor.

a

b

c
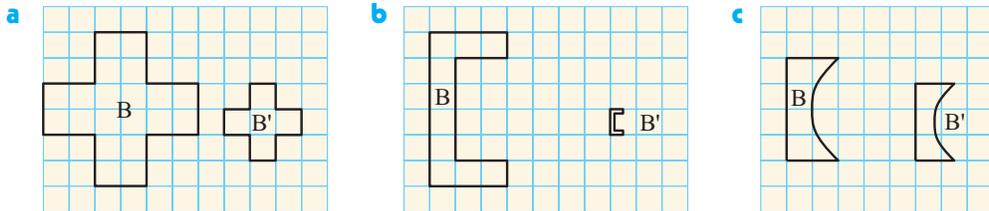
DISCUSSION   **CENTRE OF ENLARGEMENT**

Look carefully at the diagrams below. In each case triangle PQR has been enlarged to triangle P′Q′R′ using a scale factor of 3.

A

B

C
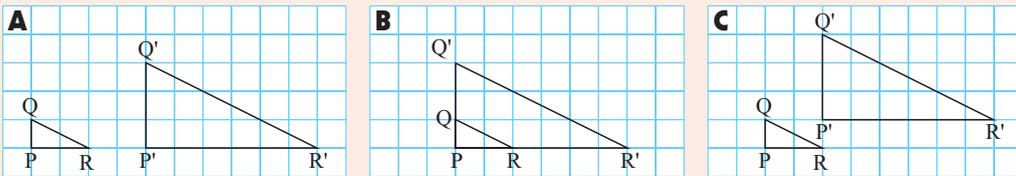
1 What is the difference between these enlargements?

2 What does 'enlargement with scale factor 3' tell you?

3 What additional instructions would you need to be able to draw an enlargement of the right size in the right position?

A

C
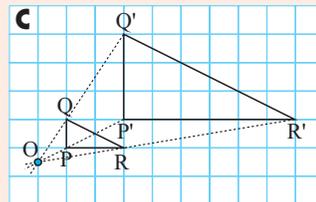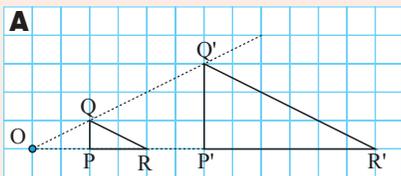
Consider again the diagrams A and C from above. If we draw lines through each original point and its corresponding point on the enlargement, the lines always meet at a point. We label this point O and call it the **centre of enlargement**.

Notice on each diagram that length OP′ = 3 times length OP, length OQ′ = 3 times length OQ and length OR′ = 3 times length OR. This is because the scale factor was 3.

4 What is the centre of enlargement in diagram B?

Example 6   ◄ŋ) **Self Tutor**

Find the image of the following figures for the centre of enlargement O and the scale factor given:

a

b

c
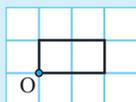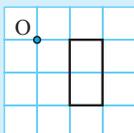
scale factor 2   scale factor 3
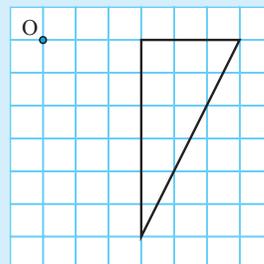
a
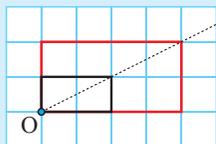
b
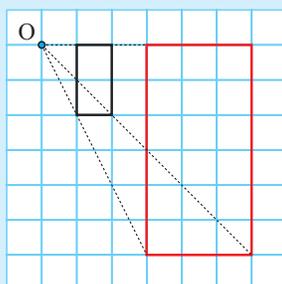
c
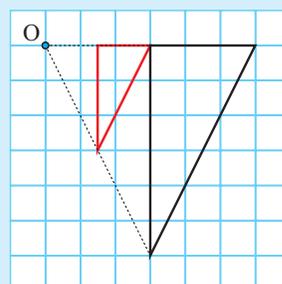
Sometimes the centre of enlargement can lie within the original figure.

Example 7   ◄ŋ) **Self Tutor**

Find the image of the following figures for the centre of enlargement O and the scale factor given:

a

b
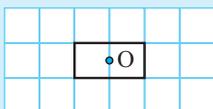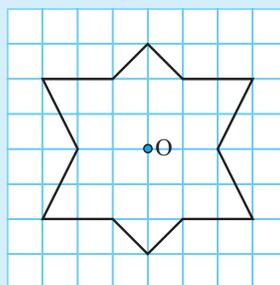
scale factor 2   scale factor $\frac{1}{2}$

Scale factor $\frac{1}{2}$ means a reduction.

a
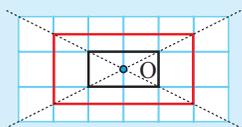
b
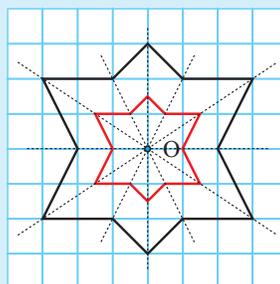
EXERCISE 15D.2

1 Copy the following diagrams. Locate the centre of enlargement for each by drawing lines on your diagrams if necessary:

a

b

c
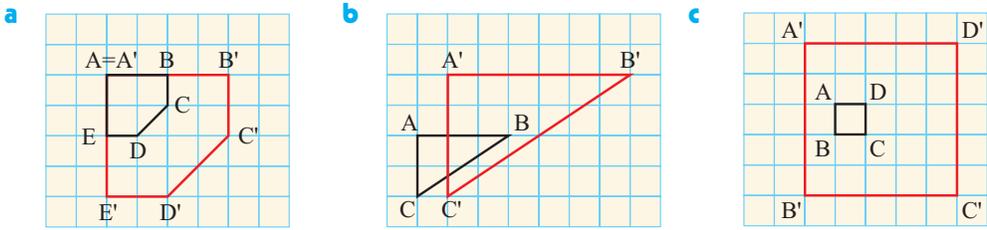
2 Copy the given figures. Enlarge or reduce them with the given scale factor and centre of enlargement:
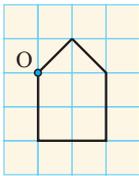
a scale factor 2

b scale factor 3
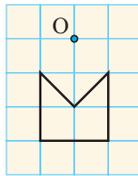
c scale factor $\frac{1}{2}$
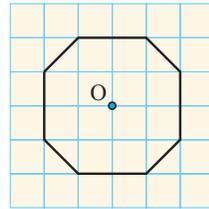
d scale factor 2
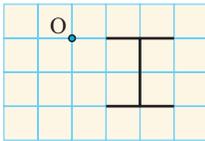
e scale factor $\frac{1}{3}$
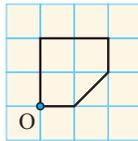
f scale factor 3
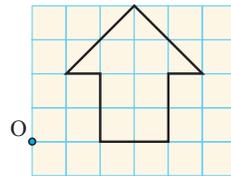
3 Copy the given figures. Enlarge or reduce them with the given scale factor and centre of enlargement:
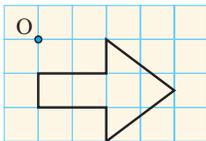
a scale factor 2

b scale factor $\frac{1}{2}$
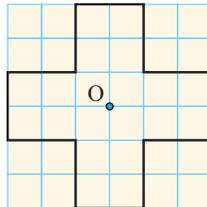
c scale factor 3
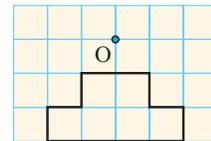
4 For the examples in question **3**, copy and complete the table alongside. The areas are in square units.

Part	Area of object	Area of image	Scale factor, k
a			
b			
c			

In each case find the value of the fraction $\dfrac{\text{area of image}}{\text{area of object}}$ and k^2.

State, in equation form, how the area of the image is connected to the area of the object.

5 Jason made a $\frac{1}{100}$ scale model of a yacht. The mainsail of his scale model had an area of 56 cm^2. What was the actual sail area, in m^2, of the full sized yacht?

6 A 10 cm by 8 cm photograph is to be enlarged so that the longer side will be 25 cm.

   a What is the value of k, the enlargement factor?

   b Find the length of the enlarged shorter side.

   c The cost of printing depends on its area. If the smaller photograph costs $3.00 to print, find the cost of printing the larger photograph.

7 A gardener wishes to double the area of his 5 m by 8 m rectangular garden bed. However, he likes its shape and so wants the new garden bed to be an enlargement of the old one. What dimensions must he make the new garden bed?

E   TESSELLATIONS
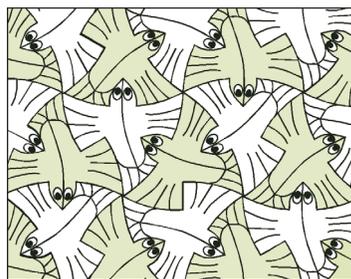
Maurits Escher, who died in 1972 aged 74, created many clever and fascinating patterns called **tessellations**. The pattern above resembles one of his better known examples. Notice how the shapes fit together with no gaps.

Here are some tessellations made from polygons:
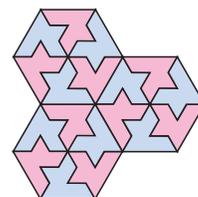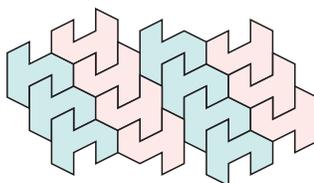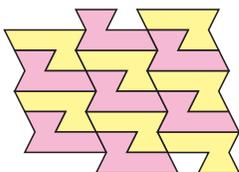
Each of the three tile patterns is made by using tiles of the same size and shape. They fit together with no gaps.

For a tessellation the shapes must fit together with no gaps.

BRICK PAVING

Simple tessellations can be formed using rectangles. We see this often with bricks and tiling.
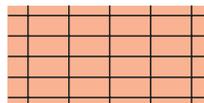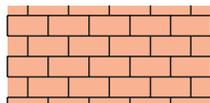
These tessellations are both formed by translations only. Can you think why brick walls are usually constructed using the first pattern rather than the second?

Two more tessellations of rectangles are shown below. These involve translations and rotations, and are common patterns for pavers.

Herringbone   Basketweave
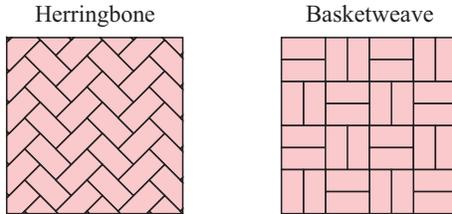
For these particular patterns to work, the length of each rectangle needs to be twice the width.

It is clear that rectangles tessellate in a variety of ways.

The only **regular** polygons to tessellate are equilateral triangles, squares, and regular hexagons:
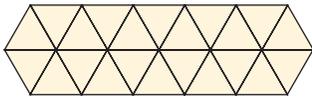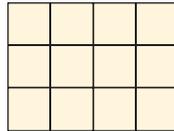
- **equilateral triangles**   - **squares**   - **hexagons**
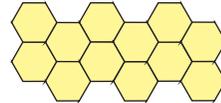
Example 8   ◀)) **Self Tutor**

Draw tessellations using the following shapes:

a   b
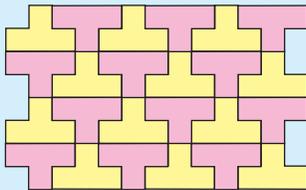
a   b
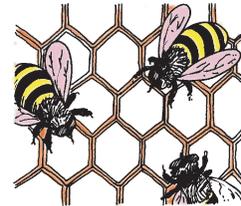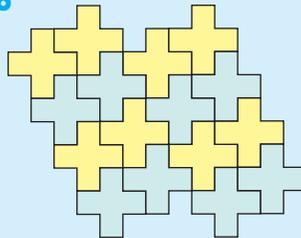
We can use the three regular polygons or other shapes which we know tessellate as a starting point to create more interesting patterns and designs. We can alter these basic shapes using translations, rotations and reflections in certain ways.

Translations can be used on parts of a regular shape to produce another more complex tessellating shape.
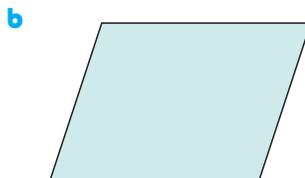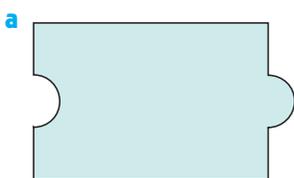
For example, we can start with a parallelogram, alter one side, then translate this alteration to the other side.
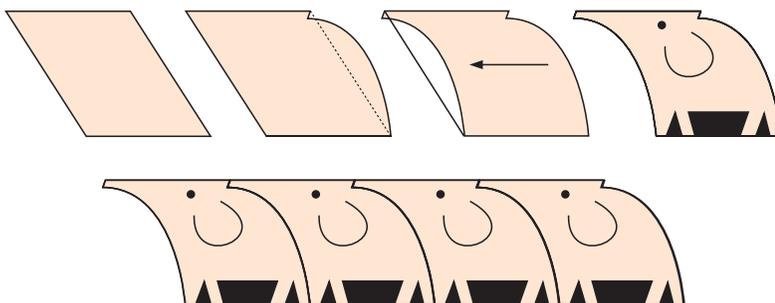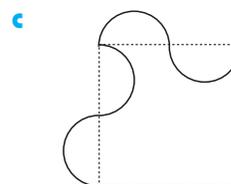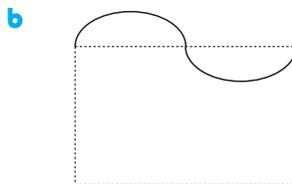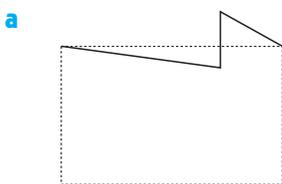
EXERCISE 15E

1 Draw tessellations using the following shapes:

 a   **b**   **c**
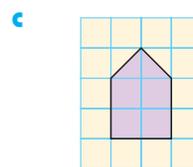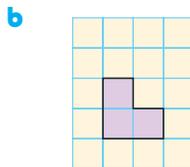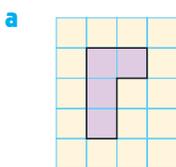
2 Would the following shapes tessellate?

 a   **b**   **c**
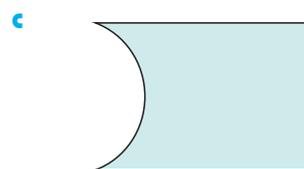
3 Some sides of the following rectangles have been altered as shown. Redraw the shapes and translate the changes to the opposite sides to create a tessellating shape.

 a   **b**   **c**

**4** The diagram shows a popular paver produced by a leading paver manufacturer.

 a Trace this design.

 b Show how it has been created by altering the sides of rectangle ABCD.

 c Draw several of these pavers to show that they tessellate.

 d Apart from their visual appeal, what advantage do these pavers have over rectangular ones?

 e The manufacturer has decided that it is time to market different pavers. He has hired you as a designer. Your job is to design two new pavers, both based on a rectangle.
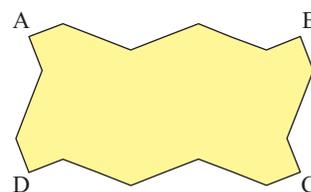
5 Create your own artistic tessellation by beginning with a simple shape which tessellates by translating.

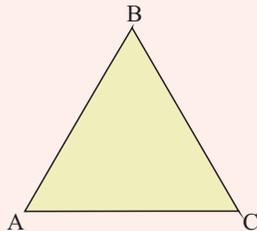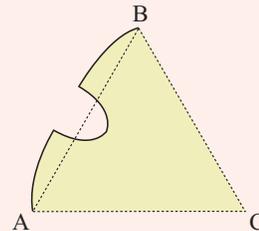## ACTIVITY   MAKE YOUR OWN 'ESCHER' TESSELLATION

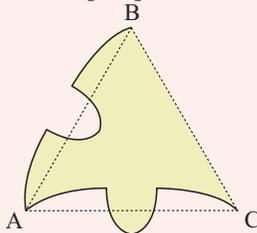To make the 'bird' tessellation shown at the start of this section, follow these steps:

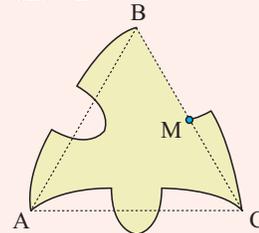*Step 1:* Begin with an equilateral triangle.
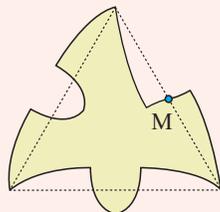
Step 2: Change side [AB] as shown.

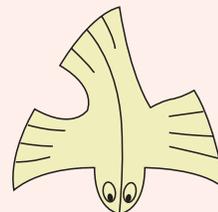*Step 3:* Rotate the change 60° about vertex A to side [AC].

Step 4: Change side [BC] as shown.

Step 5: Rotate the change 180° about the midpoint M.

Step 6: Add the final detail.

What to do:

- Photocopy your 'bird' many times and carefully cut each one out with scissors.
- Show how six of these shapes tessellate by rotating them about the point A.
- Continue the tessellation to cover a wider area.
- Repeat the exercise using figures of your own design.

WHAT DETERMINES COIN SIZES?

LINKS
click here

Areas of interaction:
Human ingenuity
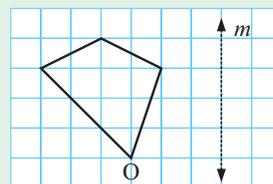
REVIEW SET 15A

1 Copy the figure alongside. On separate diagrams carry out the following transformations:

  a   a translation under $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

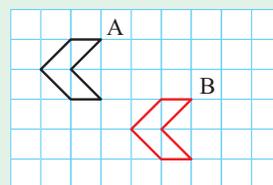 **b**   a rotation about O, 90° clockwise

  c   a reflection in the mirror line m

  d   an enlargement with scale factor 2 and centre of enlargement O.

2 Figure A has been translated to give the image B.

State the translation vector.

3 Draw triangle ABC where A is $(3, 3)$, B is $(-3, -2)$ and C is $(3, -2)$.

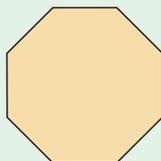 **a**   Translate the figure under translation vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

  b   State the coordinates of the image vertices A′, B′, C′.

4 By plotting points on grid paper, complete the table alongside which shows the images of the given points under a reflection in the x-axis.

	P	P'
a	$(1, 4)$	
b	$(-2, 3)$	
c	$(-3, -1)$	
d	$(4, 2)$	

5 Copy the following figures and draw in all axes of symmetry:

  a   **b**   **c**

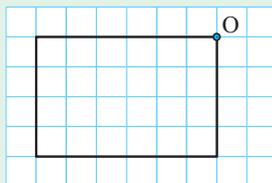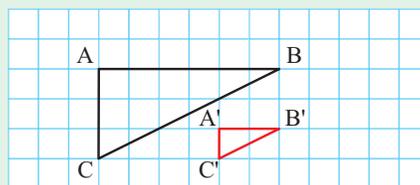**6** **a**   Reduce with scale factor $\frac{1}{2}$ and centre O.

  b   Copy the given diagram. Locate the centre of enlargement by drawing lines on your diagram.

REVIEW SET 15B

1 Copy the figure alongside. On separate diagrams carry out the following transformations:

    a a translation under $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

    b a rotation about O, $180°$ clockwise

    c a reflection in the mirror line m

    d an enlargement with scale factor 2 and centre of enlargement O.
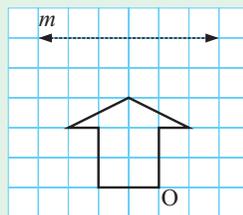
2 Figure A has been translated to B, then B has been translated to C.

    a Give the translation vector from A to B.

    b Give the translation vector from B to C.

    c What translation vector would move A to C?
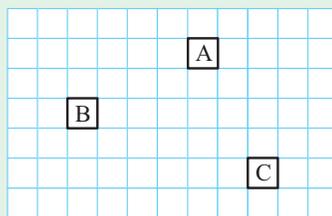
3 Find the coordinates of the images of these points under a $90°$ anticlockwise rotation about O:

    a $(0, 2)$ **b** $(-5, 0)$ **c** $(1, -3)$ **d** $(-2, -4)$ **e** $(-3, 2)$

4 By plotting points on grid paper, complete the table alongside which shows the images of the given points under a reflection in the y-axis.

	P	P'
a	$(1, 5)$	
b	$(-2, 4)$	
c	$(-4, -3)$	
d	$(3, 2)$	

5 Draw any rectangle and draw in all of its lines of symmetry.

6 A figure is enlarged with a scale factor of 2. What happens to the area of the figure?

7 Which of the following shapes will tessellate? Draw a diagram to illustrate your answer.
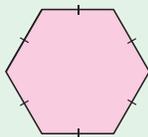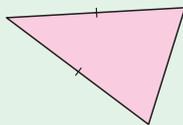
    a   **b**   **c**

Chapter **16**

Quadratic equations

Contents:

OPENING PROBLEM

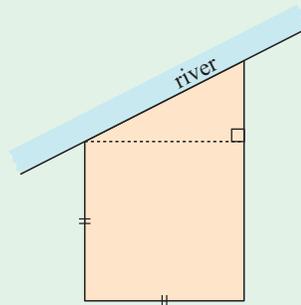Alexander has just bought a property. It is made up of a square and a triangle. He knows from the council plans that its area is $16\,800$ m^2. He needs 400 metres of fencing for three boundaries of the property, as he does not fence the river bank.

What is the length of the river bank boundary?

Equations of the form $ax + b = 0$ where $a \neq 0$ are called **linear equations** and have *only one* solution.

For example, $2x + 5 = 0$ is the linear equation with $a = 2$ and $b = 5$. It has the solution $x = -\frac{5}{2}$.

Equations of the form $ax^2 + bx + c = 0$ where $a \neq 0$ are called **quadratic equations**. They may have *two*, *one* or *zero* solutions.

Here are some simple quadratic equations which clearly show the truth of this statement:

Equation	$ax^2 + bx + c = 0$ form	a	b	c	Solutions	
$x^2 - 1 = 0$	$x^2 + 0x - 1 = 0$	1	0	-1	$x = 1$ or $x = -1$	**two**
$(x - 3)^2 = 0$	$x^2 - 6x + 9 = 0$	1	-6	9	$x = 3$	**one**
$x^2 + 1 = 0$	$x^2 + 0x + 1 = 0$	1	0	1	none as x^2 is always $\geqslant 0$	**zero**

Now consider the example $x^2 + 5x + 6 = 0$.

If $x = -2$,
$$x^2 + 5x + 6$$
$$= (-2)^2 + 5 \times (-2) + 6$$
$$= 4 - 10 + 6$$
$$= 0$$

and if $x = -3$,
$$x^2 + 5x + 6$$
$$= (-3)^2 + 5 \times (-3) + 6$$
$$= 9 - 15 + 6$$
$$= 0$$

$x = -2$ and $x = -3$ both satisfy the equation $x^2 + 5x + 6 = 0$, so we say that they are both **solutions**.

But, how do we find these solutions without using trial and error?

In this chapter we will discuss several methods for solving quadratic equations, and apply them to practical problems.

A   QUADRATIC EQUATIONS OF THE FORM $x^2 = k$

Consider the equation $x^2 = 2.$

Now $\sqrt{2} \times \sqrt{2} = 2$, so $x = \sqrt{2}$ is one solution,

and $(-\sqrt{2}) \times (-\sqrt{2}) = 2$, so $x = -\sqrt{2}$ is also a solution.

Thus, if $x^2 = 2$, then $x = \pm\sqrt{2}$.

SOLUTION OF $x^2 = k$

If $x^2 = k$ then $\begin{cases} x = \pm\sqrt{k} & \text{if } k > 0 \\ x = 0 & \text{if } k = 0 \\ \text{there are \textbf{no real solutions}} & \text{if } k < 0. \end{cases}$

> $\pm\sqrt{2}$ is read as 'plus or minus the square root of 2'

This principle can be extended to other perfect squares.

For example, if $(x - 1)^2 = k$ then

$$x - 1 = \pm\sqrt{k} \quad \text{provided} \quad k > 0.$$

Example 1   ◀) **Self Tutor**

Solve for x: **a** $x^2 + 3 = 6$ **b** $3 - 2x^2 = 7$

a $x^2 + 3 = 6$
$\therefore \quad x^2 = 3$ {subtracting 3 from both sides}
$\therefore \quad x = \pm\sqrt{3}$

b $3 - 2x^2 = 7$
$\therefore \quad -2x^2 = 4$ {subtracting 3 from both sides}
$\therefore \quad x^2 = -2$ {dividing both sides by -2}

which has no solutions as x^2 cannot be < 0.

Example 2   ◀) **Self Tutor**

Solve for x:
a $(x - 2)^2 = 25$ **b** $(x + 1)^2 = 7$

a $(x - 2)^2 = 25$
$\therefore \quad x - 2 = \pm\sqrt{25}$
$\therefore \quad x - 2 = \pm 5$
$\therefore \quad x = 2 \pm 5$
$\therefore \quad x = 7 \text{ or } -3$

b $(x + 1)^2 = 7$
$\therefore \quad x + 1 = \pm\sqrt{7}$
$\therefore \quad x = -1 \pm \sqrt{7}$

> For equations of the form $(x \pm a)^2 = k$ we do not expand the LHS.

EXERCISE 16A

1 Solve for x:

    a $x^2 = 16$
        b $2x^2 = 18$
        c $3x^2 = 27$

    d $12x^2 = 72$
        e $3x^2 = -12$
        f $4x^2 = 0$

    g $2x^2 + 1 = 19$
        h $1 - 3x^2 = 10$
        i $2x^2 + 7 = 13$

2 Solve for x:

    a $(x - 2)^2 = 9$
        b $(x + 4)^2 = 25$
        c $(x + 3)^2 = -1$

    d $(x - 4)^2 = 2$
        e $(x + 3)^2 = -7$
        f $(x - 2)^2 = 0$

    g $(2x + 5)^2 = 0$
        h $(3x - 2)^2 = 4$
        i $4(2x - 1)^2 = 8$

B   THE NULL FACTOR LAW

For quadratic equations which are not of the form $x^2 = k$, we need an alternative method of solution. One method is to factorise the quadratic and then apply the **Null Factor** law.

The Null Factor law states that:

> When the product of two (or more) numbers is zero, then *at least one* of them must be zero.
>
> So, if $ab = 0$ then $a = 0$ or $b = 0$.

Example 3   ◀) **Self Tutor**

Solve for x using the Null Factor law:

    a $2x(x - 4) = 0$
        b $(x + 3)(2x - 5) = 0$

a $2x(x - 4) = 0$
    $\therefore$ $2x = 0$ or $x - 4 = 0$
          $\therefore$ $x = 0$ or 4

b $(x + 3)(2x - 5) = 0$
    $\therefore$ $x + 3 = 0$ or $2x - 5 = 0$
        $\therefore$ $x = -3$ or $2x = 5$
               $\therefore$ $x = -3$ or $\frac{5}{2}$

EXERCISE 16B

1 What can be deduced from the following equations using the Null Factor law?

    a $2a = 0$
        b $3y = 0$
        c $-4p = 0$
        d $pq = 0$

    e $2xy = 0$
        f $-3mn = 0$
        g $\dfrac{ac}{3} = 0$
        h $wx^2 = 0$

    i $a(b - c) = 0$
        j $c(d + e) = 0$
        k $abc = 0$
        l $wxyz = 0$

2 Solve for x using the Null Factor law:

a $x(x+3) = 0$

b $2x(x-5) = 0$

c $(x-1)(x-3) = 0$

d $4x(2-x) = 0$

e $-3x(2x+1) = 0$

f $5(x+2)(2x-1) = 0$

g $(2x+3)(2x+1) = 0$

h $11(x+2)(x-7) = 0$

i $-6(x-5)(3x+2) = 0$

j $2(x-3)^2 = 0$

k $-4(2x+1)^2 = 0$

l $x^2 = 0$

m $2x(3x-5) = 0$

n $-3x(6-5x) = 0$

o $5x(1+2x) = 0$

p $8(3x+1)^2 = 0$

q $-5(3-4x) = 0$

r $-6x(4-3x) = 0$

C | SOLUTION BY FACTORISATION

Consider the equation $x^2 = 7x$.

x is a common factor on both sides of the equation, so we may be tempted to divide both sides by x.

If we do this, we get $\dfrac{x^2}{x} = \dfrac{7x}{x}$ which gives $x = 7$.

$x = 7$ is clearly a solution of the equation $x^2 = 7x$. However, it is not the only solution.

$x = 0$ is also a solution since $0^2 = 7 \times 0$. By dividing both sides by x, we have lost this solution.

From this example we conclude that:

> We should never cancel a variable that is a common factor unless we know that this factor is non-zero.

Instead, we adopt a method which uses the Null Factor law.

STEPS FOR SOLVING QUADRATIC EQUATIONS

To use the **Null Factor** law when solving equations, we must have one side of the equation *equal to zero*.

> *Step 1:* If necessary, rearrange the equation so one side is **zero**.
>
> *Step 2:* **Fully factorise** the other side (usually the LHS).
>
> *Step 3:* Use the **Null Factor** law: if $ab = 0$ then $a = 0$ or $b = 0$.
>
> *Step 4:* **Solve** the resulting linear equations.
>
> *Step 5:* **Check** at least one of your solutions.

To factorise the quadratic, the following summary may prove useful:

Expression

Take out any common factors

Recognise type

Difference of two squares

$x^2 - a^2 = (x + a)(x - a)$

Splitting the x-term

$ax^2 + bx + c, \quad a \neq 0$

- find ac
- find the factors of ac which add to b
- if these factors are p and q, replace bx by $px + qx$
- complete the factorisation

Perfect square

$x^2 + 2ax + a^2 = (x + a)^2$

$x^2 - 2ax + a^2 = (x - a)^2$

Sum and product type

$x^2 + bx + c$

$x^2 + bx + c = (x + p)(x + q)$

where $p + q = b$ and $pq = c$
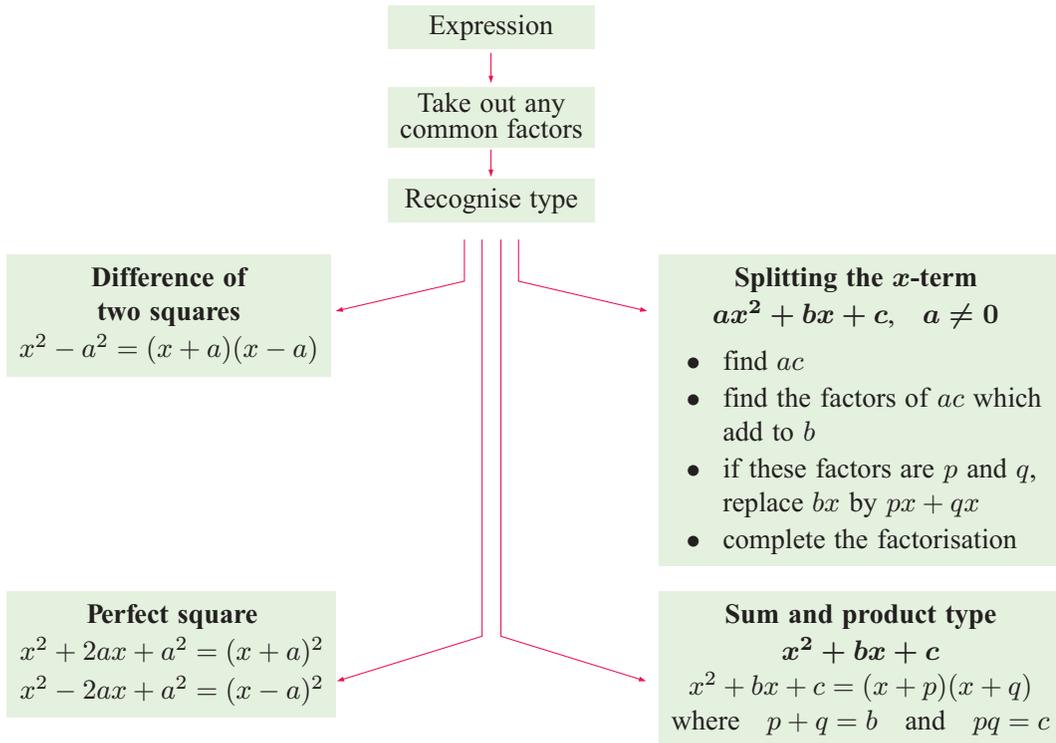
Example 4   ◄)) Self Tutor

Solve for x: $x^2 = 7x$

$$x^2 = 7x$$
$$\therefore \quad x^2 - 7x = 0 \qquad \{\text{rearranging so RHS} = 0\}$$
$$\therefore \quad x(x - 7) = 0 \qquad \{\text{factorising the LHS}\}$$
$$\therefore \quad x = 0 \quad \text{or} \quad x - 7 = 0 \qquad \{\text{Null Factor law}\}$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = 7$$
$$\therefore \quad x = 0 \quad \text{or} \quad 7$$

If $a \times b = 0$ then either $a = 0$ or $b = 0$.

Example 5   ◄)) Self Tutor

Solve for x: $x^2 + 2x = 8$

$$x^2 + 2x = 8$$
$$\therefore \quad x^2 + 2x - 8 = 0 \qquad \{\text{rearranging so RHS} = 0\}$$
$$\therefore \quad (x + 4)(x - 2) = 0 \qquad \{\text{sum} = +2 \text{ and product} = -8$$
$$\qquad \qquad \qquad \qquad \therefore \quad \text{the numbers are } +4 \text{ and } -2\}$$
$$\therefore \quad x + 4 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \{\text{Null Factor law}\}$$
$$\therefore \quad x = -4 \quad \text{or} \quad 2$$

Check: If $x = -4$ then $(-4)^2 + 2(-4) = 16 - 8 = 8$ ✓
      If $x = 2$ then $2^2 + 2(2) = 4 + 4 = 8$ ✓

Example 6
◀) **Self Tutor**

Solve for x: $2x^2 = 3x - 1$

$$2x^2 = 3x - 1$$
$$\therefore \quad 2x^2 - 3x + 1 = 0 \qquad \text{\{rearranging so RHS} = 0\}$$
$$\therefore \quad 2x^2 - 2x - x + 1 = 0 \qquad \{ac = 2, \quad b = -3$$
$$\therefore \quad \text{the numbers are } -2 \text{ and } -1\}$$
$$\therefore \quad 2x(x-1) - 1(x-1) = 0 \qquad \text{\{factorising the pairs\}}$$
$$\therefore \quad (x-1)(2x-1) = 0 \qquad \{(x-1) \text{ is a common factor}\}$$
$$\therefore \quad x - 1 = 0 \ \text{ or } \ 2x - 1 = 0 \qquad \text{\{Null Factor law\}}$$
$$\therefore \quad x = 1 \ \text{ or } \ \tfrac{1}{2}$$

Example 7
◀) **Self Tutor**

Solve for x: $\dfrac{x-2}{x} = \dfrac{6+x}{2}$

$$\frac{x-2}{x} = \frac{6+x}{2}$$
$$\therefore \quad 2(x-2) = x(6+x)$$
$$\therefore \quad 2x - 4 = 6x + x^2$$
$$\therefore \quad x^2 + 4x + 4 = 0$$
$$\therefore \quad (x+2)^2 = 0$$
$$\therefore \quad x + 2 = 0$$
$$\therefore \quad x = -2$$

Check: If $x = -2$ then

$$\text{LHS} = \frac{(-2) - 2}{(-2)} = \frac{-4}{-2} = 2$$

and $\text{RHS} = \dfrac{6 + (-2)}{2} = \tfrac{4}{2} = 2$ ✓

EXERCISE 16C

1 Solve for x:

  a $4x^2 + 12x = 0$ **b** $3x^2 + 9x = 0$ **c** $4x^2 = 16x$

  d $3x^2 = 21x$ **e** $2x^2 + 7x = 0$ **f** $2x^2 = 18x$

  g $3x^2 = 7x$ **h** $4x^2 = 9x$ **i** $0 = 3x^2 + 8x$

2 Solve for x:

  a $x^2 + 6x + 8 = 0$ **b** $x^2 + 11x + 24 = 0$ **c** $x^2 + 4x + 4 = 0$

  d $x^2 + 2x = 15$ **e** $x^2 + 2x = 48$ **f** $x^2 + 25 = 10x$

  g $x^2 = 2x + 15$ **h** $x^2 = 5x + 14$ **i** $x^2 = 7x - 12$

  j $x^2 + 60 = 17x$ **k** $x^2 = 9x + 22$ **l** $x^2 = 3x + 18$

3 Solve for x:

a $2x^2 + 11x + 5 = 0$ b $5x^2 + 21x + 4 = 0$ c $3x^2 = 11x + 4$

d $3x^2 + 10x = 8$ e $2x^2 = 13x + 7$ f $7x^2 = 11x + 6$

g $3x^2 + 10 = 17x$ h $2x^2 + 20 = 13x$ i $3x^2 + x = 10$

j $2x^2 = 13x + 15$ k $2x^2 = 7x + 15$ l $3x^2 - 7x + 2 = 0$

4 Solve for x:

a $6x^2 + 11x + 3 = 0$ b $6x^2 = 17x + 3$ c $6x^2 + x = 2$

d $10x^2 + 21x = 10$ e $12x^2 + 13x + 3 = 0$ f $6x^2 = 17x + 14$

g $14x^2 + 15x + 4 = 0$ h $6x^2 + 13x + 6 = 0$ i $12x^2 + 13x = 4$

j $25x^2 + 10x = 8$ k $15x^2 + 6 = 23x$ l $8x^2 = 10x + 3$

m $2x^2 + 5x = 12$ n $3x^2 + 2x = 16$ o $10x^2 + 5 = 27x$

5 Solve for x by first expanding brackets and then making one side of the equation zero:

a $x(x + 2) + 3(x - 1) = 11$ b $x(3 + x) + 3 = 31$

c $(x + 2)(x - 7) = 8x$ d $2x(x - 1) - 3(x + 2) = -3$

e $3x(x + 2) = 9$ f $3x(x + 4) = x - 10$

6 Solve for x by first eliminating the algebraic fractions:

a $\dfrac{x}{4} = \dfrac{1}{x}$ b $\dfrac{5}{x} = \dfrac{x}{2}$ c $\dfrac{x}{8} = \dfrac{2}{x}$

d $\dfrac{x + 1}{4} = \dfrac{1}{2x}$ e $\dfrac{x + 4}{2} = \dfrac{6}{x}$ f $\dfrac{x + 2}{x} = x$

g $\dfrac{x - 1}{x + 2} = \dfrac{2}{x}$ h $\dfrac{x}{1 + 2x} = \dfrac{1}{3x}$ i $\dfrac{3x + 1}{2x} = x + 2$

D   COMPLETING THE SQUARE

Some quadratic equations such as $x^2 + 6x + 2 = 0$ cannot be solved by the methods already practiced. This is because these quadratics have solutions which are irrational.

We therefore use a new technique where we complete a perfect square.

Consider $x^2 + 6x + 2 = 0$.

The first step is to keep the terms containing x on the LHS and write the constant term on the RHS. We get $x^2 + 6x = -2$.

We then ask the question, 'What do we need to add to the LHS to make a perfect square?'

We need to add 9 as $x^2 + 6x + 9 = (x + 3)^2$.   However, we need to add 9 to the RHS as well to keep the equation balanced.

So, $x^2 + 6x + 9 = -2 + 9$

$\therefore$ $(x + 3)^2 = 7$

$\therefore$ $x + 3 = \pm\sqrt{7}$

$\therefore$ $x = -3 \pm \sqrt{7}$

WHAT DO WE ADD ON TO MAKE A PERFECT SQUARE?

Halve the coefficient of x. Add the square of this number to both sides of the equation.

In the above example, the coefficient of x is 6, so half this number is 3.
We added 3^2 or 9 to both sides of the equation.

To complete the square on $x^2 + 8x$ we add on $4^2 = 16$.

To complete the square on $x^2 - 8x$ we add on $(-4)^2 = 16$.

To complete the square on $x^2 + 20x$ we add on $10^2 = 100$.

Example 8 ◄» **Self Tutor**

Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2 + 4x - 4 = 0$

b $x^2 - 2x + 7 = 0$

a $\qquad x^2 + 4x - 4 = 0$

$\therefore \quad x^2 + 4x \qquad = 4$ {move constant term to RHS}

$\therefore \quad x^2 + 4x + 2^2 = 4 + 2^2$ {add $(\frac{4}{2})^2 = 2^2$ to both sides}

$\therefore \quad (x + 2)^2 = 8$ {factorise LHS, simplify RHS}

$\therefore \quad x + 2 = \pm\sqrt{8}$

$\therefore \quad x = -2 \pm 2\sqrt{2}$

b $\qquad x^2 - 2x + 7 = 0$

$\therefore \quad x^2 - 2x \qquad = -7$ {move constant term to the RHS}

$\therefore \quad x^2 - 2x + 1^2 = -7 + 1^2$ {add $(\frac{-2}{2})^2 = 1^2$ to both sides}

$\therefore \quad (x - 1)^2 = -6$ {factorise LHS, simplify RHS}

which is impossible as no perfect square can be negative.

$\therefore$ no real solutions exist.

Remember that if
$x^2 = k$ where
$k > 0$, then
$x = \pm\sqrt{k}$.

EXERCISE 16D

1 What needs to be added to the following to create a perfect square?

a $x^2 + 4x$ **b** $x^2 - 4x$ **c** $x^2 - 6x$ **d** $x^2 + 10x$

e $x^2 - 10x$ **f** $x^2 + 16x$ **g** $x^2 - 18x$ **h** $x^2 - 36x$

i $x^2 + 3x$ **j** $x^2 - 3x$ **k** $x^2 + 5x$ **l** $x^2 - 5x$

2   Write the following in the form $(x + a)^2 = k$ or $(x - a)^2 = k$:

   a $x^2 + 2x = 2$ **b** $x^2 - 2x = 4$ **c** $x^2 + 6x = -4$

   d $x^2 - 6x = 7$ **e** $x^2 + 10x = -23$ **f** $x^2 + 8x = -13$

   g $x^2 + 6x = 2$ **h** $x^2 + 12x = -1$ **i** $x^2 - 12x = 5$

3   **a** How many solutions does $(x + 3)^2 = 11$ have? What are they?

   b Explain why $(x + 3)^2 = -11$ has no solutions.

4   If possible, solve for x by completing the square. Leave your answers in simplest radical form.

   a $x^2 - 4x - 2 = 0$ **b** $x^2 - 2x - 6 = 0$ **c** $x^2 - 2x + 3 = 0$

   d $x^2 + 2x + 3 = 0$ **e** $x^2 + 4x + 2 = 0$ **f** $x^2 + 4x - 1 = 0$

   g $x^2 + 6x + 11 = 0$ **h** $x^2 - 6x + 1 = 0$ **i** $x^2 + 8x + 5 = 0$

5   Solve by completing the square, leaving answers in simplest radical form:

   a $x^2 + 3x - 2 = 0$ **b** $x^2 = 6x + 1$ **c** $x^2 - 5x + 5 = 0$

   d $x^2 + x - 1 = 0$ **e** $x^2 + 3x + 1 = 0$ **f** $x^2 + 5x + 3 = 0$

6   **a** To solve $2x^2 + 6x - 1 = 0$ by completing the square, our first step is to divide both sides by 2. Explain why this is so.

   b Use completing the square to solve:

     i $2x^2 + 8x + 1 = 0$ **ii** $3x^2 - 9x + 11 = 0$ **iii** $5x^2 + 10x + 1 = 0$

INVESTIGATION   THE QUADRATIC FORMULA

What to do:

1   Use factorisation techniques or completing the square to solve:

   a $x^2 + 3x = 10$ **b** $2x^2 = 7x + 4$

   c $x^2 = 4x + 3$ **d** $4x^2 - 12x + 9 = 0$

   e $x^2 + 6x + 11 = 0$ **f** $4x^2 + 1 = 8x$

2   The **quadratic formula** for solving $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Check that this formula gives the correct answer for each equation in **1**.

3   What is the significance of $b^2 - 4ac$ in determining the solutions of the quadratic equation $ax^2 + bx + c = 0$?

Hint: Make observations from **1** and **2**.

4   Establish the quadratic formula by completing the square on $ax^2 + bx + c = 0$.

Hint: Divide each term by a to begin with.

5   Use the quadratic formula to solve these quadratic equations:

   a $x^2 - 3x + 1 = 0$ **b** $x^2 - 7x + 2 = 0$ **c** $x^2 + 2x - 5 = 0$

   d $x^2 - 4x + 6 = 0$ **e** $x^2 + 10x + 25 = 0$ **f** $x^2 - 4x - 5 = 0$

g $x^2 - 10x + 22 = 0$ **h** $2x^2 - 4x - 7 = 0$ **i** $x^2 + x + 2 = 0$

j $4x^2 - 12x + 9 = 0$ **k** $9x^2 - 12x + 11 = 0$ **l** $5x^2 - 6x + 1 = 0$

6 Use the quadratic formula to help solve:

a $x + \dfrac{1}{x} = 3$ **b** $x + \dfrac{2}{x} = 5$ **c** $(2x - 1)^2 = x + 4$

E   PROBLEM SOLVING

The problems in this section can all be converted to algebraic form as **quadratic equations**. They can all be solved using **factorisation** or **completing the square**.

PROBLEM SOLVING METHOD

Step 1:   Carefully read the question until you understand the problem. A rough sketch may be useful.

Step 2:   Decide on the unknown quantity and label it x, say.

Step 3:   Use the information given to find an equation which contains x.

Step 4:   Solve the equation.

Step 5:   Check that any solutions satisfy the equation and are realistic to the problem.

Step 6:   Write your answer to the question in sentence form.

Example 9   ◀) **Self Tutor**

The difference between a number and its square is 20. Find the number.

Let the number be x.   Therefore its square is x^2.

$$x^2 - x = 20$$
$$\therefore \quad x^2 - x - 20 = 0 \qquad \{\text{rearranging}\}$$
$$\therefore \quad (x - 5)(x + 4) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad x = 5 \quad \text{or} \quad x = -4$$

So, the number is -4 or 5.   *Check:*   If $x = -4$, $(-4)^2 - (-4) = 16 + 4 = 20$ ✓

If $x = 5$, $5^2 - 5 = 25 - 5 = 20$ ✓

Example 10   ◀) **Self Tutor**

Use Pythagoras' theorem to find x given:

$(2x - 2)$ cm

x cm

$(x + 2)$ cm
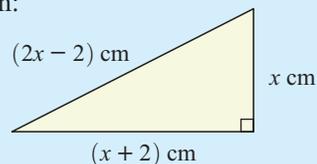
Using Pythagoras' theorem, $(x + 2)^2 + x^2 = (2x - 2)^2$

$$\therefore \quad x^2 + 4x + 4 + x^2 = 4x^2 - 8x + 4$$
$$\therefore \quad 2x^2 + 4x = 4x^2 - 8x$$
$$\therefore \quad 2x^2 - 12x = 0$$
$$\therefore \quad 2x(x - 6) = 0$$
$$\therefore \quad x = 0 \ \text{ or } \ 6$$

But $x > 0$ as lengths are positive quantities.

$$\therefore \quad x = 6$$

Example 11   ◀)) Self Tutor

A rectangular housing block has perimeter 64 m and area 220 m².
What are its dimensions?

Suppose the block is x m long and y m wide.

$$\therefore \quad 2x + 2y = 64 \qquad \{\text{perimeter } = \ 64 \text{ m}\}$$
$$\therefore \quad x + y = 32$$
$$\therefore \quad y = 32 - x$$

The area is 220 m², so $xy = 220$

$$\therefore \quad x(32 - x) = 220$$
$$\therefore \quad 32x - x^2 = 220$$
$$\therefore \quad x^2 - 32x + 220 = 0$$
$$\therefore \quad (x - 10)(x - 22) = 0$$
$$\therefore \quad x = 10 \ \text{ or } \ 22$$
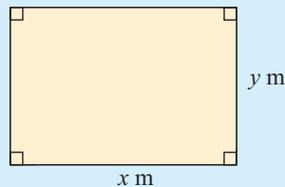
So, the block is 10 m × 22 m {when $x = 10$, $y = 22$ and when $x = 22$, $y = 10$}

EXERCISE 16E

1 The product of a number and the number increased by 3 is 54. Find the two possible values the number could be.

2 The difference between a number and its square is 42. Find the number.

3 When 15 is added to the square of a number, the result is eight times the original number. Find the number.

4 Two numbers add up to 5. The sum of their squares is 73. What are the numbers?

5 Two numbers differ by 13 and the sum of their squares is 125. Find the numbers.

6 Use the theorem of Pythagoras to find x given:

a

$(x - 7)$ cm
$(x + 1)$ cm
x cm

b

$(x + 2)$ cm
x cm
$(x + 7)$ cm

c

$(x + 1)$ m
x m
5 m

7 A right angled triangle has sides 2 cm and 7 cm respectively less than its hypotenuse. Find the length of each side of the triangle to the nearest millimetre.

8 A rectangle has length 3 cm greater than its width. Find its length given that its area is 40 cm².

9 A rectangular paddock has perimeter 600 m and area 21 600 m². What are its dimensions?
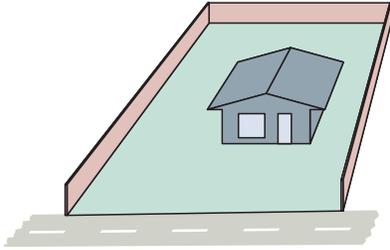
10 A new fence is constructed on three sides of a housing block with area 240 m².
The fourth side facing the road is left open.
If 52 m of fencing is used, what are the dimensions of the block?

11 A rectangular chicken yard was built against an existing shed wall. 30 m of fencing was used to enclose 108 m². Find the dimensions of the yard.
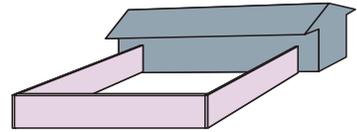
12 A gardener plants 462 seedlings. The number of seedlings in each row is 20 more than twice the number of rows. If equal numbers of seedlings were planted in each row, how many rows did the gardener plant?

13 A manufacturer has determined that if his company sells x items per day then their profit is given by $P = -x^2 + 700x - 100\,000$ euros.
How many items must be sold each day in order to make a profit?

14 Answer the **Opening Problem** on page **340**.

REVIEW SET 16A

1 Solve for x:

   a $-x^2 + 11 = 0$ **b** $(x - 4)^2 = 25$ **c** $(x + 1)^2 = -1$

2 Solve for x:

   a $x^2 - 4x - 21 = 0$ **b** $4x^2 - 25 = 0$ **c** $3x^2 - 6x - 72 = 0$

   d $6x^2 + x - 2 = 0$ **e** $x^2 + 24 = 11x$ **f** $10x^2 - 11x - 6 = 0$

3 Solve by completing the square: $x^2 + 6x + 11 = 0$

4 The width of a rectangle is 7 cm less than its length. Its area is 260 cm². Find the dimensions of the rectangle.

5 The sum of a number and its reciprocal is $2\frac{1}{6}$. Find the number.

6 When the square of a number is increased by one, the result is four times the original number. Find the number.

7 Use the quadratic formula to solve for x:

   a $2x^2 - 3x - 2 = 0$ **b** $3x^2 + 4x + 5 = 0$

REVIEW SET 16B

1 Solve for x:

   a $x^2 = 169$ **b** $(x+3)^2 - 11 = 0$ **c** $(x-4)^2 + 16 = 0$

2 Solve for x:

   a $x^2 - 8x - 33 = 0$ **b** $x^2 + 4x - 32 = 0$ **c** $x^2 - 10x + 25 = 0$

   d $x^2 + 5x = 24$ **e** $2x^2 - 18 = 0$ **f** $8x^2 + 2x - 3 = 0$

3 Solve by completing the square: $x^2 - 2x = 100$

4 A rectangle has length 3 cm greater than its width. If it has an area of 108 cm^2, find the dimensions of the rectangle.

5 When the square of a number is increased by 10, the result is seven times the original number. Find the number.

6 The hypotenuse of a right angled triangle is one centimetre more than twice the length of the shortest side. The other side is 7 cm longer than the shortest side. Find the length of each side of the triangle.

7 Use the quadratic formula to solve for x:

   a $2x^2 + 3x + 6 = 0$ **b** $5x^2 - x - 5 = 0$

8 A number minus 12 times its reciprocal is equal to 1. Find the number.

HISTORICAL NOTE   CAROLINE LUCRETIA HERSCHEL 1750 - 1848

Caroline Herschel was born in 1750 in Hanover. Her father was an army musician and her brother William became famous for his discovery of the planet Uranus.

Caroline was a very determined and independent young woman and after her father died she planned to pursue a career as a governess. Before she gained a position, her brother William, who had moved to England, asked her to go and join him as his housekeeper. After she arrived, Caroline studied a variety of subjects and commenced music lessons in singing and violin and became a capable singer.

Whilst living with William she became fascinated with his work in astronomy and became his assistant. Between them, they built and operated the largest telescopes of the time. Caroline began to achieve fame in her own right as an astronomer. William supervised the construction of the telescopes which they built and sold to help support themselves while Caroline summarised and collated the results of their observations. She taught herself geometry and logarithms to assist with the difficult calculations involved.

After William's death, Caroline produced a number of books on astronomy from the notes they had collated over the years. Such was her contribution to astronomy that she was awarded honours by the kings of both Denmark and Prussia and was awarded membership of the Royal Irish Academy and the Royal Astronomical Society. She died in 1848, at the age of 97.

Chapter **17**

Simultaneous equations

Contents:

INVESTIGATION   THE CYCLES PROBLEM

In the display window of the local cycle store there are bicycles and tricycles. Altogether there are 7 cycles and 16 wheels. How many bicycles and tricycles are in the display?

What to do:

1 Copy and complete the following table:

Number of bicycles	0	1	2	3	4	5	6	7
Number of bicycle wheels	0							
Number of tricycles	7	6						
Number of tricycle wheels	21							
Total Wheels	21							

2 Use the table to find the solution to the problem.

3 Suppose there were x bicycles and y tricycles in the window.

   a By considering the number of cycles seen, explain why $x + y = 7$.

   b By considering the total number of wheels seen, explain why $2x + 3y = 16$.

4 You should have found that there were five bicycles and two tricycles in the window.

   a Substitute $x = 5$ and $y = 2$ into $x + y = 7$. What do you notice?

   b Substitute $x = 5$ and $y = 2$ into $2x + 3y = 16$. What do you notice?

The solution to the bicycles problem, $x = 5$ and $y = 2$, was found by **trial and error**.

However, the solution can also be found by graphical or by algebraic means. We have already seen the graphical approach in **Chapter 6** where we considered the point of intersection where two lines meet. In this chapter we consider the algebraic approach.

A   LINEAR SIMULTANEOUS EQUATIONS

If we have two equations and we wish to make both equations true at the same time, we require values for the variables which satisfy both equations. These values are the **simultaneous solution** to the pair of equations.

Notice that if $x = 5$ and $y = 2$ then:

- $x + y = (5) + (2) = 7$
- $2x + 3y = 2(5) + 3(2) = 10 + 6 = 16$

So, $x = 5$ and $y = 2$ is the **solution** to the simultaneous equations $\begin{cases} x + y = 7 \\ 2x + 3y = 16. \end{cases}$

SOLUTION BY SUBSTITUTION

The method of **solution by substitution** is used when at least one equation is given with either x or y as the **subject** of the formula. We substitute an expression for this variable into the other equation.

Example 1

Self Tutor

Solve simultaneously by substitution: $y = x + 5$
$3x - y = 1$

$y = x + 5$ ......(1)
$3x - y = 1$ ......(2)

Since $y = x + 5$, $3x - (x + 5) = 1$
$\therefore \quad 3x - x - 5 = 1$
$\therefore \quad 2x = 6$
$\therefore \quad x = 3$

> Notice that $(x + 5)$ is substituted for y in the other equation.

When $x = 3$, $y = 3 + 5$ {substituting $x = 3$ into (1)}
$\therefore \quad y = 8$

The solution is $x = 3$, $y = 8$.

Check: (1) $8 = 3 + 5$ ✓
(2) $3(3) - 8 = 9 - 8 = 1$ ✓

Example 2

Self Tutor

Solve simultaneously by substitution: $-2y + x = 5$
$x = 7 + 3y$

$-2y + x = 5$ ......(1)
$x = 7 + 3y$ ......(2)

Substituting (2) into (1) gives

$-2y + (7 + 3y) = 5$
$\therefore \quad -2y + 7 + 3y = 5$
$\therefore \quad y + 7 = 5$
$\therefore \quad y = -2$

Substituting $y = -2$ into (2) gives
$x = 7 + 3(-2) = 1$.

> There are infinitely many points (x, y) which satisfy the first equation. Likewise there are infinitely many which satisfy the second. However, only one point satisfies both equations at the same time.

The solution is $x = 1$, $y = -2$.

Check: (1) $-2(-2) + 1 = 4 + 1 = 5$ ✓
(2) $7 + 3(-2) = 7 - 6 = 1$ ✓

EXERCISE 17A.1

1 Solve simultaneously:

    a $x = 8 - 2y$
        $2x + 3y = 13$

    b $y = 4 + x$
        $5x - 3y = 0$

    c $x = -10 - 2y$
        $3y - 2x = -22$

    d $x = -1 + 2y$
        $x = 9 - 2y$

    e $3x - 2y = 8$
        $x = 3y + 12$

    f $x + 2y = 8$
        $y = 7 - 2x$

2 Solve simultaneously:

    a $x = -1 - 2y$
        $2x - 3y = 12$

    b $y = 3 - 2x$
        $y = 3x + 1$

    c $x = 3y - 9$
        $5x + 2y = 23$

    d $y = 5x$
        $7x - 2y = 3$

    e $x = -2 - 3y$
        $3x - 2y = -17$

    f $3x - 5y = 26$
        $y = 4x - 12$

3 **a** Try to solve by substitution: $y = 3x + 1$ and $y = 3x + 4$.

    b What is the simultaneous solution for the equations in **a**?

4 **a** Try to solve by substitution: $y = 3x + 1$ and $2y = 6x + 2$.

    b How many simultaneous solutions do the equations in **a** have?

SOLUTION BY ELIMINATION

In many problems which require the simultaneous solution of linear equations, each equation will be of the form $ax + by = c$. Solution by substitution is often tedious in such situations and the method of **elimination** of one of the variables is preferred.

In this method, we make the coefficients of x (or y) the **same size** but **opposite in sign** and then **add** the equations. This has the effect of **eliminating** one of the variables.

The method of elimination uses the fact that: if $a = b$ and $c = d$ then $a + c = b + d$.

Example 3   ◀) **Self Tutor**

Solve simultaneously, by elimination: $3x + 2y = 5$ ...... (1)
                             $x - 2y = 3$ ...... (2)

Notice that the coefficients of y are the same size but opposite in sign.

We **add** the LHSs and the RHSs to get an equation which contains x only.

$$
\begin{aligned}
& \quad 3x + 2y = 5 \\
& +\quad x - 2y = 3 \\
\hline
& \therefore \quad 4x = 8 \qquad \text{\{adding the equations\}} \\
& \quad\quad \therefore \quad x = 2 \qquad \text{\{dividing both sides by 4\}}
\end{aligned}
$$

Substituting $x = 2$ into equation (1) gives

$3(2) + 2y = 5$

$\therefore \quad 6 + 2y = 5$

$\therefore \quad 2y = 5 - 6 \qquad$ {subtracting 6 from both sides}

$\therefore \quad 2y = -1$

$\therefore \quad y = -\frac{1}{2} \qquad$ {dividing both sides by 2}

The solution is $x = 2$ and $y = -\frac{1}{2}$.

Check: in (2): $(2) - 2(-\frac{1}{2}) = 2 + 1 = 3$ ✓

In problems where the coefficients of x (or y) are **not** the **same size** or **opposite in sign**, we may first have to **multiply** each equation by a number.

Example 4   ◀) **Self Tutor**

Solve simultaneously, by elimination: $3x + 2y = 8$
$ 2x - 3y = 1$

$3x + 2y = 8 \qquad$ ...... (1)
$2x - 3y = 1 \qquad$ ...... (2)

We can eliminate y by multiplying (1) by 3 and (2) by 2.

$$\therefore \quad 9x + 6y = 24 \qquad \{(1) \times 3\}$$
$$+ \quad 4x - 6y = 2 \qquad \{(2) \times 2\}$$
$$\therefore \quad 13x = 26 \qquad \{\text{adding the equations}\}$$
$$\therefore \quad x = 2 \qquad \{\text{dividing both sides by 13}\}$$

Substituting $x = 2$ into equation (1) gives

$3(2) + 2y = 8$

$\therefore \quad 6 + 2y = 8$

$\therefore \quad 2y = 2$

$\therefore \quad y = 1$

So, the solution is: $x = 2,\ y = 1$.   *Check:* $3(2) + 2(1) = 6 + 2 = 8$ ✓
$ 2(2) - 3(1) = 4 - 3 = 1$ ✓

There is always a choice whether to eliminate x or y, so our choice depends on which variable is easier to eliminate.

In **Example 4**, try to solve by multiplying (1) by 2 and (2) by -3. This eliminates x rather than y. The final solutions should be the same.

EXERCISE 17A.2

1 What equation results when the following are added vertically?

a $5x + 3y = 12$
$x - 3y = -6$

b $2x + 5y = -4$
$-2x - 6y = 12$

c $4x - 6y = 9$
$x + 6y = -2$

d $12x + 15y = 33$
$-18x - 15y = -63$

e $5x + 6y = 12$
$-5x + 2y = -8$

f $-7x + y = -5$
$7x - 3y = -11$

2 Solve the following using the method of elimination:

a $2x + y = 3$
$3x - y = 7$

b $4x + 3y = 7$
$6x - 3y = -27$

c $2x + 5y = 16$
$-2x - 7y = -20$

d $3x + 5y = -11$
$-3x - 2y = 8$

e $4x - 7y = 41$
$3x + 7y = -6$

f $-4x + 3y = -25$
$4x - 5y = 31$

3 Give the equation that results when both sides of the equation:

a $3x + 4y = 2$ are multiplied by 3

b $x - 4y = 7$ are multiplied by -2

c $5x - y = -3$ are multiplied by 5

d $7x + 3y = -4$ are multiplied by -3

e $-2x - 5y = 1$ are multiplied by -4

f $3x - y = -1$ are multiplied by -1

4 Solve the following using the method of elimination:

a $4x - 3y = 6$
$-2x + 5y = 4$

b $2x - y = 9$
$x + 4y = 36$

c $3x + 4y = 6$
$x - 3y = -11$

d $2x + 3y = 7$
$3x - 2y = 4$

e $4x - 3y = 6$
$6x + 7y = 32$

f $7x - 3y = 29$
$3x + 4y + 14 = 0$

g $2x + 5y = 20$
$3x + 2y = 19$

h $3x - 2y = 10$
$4x + 3y = 19$

i $3x + 4y + 11 = 0$
$5x + 6y + 7 = 0$

5 Use the method of elimination to attempt to solve:

a $3x + y = 8$
$6x + 2y = 16$

b $2x + 5y = 8$
$4x + 10y = -1$

Comment on your results.

B   PROBLEM SOLVING

Many problems can be described mathematically by a **pair of linear equations**, or two equations of the form $ax + by = c$, where x and y are the two variables or unknowns.

We have already seen an example of this in the **Investigation** on page **354**.

Once the equations are formed, they can then be solved simultaneously and thus the original problem solved. The following method is recommended:

Step 1:   Decide on the two unknowns; call them x and y, say. Do not forget the units.

Step 2:   Write down **two** equations connecting x and y.

Step 3:   Solve the equations simultaneously.

Step 4:   Check your solutions with the original data given.

Step 5:   Give your answer in sentence form.

The form of the original equations will help you decide whether to use the substitution method, or the elimination method.

Example 5   ◄)) Self Tutor

Two numbers have a difference of 7 and an average of 4. Find the numbers.

Let x and y be the unknown numbers, where $x > y$.

So, $x - y = 7$ ......(1)   {'difference' means subtract}

$\dfrac{x + y}{2} = 4$ ......(2)   {'average' is a half of their sum}

Multiplying (2) $\times$ 2 gives

$$\begin{aligned}
x + y &= 8 \\
x - y &= 7 \quad \text{\{from (1)\}} \\
\therefore \quad 2x \ \ \ &= 15 \quad \text{\{adding\}} \\
\therefore \quad x &= \tfrac{15}{2} \quad \text{\{dividing both sides by 2\}}
\end{aligned}$$

Substituting into (1) gives

$\tfrac{15}{2} - y = 7$

$\therefore \quad y = \tfrac{15}{2} - 7$ *Check:*   (1) $\tfrac{15}{2} - \tfrac{1}{2} = 7$ ✓

$\therefore \quad y = \tfrac{1}{2}$ (2) $\dfrac{\tfrac{15}{2} + \tfrac{1}{2}}{2} = 4$ ✓

The numbers are $\tfrac{1}{2}$ and $\tfrac{15}{2}$.

> When solving problems with simultaneous equations we must find two equations containing two unknowns.

EXERCISE 17B

1 The sum of two numbers is 47 and their difference is 14. Find the numbers.

2 Find two numbers whose difference is 3 and average is 5.

3 The larger of two numbers is four times the smaller, and their sum is 85. Find the two numbers.

Example 6   ◄)) Self Tutor

While shopping, Ken noticed that 8 peaches and 4 plums would cost £4.60 whereas 6 peaches and 7 plums would cost £4.85.

Find the cost of each peach and each plum.

Let each peach cost x pence and each plum cost y pence.

$\therefore \quad 8x + 4y = 460$ .....(1)

$ 6x + 7y = 485$ .....(2)

Note: The units must be the same on both sides of each equation, i.e., cents.

To eliminate y, we multiply (1) by 7 and (2) by -4.

$$\therefore \quad 56x + 28y = 3220 \quad ..... (3)$$
$$-24x - 28y = -1940 \quad ..... (4)$$
$$\therefore \quad 32x \qquad = 1280 \qquad \{\text{adding (3) and (4)}\}$$
$$\therefore \quad x = 40 \qquad \{\text{dividing both sides by 32}\}$$

Substituting in (1) gives $\quad 8(40) + 4y = 460$
$$\therefore \quad 320 + 4y = 460$$
$$\therefore \quad 4y = 140$$
$$\therefore \quad y = 35$$

Check: $\quad 8 \times 40 + 4 \times 35 = 320 + 140 = 460 \ \checkmark$
$$6 \times 40 + 7 \times 35 = 240 + 245 = 485 \ \checkmark$$

$\therefore \quad$ the peaches cost 40 pence each and the plums cost 35 pence each.

4 At the local stationery shop, five pencils and six biros cost a total of €4.64, and seven pencils and three biros cost a total of €3.58. Find the cost of each item.

5 Seven toffees and three chocolates cost a total of $1.68, whereas four toffees and five chocolates cost a total of $1.65. Find the cost of each of the sweets.

Example 7   ◄) Self Tutor

Colin has a jar containing only 5-cent and 20-cent coins. In total there are 31 coins with a total value of $3.50. How many of each type of coin does Colin have?

Let x be the number of 5-cent coins and y be the number of 20-cent coins.

$$\therefore \quad x + y = 31 \qquad ...... (1) \quad \{\text{the total number of coins}\}$$
$$\text{and} \quad 5x + 20y = 350 \qquad ...... (2) \quad \{\text{the total value of coins}\}$$

To eliminate x, we multiply (1) by -5.

$$\therefore \quad -5x - 5y = -155 \qquad ...... (3)$$
$$5x + 20y = 350 \qquad ...... (2)$$
$$\therefore \quad 15y = 195 \qquad \{\text{adding (3) and (2)}\}$$
$$\therefore \quad y = 13 \qquad \{\text{dividing both sides by 15}\}$$

Substituting $\ y = 13 \ $ into (1) gives $\quad x + 13 = 31$
$$\therefore \quad x = 18$$

Check: $\qquad 18 + 13 = 31 \ \checkmark$
$$5 \times 18 + 20 \times 13 = 350 \ \checkmark$$

Colin has 18 5-cent coins and 13 20-cent coins.

6 I have only 50-cent and $1 coins in my purse. There are 43 coins and their total value is $35. How many of each coin type do I have?

7 Amy and Michelle have €29.40 between them. Amy's money totals three quarters of Michelle's. How much money does each have?

8   Margarine is sold in either 250 g or 400 g packs. A hotel ordered 19.6 kg of margarine and received 58 packs. How many of each type did the hotel receive?

9   Given that the triangle alongside is equilateral, find a and b.

$(b + 2)$ cm $(a + 4)$ cm

$(4a - b)$ cm

10   A rectangle has perimeter 32 cm. If 3 cm is taken from the length and added to the width, the rectangle becomes a square. Find the dimensions of the original rectangle.

Example 8   ◀) **Self Tutor**

A boat travels 24 km upstream in 4 hours. The return trip downstream takes only 3 hours. The speed of the current is constant throughout the entire trip.
Find:

a   the speed of the current   **b**   the speed of the boat in still water.

Let the speed of the boat in still water be x km per hour, and the speed of the current be y km per hour.

$$\text{speed upstream} = \frac{24 \text{ km}}{4 \text{ h}} = 6 \text{ km h}^{-1}$$

$$\text{speed downstream} = \frac{24 \text{ km}}{3 \text{ h}} = 8 \text{ km h}^{-1}$$

When the boat is travelling upstream,

boat speed $-$ current speed $=$ actual speed

$$\therefore \quad x - y = 6 \quad \text{..... (1)}$$

When the boat is travelling downstream,

boat speed $+$ current speed $=$ actual speed

$$\therefore \quad x + y = 8 \quad \text{..... (2)}$$

Adding (1) and (2), $2x = 14$

$$\therefore \ x = 7$$

Substituting $x = 7$ in (1), $7 - y = 6$

$$\therefore \ y = 1$$

Check: $7 - 1 = 6$ ✓   and $7 + 1 = 8$ ✓

a   The current speed is 1 km h^{-1}.

b   The boat speed in still water is 7 km h^{-1}.

11 A motor boat travels at 12 km h^{-1} upstream against the current and 18 km h^{-1} downstream with the current. Find the speed of the current and the speed of the motor boat in still water.

12 A jet plane made a 4000 km trip downwind in 4 hours, but required 5 hours to make the return trip. Assuming the speed of the wind was constant throughout the entire journey, find the speed of the wind and the average speed of the plane in still air.

13 A man on foot covers the 25 km between two towns in $3\frac{3}{4}$ hours. He walks at 4 km h^{-1} for the first part of the journey and runs at 12 km h^{-1} for the remaining part.

    a How far did he run?   **b** For how long was he running?

14   **a** Explain why any two digit number can be written in the form $10a + b$.

    b A number consists of two digits which add up to 9. When the digits are reversed, the original number is decreased by 45. What was the original number?

C   NON-LINEAR SIMULTANEOUS EQUATIONS

A QUADRATIC MEETING A LINEAR FUNCTION

So far we have only considered simultaneous equations that are linear. This means that both equations can be written in the form $ax + by = c$ where a, b and c are constants.

In this section we consider non-linear equations. In some situations we can still solve the equations simultaneously by substitution. In other situations we can solve the equations simultaneously using a **computer graphing package**.

Consider the graphs of a quadratic function and a linear function on the same set of axes.

There are three possible scenarios:
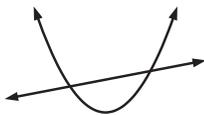
cutting	touching	missing
(2 points of intersection)	(1 point of intersection)	(no points of intersection)
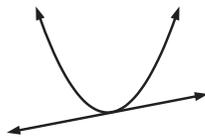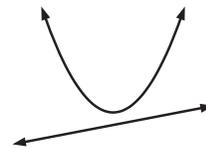
The coordinates of any points of intersection can be found by solving the two equations simultaneously.

Example 9   ◆)) **Self Tutor**

Find algebraically the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore \ x^2 - 2x - 15 = 0 \qquad \text{\{writing with RHS} = 0\}$$
$$\therefore \ (x - 5)(x + 3) = 0 \qquad \text{\{factorising\}}$$
$$\therefore \ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5, 2)$ and $(-3, -6)$.

EXERCISE 17C.1

1 Find algebraically the coordinates of the points of intersection of the graphs with equations:

    a $y = x^2 - 3x + 2$ and $y = 2x - 4$

    b $y = -x^2 + 2x + 3$ and $y = 9x - 5$

    c $y = x^2 + 9x + 7$ and $y = 3x - 2$

    d $y = x^2 - x - 5$ and $y = 3x - 9$

2 Show that the graphs of $y = x^2 + 7$ and $y = -6x - 3$ do not intersect.
    Hint: Use 'completing the square'.

SOLVING NON-LINEAR EQUATIONS BY SUBSTITUTION

The following questions can all be solved by substituting an expression for a variable from one equation into the other equation.

EXERCISE 17C.2

1 Find algebraically the coordinates of the points of intersection of the graphs with equations:

    a $y = 2x^2 - 4x + 3$ and $y = x^2 + 2x - 6$

    b $y = x^2 + 2x - 2$ and $y = 2x^2 + 5x - 6$

    c $y = 2x + 2$ and $y = \dfrac{4}{x}$

    d $y = 3x - 1$ and $y = 3 + \dfrac{2}{x - 1}$

> Do not forget to check your answers by substituting them in both equations.

2 Solve simultaneously:

    a $y - y^2 = x$ and $x - y = -1$

    b $x^2 = 1 + y^2 - 4y$ and $y + x = 2$

    c $2x^2 + y^2 = 24$ and $y = x^2$

    d $(x - y)^2 = 4y$ and $2x + y = 7$

    e $y = \dfrac{2}{x}$ and $y = \dfrac{x - 1}{x + 2}$

SOLVING SIMULTANEOUS EQUATIONS USING TECHNOLOGY

Graphics calculators can be used to solve equations simultaneously. However, this is generally restricted to equations which can be written in the form $y = ...$

For example, in **Exercise 17C.2** we could solve question **1** using a graphics calculator, but we could not solve question **2**.

Instructions for drawing graphs and finding the coordinates of their points of intersection can be found on pages **21** to **24**.

Some computer graphing packages are able to handle equations given in forms other than $y = ...$ You can obtain one by clicking on the icon.

GRAPHING PACKAGE

For example, the screen-shot below shows the points of intersection of the circle $x^2 + y^2 = 4$ and the line $y = -x$.
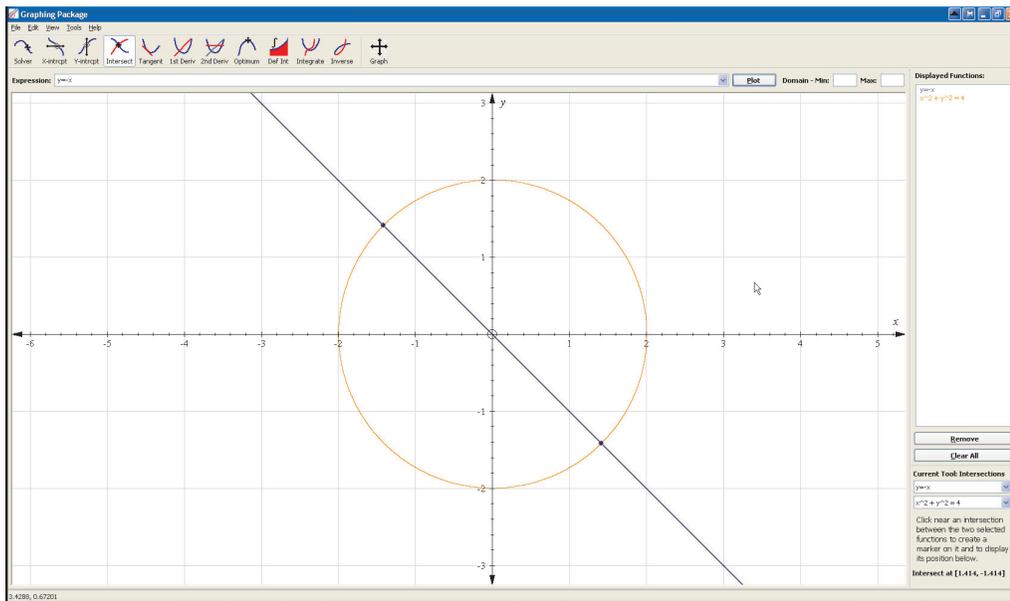
EXERCISE 17C.3

1 Find the coordinates of the points of intersection of the graphs with equations:

   a $y = x^2$ and $y = x + 3$ **b** $y = x^3 + 2x$ and $y = x + 1$

   c $y = \dfrac{2}{x}$ and $y = 2x + 3$ **d** $y = x^2 - x + 1$ and $y = -\dfrac{3}{x}$

   e $y = \dfrac{3}{x(x-2)}$ and $y = x^2 + 2x - 8$ **f** $x^2 + y^2 = 4$ and $x^2 + 2y^2 = 6$

2 Solve simultaneously:

   a $y = \sqrt{5 - x}$ and $x^2 + y^2 = 10$ **b** $y = x^2$ and $x = y^2 + 1$

LINKS
click here

SOLVING 3 BY 3 SYSTEMS

Areas of interaction:
Human ingenuity

REVIEW SET 17A

1 Solve by substitution: $y = 2x - 7$ and $2x + 3y = 11$.

2 Solve simultaneously: **a** $y = 2x - 5$

$3x - 2y = 11$

b $3x + 5y = 1$

$4x - 3y = 11$

3 Flour is sold in 5 kg and 2 kg packets. The 5 kg packets cost €2.75 each and the 2 kg packets cost €1.25 each. If I bought 67 kg of flour and the total cost was €38.50, how many of each size of packet did I buy?

4 Find the coordinates of any point where these line pairs meet:

a $y = 3x + 2$

$y = 3x - 1$

b $2x - 3y = 18$

$4x + 5y = -8$

5 Find the perimeter of the given rectangle. Your answer must not contain x or y.

$2x + 3y + 5$

$x + y + 3$

$2x - y + 6$

$3x - y + 16$

6 Sally has only 10-pence and 50-pence coins in her purse. She has 21 coins altogether with a total value of £5.30. How many of each coin type does she have?

7 Find algebraically where the line with equation $y = 2x - 3$ meets the parabola with equation $y = 2x^2 - 3x - 10$.

REVIEW SET 17B

1 Solve by substitution: $y = 11 - 3x$

$4x + 3y = -7$

2 Solve simultaneously: **a** $3x - 2y = 16$

$y = 2x - 10$

b $3x - 5y = 11$

$4x + 3y = 5$

3 A bus company uses two different sized buses. The company needs 7 small buses and 5 large buses to transport 331 people, but needs 4 small buses and 9 large buses to carry 398 people. In these situations all of the buses are full. Determine the number of people each bus can carry.

4 Find the coordinates of any point where these line pairs meet:

a $y = 2x + 5$

$y = 2x + 15$

b $3x + 7y = -6$

$6x + 5y = 15$

5 Orange juice can be purchased in 2 L cartons or in 600 mL bottles. The 2 L cartons cost \$1.50 each and the 600 mL bottles cost \$0.60 each. A consumer purchases 73 L of orange juice for a total cost of \$57. How many of each container did the consumer buy?

6

$x + y + 1$

$2x - y + 4$

$3x - 2y + 14$
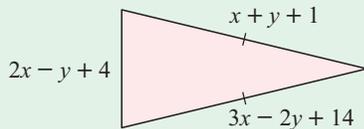
The perimeter of this *isosceles* triangle is 29 cm.

Find the length of the equal sides. Your answer must not contain x or y.

7 Find algebraically the points of intersection of the parabolas with equations $y = 2x^2 + 3x - 1$ and $y = 5 - x^2$.

HISTORICAL NOTE   **GRACE CHISHOLM YOUNG 1868 - 1944**

Grace Chisholm was born in 1868 as the youngest of a family of four. As her father held the prestigious position of Warden of the Standards - in charge of the Department of Weights and Measures for the British Government - she was brought up in a well educated upper-class environment.

In early childhood, Grace suffered from headaches and nightmares. Subsequently, she only studied mental arithmetic and music with her mother as tutor. Initially she wished to study medicine but her mother forbade this and with the encouragement of her father she won a scholarship to study mathematics at Girton College, Cambridge.

Students in those days worked very closely with their tutors who determined what aspects of their chosen subject they should study. Grace found the mathematical atmosphere at Cambridge stifling but she still managed to qualify for a first class degree. Despite her outstanding performance, her hopes of pursuing a career as a mathematician were small as women were not, at that time, able to be admitted to graduate schools in England.

After completing another final honours year at Oxford, Grace moved to the university of Gottingen in Germany. Here she qualified for her doctorate and was the first woman to be officially awarded that degree in any subject in Germany.

Subsequently, one of her former tutors, William Young read her thesis and suggested that they write an astronomy book together. Eventually, they married and continued to work together with Grace performing research and William teaching mathematics. Despite Grace doing most of the research, William convinced her that most of the papers should appear under his name.

Whilst raising six children she continued to produce high quality mathematical research and had she not been stifled by William she may have gained even greater reknown. At the time of her death in 1944, Grace had been proposed as an honourary fellow of Girton College.

Chapter 18

Matrices

Contents:

OPENING PROBLEM

Gerty has invested in real estate. She owns a block of 10 apartments, 5 units, and 3 houses. She wishes to put new furniture in each of them.

In each apartment she puts 4 chairs, 1 table and 2 beds.

In each unit she puts 6 chairs, 1 table and 3 beds.

In each house she puts 8 chairs, 2 tables and 4 beds.

If each chair costs €35, each table €175 and each bed €235, how much will the furniture cost in total?

One way to solve the **Opening Problem** is to use a **matrix**, or rectangular array of numbers.

You have been using matrices for many years without realising it.

For example, you may have used them to display your school results:

	Mid-year	Final
Science	84	85
English	78	79
Mathematics	81	86
French	73	74
Geography	69	77

A   MATRIX SIZE AND CONSTRUCTION

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

January 2008						
Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

is not a matrix because it has rows and columns that are not complete.

In this course we will write our matrices with round brackets around the numbers. Elsewhere, square brackets are often used.

The information in the **Opening Problem** can be summarised in the tables:

Furniture Prices	
	cost (€)
chair	35
table	175
bed	235

Furniture inventory			
	chairs	tables	beds
flat	4	1	2
unit	6	1	3
house	8	2	4

We could also write the information in detailed matrix form as:

$$\begin{array}{c} \text{cost} \\ \begin{array}{c} C \\ T \\ B \end{array} \begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{array}{ccc} C & T & B \end{array} \\ \begin{array}{c} F \\ U \\ H \end{array} \begin{pmatrix} 4 & 1 & 2 \\ 6 & 1 & 3 \\ 8 & 2 & 4 \end{pmatrix} \end{array}$$

Provided we can remember what each row and column stands for, we can omit the labels:

$$\begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 1 & 2 \\ 6 & 1 & 3 \\ 8 & 2 & 4 \end{pmatrix}.$$

$$\begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix}$$ has 3 rows and 1 column and we say that this is a 3×1 **column matrix** or **column vector**.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$ has 1 row and 4 columns and is called a 1×4 **row matrix** or **row vector**.

column 2 ⟶

row 3 ⟶
$$\begin{pmatrix} 4 & 1 & 2 \\ 6 & 1 & 3 \\ 8 & 2 & 4 \end{pmatrix}$$ has 3 rows and 3 columns and is called a 3×3 **square matrix**.

this element, 2, is in row 3, column 2.

> An $m \times n$ matrix has m rows and n columns.
>
> The numbers in the matrix are called its **elements**.
>
> $m \times n$ is called the **order** of the matrix.

We often use capital letters to denote matrices.

For example: $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}.$

DISCUSSION

Why do we agree to state rows first, then columns when we specify a matrix size?

EXERCISE 18A

1 State the order of:

   a $\begin{pmatrix} 6 & 2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$

2 For the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 0 \\ 4 & 1 \end{pmatrix}$ what element is found in:

   a row 1, column 2   **b** row 3, column 1   **c** row 2, column 2?

Example 1   ◀) **Self Tutor**

Peter goes to the local newsagent to buy 3 pens at 60 pence each, 2 notepads at £1.20 each, and one folder at £2.30.

  a Represent the quantities purchased in a row matrix.

  b Represent the costs in a column matrix.

  c When Peter is next at the supermarket, he finds the same items for sale. The prices are 45 pence for each pen, £1.10 per each pad, and £2.50 for the folder. Write one cost matrix which shows prices from both stores.

  a The quantities matrix is $\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$

                    pens notepads folder
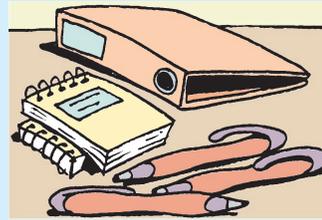
  b The costs matrix is $\begin{pmatrix} 0.60 \\ 1.20 \\ 2.30 \end{pmatrix}$ ← pens
 ← notepads
 ← folder

  c The costs matrix is now: $\begin{pmatrix} 0.60 & 0.45 \\ 1.20 & 1.10 \\ 2.30 & 2.50 \end{pmatrix}$ ← pens
 ← notepads
 ← folder

               newsagent supermarket

3 Claude went to a hardware store to purchase items for his home construction business. He bought 6 hammers, 3 pinchbars, 5 screwdrivers, and 4 drill sets. The costs of the items were $12, $35, $3, and $8 respectively.

  a Construct a row matrix showing the quantities purchased.

  b Construct a column matrix showing the prices of the items in the appropriate order.

4 A baker produces cakes, slices and scones. On Saturday she produced 30 dozen cakes, 40 dozen slices, and 25 dozen scones. On Sunday she produced 25 dozen cakes, 30 dozen slices, and 35 dozen scones. On the Monday she produced 25 dozen of all three items. Construct a 3×3 matrix to display the items made.

5 Seafresh Tuna cans are produced in three sizes: 100 g, 250 g and 500 g.
  In February they produced respectively: 2500, 4000 and 6500 cans in week 1
                                        3000, 3500 and 7000 cans in week 2
                                        3500, 3500 and 6500 cans in week 3
                                        4500   of each type in week 4.

  a Construct a 3×4 matrix to display production levels.

  b Construct a 4×3 matrix to display production levels.

RESEARCH   PRODUCT DETAILS

On the packaging of most consumable food items you will find a matrix showing the ingredients and nutritional content. Collect three of these matrices for similar products. Combine them into a single matrix.

In a brief report, outline how information like this could be used.

DISCUSSION

Discuss the usefulness of organising data in matrix form.

B   MATRIX EQUALITY

Two matrices are **equal** if and only if:

- the matrices have the same shape or order, and
- elements in corresponding positions are equal.

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, the $a = 2$, $b = 1$, $c = 3$, and $d = 5$.

Example 2   ◀) Self Tutor

If $\begin{pmatrix} j & k+1 \\ 2l & m \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 6 & 0 \end{pmatrix}$, find j, k, l and m.

Since the matrices are equal, $j = 2$, $k+1 = -1$, $2l = 6$, $m = 0$

$\therefore$ $j = 2$, $k = -2$, $l = 3$, and $m = 0$

EXERCISE 18B

1 Explain why the following pairs of matrices are not equal:

a $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$

b $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$

2 Find the unknowns if:

a $\begin{pmatrix} 1 & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$

b $\begin{pmatrix} x & 3 \\ 4 & 2y \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 4 & 4 \end{pmatrix}$

c $\begin{pmatrix} 1 & y-1 \\ x & 3 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 2 & 3 \end{pmatrix}$

d $\begin{pmatrix} x^2 & y \\ z & 1-x \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ y-2 & 0 \end{pmatrix}$

C   ADDITION AND SUBTRACTION OF MATRICES

ADDITION

Harry has three stores (A, B and C). His stock levels for TV sets, microwave ovens and refrigerators are given by the matrix:

$$\begin{array}{c} \text{Store} \\ \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{array}{c} \text{TVs} \longrightarrow \\ \text{microwaves} \longrightarrow \\ \text{fridges} \longrightarrow \end{array} \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} \end{array}$$

Some newly ordered stock has just arrived. For each store 30 TVs, 40 microwaves and 50 refrigerators must be added to stock levels.

His stock order is given by the matrix:

$$\begin{pmatrix} 30 & 30 & 30 \\ 40 & 40 & 40 \\ 50 & 50 & 50 \end{pmatrix}$$

Clearly the new levels are shown as:

$$\begin{pmatrix} 23+30 & 41+30 & 68+30 \\ 28+40 & 39+40 & 79+40 \\ 46+50 & 17+50 & 62+50 \end{pmatrix}$$

or $$\begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 30 & 30 & 30 \\ 40 & 40 & 40 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 53 & 71 & 98 \\ 68 & 79 & 119 \\ 96 & 67 & 112 \end{pmatrix}$$

To **add** two matrices of the same shape, we **add corresponding elements**.

We cannot add matrices of different shape.

ZERO MATRIX

A **zero matrix** is one which has all elements zero.

For example, $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the 2×2 zero matrix.

For any matrix **A**, $$\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$$

SUBTRACTION

If Harry's stock levels were $\begin{pmatrix} 39 & 61 & 29 \\ 41 & 38 & 42 \\ 50 & 27 & 39 \end{pmatrix}$ and his sales matrix for the week is

$\begin{pmatrix} 16 & 13 & 7 \\ 21 & 17 & 20 \\ 20 & 9 & 15 \end{pmatrix}$ what are the current stock levels?

It is clear that we must subtract corresponding elements:

$$\begin{pmatrix} 39 & 61 & 29 \\ 41 & 38 & 42 \\ 50 & 27 & 39 \end{pmatrix} - \begin{pmatrix} 16 & 13 & 7 \\ 21 & 17 & 20 \\ 20 & 9 & 15 \end{pmatrix} = \begin{pmatrix} 23 & 48 & 22 \\ 20 & 21 & 22 \\ 30 & 18 & 24 \end{pmatrix}$$

To **subtract** matrices of the same shape, we **subtract corresponding elements**.
We cannot subtract matrices of different shape.

Example 3   ◀) Self Tutor

If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$

a $\mathbf{A} + \mathbf{B}$
$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$= \begin{pmatrix} 2+1 & 1+0 \\ -1+2 & 0+-1 \end{pmatrix}$
$= \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

b $\mathbf{A} - \mathbf{B}$
$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$= \begin{pmatrix} 2-1 & 1-0 \\ -1-2 & 0--1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$

Example 4   ◀) Self Tutor

At the end of March, Harry's stock matrix was:

Store
A   B   C
TVs ⟶ $\begin{pmatrix} 28 & 13 & 17 \\ 19 & 24 & 22 \\ 25 & 16 & 10 \end{pmatrix}$
microwaves ⟶
fridges ⟶

His stock order matrix in early April was

$\begin{pmatrix} 20 & 37 & 23 \\ 21 & 16 & 18 \\ 15 & 24 & 30 \end{pmatrix}$ and his sales during April were $\begin{pmatrix} 31 & 28 & 26 \\ 20 & 31 & 19 \\ 25 & 17 & 32 \end{pmatrix}$.

What is Harry's stock matrix at the end of April?

At the beginning of April the stock matrix was:
$\begin{pmatrix} 28 & 13 & 17 \\ 19 & 24 & 22 \\ 25 & 16 & 10 \end{pmatrix} + \begin{pmatrix} 20 & 37 & 23 \\ 21 & 16 & 18 \\ 15 & 24 & 30 \end{pmatrix} = \begin{pmatrix} 48 & 50 & 40 \\ 40 & 40 & 40 \\ 40 & 40 & 40 \end{pmatrix}$

At the end of April the stock matrix is:
$\begin{pmatrix} 48 & 50 & 40 \\ 40 & 40 & 40 \\ 40 & 40 & 40 \end{pmatrix} - \begin{pmatrix} 31 & 28 & 26 \\ 20 & 31 & 19 \\ 25 & 17 & 32 \end{pmatrix} = \begin{pmatrix} 17 & 22 & 14 \\ 20 & 9 & 21 \\ 15 & 23 & 8 \end{pmatrix}$

EXERCISE 18C

1 If possible, find:

a $(1 \quad 0 \quad 4) + (2 \quad 3 \quad -2)$

b $\begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

c $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 3 & -1 \end{pmatrix}$

d $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 3 \\ -3 & 2 & 0 \end{pmatrix}$

e $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

f $\begin{pmatrix} 6 & 1 & 6 \\ 0 & 2 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ -1 & -5 \end{pmatrix}$

2 If $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$

b $\mathbf{A} - \mathbf{C}$

c $\mathbf{A} - \mathbf{B} + \mathbf{C}$

3 At the end of April, Harry's stock matrix is $\begin{pmatrix} 17 & 22 & 14 \\ 20 & 9 & 21 \\ 15 & 23 & 8 \end{pmatrix}$.

a What will be his stock order matrix if he wishes to have 50 of each item in each store at the start of May?

b During May, Harry's sales matrix is $\begin{pmatrix} 22 & 17 & 28 \\ 19 & 22 & 30 \\ 41 & 7 & 44 \end{pmatrix}$.

What is his stock matrix at the end of May?

4 A restaurant served 72 men, 84 women and 49 children on Friday night. On Saturday night they served 86 men, 72 women and 46 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women and children served over the two day period.

5 On Tuesday Keong bought shares in five companies and on Friday he sold them. The details are:

a Find Keong's:

    i cost price column matrix

    ii selling price column matrix.

	Cost price per share	Selling price per share
A	RM 1.23	RM 1.38
B	RM 22.15	RM 22.63
C	RM 0.72	RM 0.69
D	RM 3.75	RM 3.68
E	RM 4.96	RM 5.29

b What matrix operation is needed to find Keong's profit or loss matrix?

c Find Keong's profit or loss matrix.

6 If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{A} + \mathbf{B} = \mathbf{O}$.

D   SCALAR MULTIPLICATION

A matrix can be multiplied by a number or **scalar** to produce another matrix.

For example, in the **Opening Problem**, each of Gerty's 10 apartments has 4 chairs, 1 table and 2 beds.

If $\mathbf{F} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ represents the furniture in one each apartment, then the matrix representing

the furniture in *all* apartments can be writen as $10\mathbf{F} = \begin{pmatrix} 10 \times 4 \\ 10 \times 2 \\ 10 \times 1 \end{pmatrix}$ or $\begin{pmatrix} 40 \\ 20 \\ 10 \end{pmatrix}$.

> If a scalar k is multiplied by a matrix $\mathbf{A}$ the result is matrix $k\mathbf{A}$ obtained by multiplying every element of $\mathbf{A}$ by k.

Example 5   ◄) **Self Tutor**

If $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, find:

a $2\mathbf{A}$ **b** $\mathbf{A} - 2\mathbf{B}$

a $2\mathbf{A} = 2\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ **b** $\mathbf{A} - 2\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix}$ $= \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} -2 & 1 \\ 4 & -3 \end{pmatrix}$

EXERCISE 18D

1 If $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$, find:

    a $2\mathbf{A}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{A} + 2\mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

2 Bob's order for hardware items is shown in matrix form as

$\mathbf{H} = \begin{pmatrix} 10 \\ 20 \\ 100 \\ 40 \end{pmatrix}$ ← hammers
← screw driver sets
← packets of nails
← packets of screws

Find the matrix if:

    a Bob doubles his order

    b Bob halves his order

    c Bob increases his order by 50%.

3 Sarah sells dresses made by four different companies which we will call A, B, C and D. Her usual monthly order is shown in the matrix opposite.

Find her order if:

   a she increases her total order by 20%

   b she decreases her total order by 20%.

$$\begin{array}{c} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} \begin{array}{cccc} A & B & C & D \\ \left(\begin{array}{cccc} 40 & 30 & 30 & 60 \\ 50 & 40 & 30 & 60 \\ 40 & 60 & 50 & 40 \\ 20 & 20 & 20 & 20 \end{array} \right) \end{array}$$

4 Colin's stock matrix is **M**. His normal weekly order matrix is **N** and his sales matrix is **S**. Unfortunately this time Colin ordered twice. Which of the following will be his actual stock holding at the end of this period?

   A $M + N + S$ **B** $M + N - S$ **C** $M + 2N - S$ **D** $M + N - 2S$

E   MATRIX MULTIPLICATION

In **Example 1**, Peter bought 3 pens, 2 notepads and 1 folder. At the local newsagent they cost 60 pence, £1.20, and £2.30 each respectively. At the supermarket they cost 45 pence, £1.10, and £2.50 each respectively.

We can represent this information by using a quantities matrix $\mathbf{Q} = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$

and a costs matrix $\mathbf{C} = \begin{pmatrix} 0.60 & 0.45 \\ 1.20 & 1.10 \\ 2.30 & 2.50 \end{pmatrix}$.

The total cost at the newsagent is $(3 \times 0.60 + 2 \times 1.20 + 1 \times 2.30)$ pounds $= £6.50$.
The total cost at the supermarket is $(3 \times 0.45 + 2 \times 1.10 + 1 \times 2.50)$ pounds $= £6.05$.

We can write this using the **matrix multiplication**

$$\mathbf{QC} = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0.60 & 0.45 \\ 1.20 & 1.10 \\ 2.30 & 2.50 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 0.60 + 2 \times 1.20 + 1 \times 2.30 & 3 \times 0.45 + 2 \times 1.10 + 1 \times 2.50 \end{pmatrix}$$

Matrix multiplication has a different meaning to that used for multiplying numbers. In this case we multiply each element of a row by the corresponding element of a column, and then add the results.

> In matrix multiplication, two or more matrices are 'multiplied' to produce a final matrix.

If matrix **A** is $m \times n$ and matrix **B** is $n \times p$ then their product **AB** exists and is an $m \times p$ matrix.

We multiply each member of a row in **A** by each member of a column of **B** and add the results. We do this for all possible rows and columns.

If the number of columns in **A** does not equal the number of rows in **B**, then **AB** does not exist.

For example, $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \times 3 + 2 \times 1 + 1 \times 2 \\ 2 \times 3 + 1 \times 1 + 0 \times 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

$2 \times 3 \qquad 3 \times 1 = \qquad 2 \times 1$

the same

Example 6

◀) Self Tutor

If $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 4 \end{pmatrix}$, find AB.

A is 2×3 and B is 3×2 $\quad \therefore \quad AB = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 4 \end{pmatrix}$

$\therefore \quad AB$ is 2×2

$= \begin{pmatrix} 1 + 6 + 15 & 1 + 0 + 20 \\ 2 + 8 + 18 & 2 + 0 + 24 \end{pmatrix}$

$= \begin{pmatrix} 22 & 21 \\ 28 & 26 \end{pmatrix}$

EXERCISE 18E

1 If **A** is 3×2 and **B** is 2×1:

   a explain whether **AB** can be found

   b find the shape of **AB**

   c explain why **BA** cannot be found.

2 For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, find **AB**.

3 For $A = \begin{pmatrix} 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find: **a** **AB**   **b** **BA**.

4 Find:

   a $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$

   b $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

   c $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}$

   d $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

5 At the local fair it costs \$3 for a pony ride and \$5 for a go-cart ride. On day 1 there are 43 pony rides and 87 go-cart rides. On day 2 there are 48 pony rides and 66 go-cart rides.

   a Write the costs in the form of a 2×1 cost matrix **C**.

   b Write down the 2×2 numbers matrix **N**.

   c Find **NC** and interpret your result.

   d Find the total income for both rides over the two days.

6 You and your friend each go to your local supermarkets A and B to price items you wish to purchase. You want to buy 1 leg of ham, 1 Christmas Pudding and 2 litres of cola. Your friend wants 1 leg of ham, 2 Christmas puddings and 3 litres of cola. The prices of these goods are:

	Leg of ham	*Christmas pudding*	*Litre of cola*
A	\$43	\$7	\$3
B	\$39	\$8	\$4

   a Write the requirements matrix **R** as a 3×2 matrix.

   b Write the prices matrix **P** as a 2×3 matrix.

   c Find **PR**.

   d What are your costs at A and your friend's costs at B?

   e Should you buy from A or B?

7 Show how to solve the **Opening Problem** on page **368** by using matrix multiplication.

8 Suppose **A** is any 2×2 matrix. Find a matrix **I** such that $\quad \mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

9 Suppose $\mathbf{A} = \begin{pmatrix} a & b & c & d \end{pmatrix}$ where a, b, c and d are four numbers.

   a What matrix do we need to multiply **A** by to find the sum of the four numbers?

   b Use matrix multiplication to represent the mean of:

      i a, b and c **ii** a, b, c and d.

10 $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$. If $\mathbf{A}^2 = 2\mathbf{A}$ and $\mathbf{A} = \begin{pmatrix} x & a \\ b & x \end{pmatrix}$ where a and b are non-zero integers, find the possible values of x, a and b.

F   MATRICES USING TECHNOLOGY

Computer packages and graphics calculators can be used to store matrices and perform matrix operations. Instructions for graphics calculators can be found on pages **25** to **26**.

EXERCISE 18F

Using technology to answer the following questions:

1 $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 0 & 5 \\ 2 & 1 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & 9 \\ 0 & 2 & 7 \\ 1 & 5 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 5 & 6 \end{pmatrix}$ and $D = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}$.

Use technology to find:

a $37A$　　　　　**b** $46B$　　　　　**c** $23A + 97B$　　　　**d** $39A - 17B$

e AC　　　　　**f** CA　　　　　**g** BD　　　　　**h** DB

2 The selling prices of five white goods items are given by the matrix $S = \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \begin{pmatrix} 238 \\ 167 \\ 529 \\ 381 \\ 267 \end{pmatrix}$ €

The costs incurred by the store in the sale of these items are given by the matrix $C = \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \begin{pmatrix} 164 \\ 180 \\ 398 \\ 306 \\ 195 \end{pmatrix}$ €.

a Find the profit matrix $S - C$ for the five items.

b Suppose 48 of item A, 67 of B, 103 of C, 89 of D and 114 of E are sold.
Write down the corresponding 1×5 sales matrix N.

c Find the total profit made on the sale of these items using matrix multiplication.

3

	P	Q	R	S	T
Day 1	17	3	0	14	9
Day 2	31	5	1	16	14
Day 3	28	8	2	22	26
Day 4	32	7	1	9	32
Day 5	19	6	3	11	17
Day 6	31	9	0	18	18

is a sales matrix and $\begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \begin{pmatrix} 33 \\ 18 \\ 22 \\ 44 \\ 43 \end{pmatrix}$ is a price matrix in pounds.

Find the 6×1 matrix for the total income on each day.

4 Four supermarkets owned by Jason sell 600 g cans of peaches. Store A buys them for $2.00 and sells them for $3.00; Store B buys them for $1.80 and sells them for $2.90; Store C buys them for $2.10 and sells them for $3.15; Store D buys them for $2.25 and sells them for $2.95.

a Set up two column matrices, one for costs C and one for selling prices S.

b What is the profit matrix for each store?

c If A sells 367, B sells 413, C sells 519 and D sells 846 cans in one week, determine the total profit made on the cans during the week.
Use a product of matrices to find your answer.

5 Sadi owns five stores. She keeps track of her six different sales items using the stock matrix **S** alongside:

Sadi makes a profit of $117 on item A, $85 on B, $106 on C, $128 on D, $267 on E and $179 on F.

$$\text{Item}$$

$$S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{cccccc} A & B & C & D & E & F \\ \left(\begin{array}{cccccc} 42 & 24 & 26 & 52 & 39 & 68 \\ 38 & 27 & 32 & 60 & 50 & 68 \\ 57 & 27 & 40 & 58 & 40 & 73 \\ 39 & 30 & 38 & 47 & 39 & 79 \\ 62 & 31 & 32 & 71 & 52 & 80 \end{array}\right) \end{array}$$

  a Noting that **S** is a 5×6 matrix, should Sadi write down the profits matrix with order 6×1 or 1×6? Explain your answer.

  b Find a matrix which shows the *total* profit for each store.

REVIEW SET 18A

1 Consider the matrix $\begin{pmatrix} 4 & 2 & 7 & 4 \\ 3 & 5 & -1 & 4 \\ 0 & 2 & 6 & 7 \end{pmatrix}$.

  a What is its order?

  b What element is in row 3, column 2?

2 Find a, b and c given that: $\begin{pmatrix} a & a^2 \\ 2b & b+c \end{pmatrix} = \begin{pmatrix} a^2 & 1 \\ b & b \end{pmatrix}$

3 If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & -2 \\ 0 & 6 \end{pmatrix}$ find:

  a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

  e $3\mathbf{A}$ **f** $\frac{1}{2}\mathbf{B}$ **g** $-2\mathbf{B}$ **h** $\mathbf{B} - \frac{1}{2}\mathbf{A}$

4 If **A** is $m \times n$ and **B** is $r \times s$, what can be said if:

  a **AB** can be found   **b** **BA** can be found?

5 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & -2 \end{pmatrix}$. Find, if possible:

  a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$

6 Peter bought 15 drills, 14 sanders, 8 bench saws, and 18 circular saws for his construction business. Their individual costs were $45, $67, $315, and $56 respectively.

  a Write the item numbers as a 1×4 matrix **N**.

  b Write the costs as a 4×1 matrix **C**.

  c Use matrix methods to find the total cost of the power tools.

7 After the football match on Saturday afternoon, the four friends Jack, Angus, Karaline and Joan decided on a snack. Their order is shown in the matrix opposite.

A piece of fish costs £2.30. A serve of chips is £1.80. Dim Sims are 50 pence each. Pineapple fritters are £0.65 and drinks are £1.95 each.

$$\begin{array}{cccc} \text{Ja} & \text{An} & \text{Ka} & \text{Jo} \end{array}$$
$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \text{Fish} \\ \text{Chips} \\ \text{Dim Sims} \\ \text{Pineapple Fritter} \\ \text{Drink} \end{array}$$

Represent this information as a cost matrix and hence find:

    a each person's cost   **b** the total cost.

REVIEW SET 18B

1 Write down the order of the following matrices:

  a $\begin{pmatrix} 16 & -2 & 5 \\ 19 & 8 & -7 \end{pmatrix}$ **b** $\begin{pmatrix} -6 \\ 3 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 3 & -4 & 5 \\ 10 & 8 & -1 & 0 \\ 0 & 3 & 6 & 2 \end{pmatrix}$

2 Find a, b and c if $\begin{pmatrix} a & b \\ c^2 & -5 \end{pmatrix} = \begin{pmatrix} 3a - 4 & 2(a+b) \\ 9 & c-2 \end{pmatrix}$

3 If $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & -1 \\ 2 & 0 \end{pmatrix}$, find:

  a $\mathbf{A} + \mathbf{B}$ **b** $-\frac{1}{2}\mathbf{A}$ **c** $2\mathbf{A} - 5\mathbf{B}$ **d** $\mathbf{AB}$

4 Find, if possible:

  a $\begin{pmatrix} -2 & 0 & 3 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & -2 \\ 3 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 6 & -2 \\ 4 & -1 \end{pmatrix}$

  c $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \begin{pmatrix} 5 & -1 & 4 \end{pmatrix}$

5 A café sells two types of cola drinks, A and B. The drinks each come in three sizes: small, medium and large. At the beginning of the day the fridge was stocked with the number of units shown in the matrix below. At the end of the day the stock was again counted.

    Start of the day   *End of the day*

$$\begin{array}{l} \text{small} \longrightarrow \\ \text{medium} \longrightarrow \\ \text{large} \longrightarrow \end{array} \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{pmatrix} 42 & 54 \\ 36 & 27 \\ 34 & 30 \end{pmatrix}$$

$$\begin{array}{l} \text{small} \longrightarrow \\ \text{medium} \longrightarrow \\ \text{large} \longrightarrow \end{array} \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{pmatrix} 27 & 31 \\ 28 & 15 \\ 28 & 22 \end{pmatrix}$$

The profit in dollars for each item is given by: $\begin{array}{ccc} \text{small} & \text{medium} & \text{large} \end{array}$ $\begin{pmatrix} 0.75 & 0.55 & 1.20 \end{pmatrix}$

Use matrix methods to calculate the total profit made for the day from the sale of these drinks.

6 Matrix **A** is 3×7 and matrix **B** is $n \times 4$.

   a When can matrix **AB** be calculated?

   b If **AB** can be found, what is its order?

   c Can **BA** be calculated?

7 Kelly has five women's clothing shops: A, B, C, D and E. At these shops she sells standard items: skirts, dresses, suits and slacks. At the end of week 1 in May her stock holding is given by the matrix:

$$
\begin{array}{c}
 \\ \text{skirts} \\ \text{dresses} \\ \text{suits} \\ \text{slacks}
\end{array}
\begin{array}{ccccc}
\text{A} & \text{B} & \text{C} & \text{D} & \text{E} \\
\left(\begin{array}{ccccc}
27 & 38 & 14 & 59 & 26 \\
31 & 42 & 29 & 16 & 35 \\
20 & 23 & 25 & 17 & 32 \\
26 & 59 & 40 & 31 & 17
\end{array}\right)
\end{array}
$$

   a Her sales matrix for week 2 was $\begin{pmatrix} 15 & 17 & 11 & 22 & 20 \\ 28 & 31 & 26 & 10 & 30 \\ 17 & 9 & 11 & 11 & 22 \\ 13 & 32 & 30 & 27 & 8 \end{pmatrix}$.

   Find her stock matrix at the end of week 2.

   b Her order matrix at the end of week 2 was $\begin{pmatrix} 22 & 22 & 22 & 22 & 22 \\ 24 & 26 & 28 & 28 & 30 \\ 15 & 20 & 15 & 20 & 15 \\ 17 & 21 & 16 & 19 & 30 \end{pmatrix}$.

   Find her stock matrix at the start of week 3.

   c If skirts cost her \$38, dresses \$75, suits \$215 and slacks \$48, how much will she have to pay for her end of week 2 order?

Chapter

19

Quadratic functions

In this chapter, we study the relationship between two variables which are linked by a **quadratic function** of the form $y = ax^2 + bx + c$, $a \neq 0$.

The graphs of all quadratic functions are known as **parabolas**.

The parabola is the shape used in the mirror surfaces of motor vehicle headlights, torches, and satellite dishes. Parabolas are also used as arches in buildings and spans of bridges.
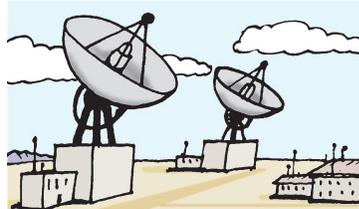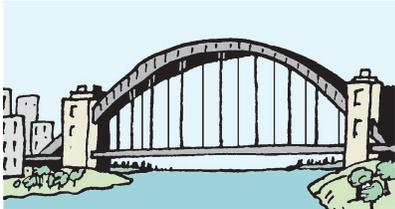
RESEARCH   THE USE OF PARABOLIC CURVES

Research the use of parabolic curves. Your work must include sketches, properties of the parabola including its focus, and reasons for using this shape rather than other shapes.

OPENING PROBLEM

Tomas has a garden hose with a nozzle that allows water from shoot out in a continuous fine line.

A set of axes are drawn on the wall and coordinates allocated to the marked points. Tomas stands with the nozzle about 5 cm from the wall and fires the water jet parallel to the wall. While Tomas holds the hose rigidly in position, his friend Troy marks the height of the water jet at various positions along the wall with chalk. The tap is then switched off.
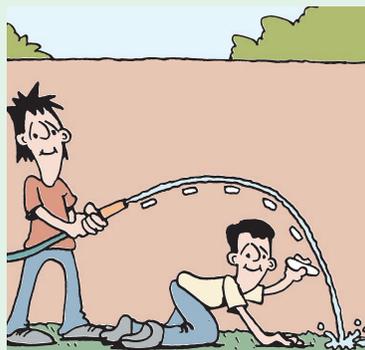
Things to think about:

- Does the water jet form part of a parabola (or approximately so)?
- Can we find a formula of the form $y = ax^2 + bx + c$ which models the shape of the water jet?

VIDEO CLIP

A   QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables which can be written in the form $y = ax^2 + bx + c$ where x and y are the variables and a, b, and c represent constants with $a \neq 0$.

For example:

$y = 3x^2 - 2x + 1$ is a quadratic function with $a = 3$, $b = -2$, $c = 1$

$y = 2x^2 + 3x$ is a quadratic function with $a = 2$, $b = 3$, $c = 0$.

Example 1 ◀)) **Self Tutor**

Are the following quadratic functions?

a $y = -x^2 + 3x - 4$ **b** $y = 2x^2 + 1$ **c** $y = x^3 + 2x + 7$

a Yes, with $a = -1$, $b = 3$, $c = -4$
b Yes, with $a = 2$, $b = 0$, $c = 1$
c No, as it has an x^3 term rather than an x^2 term.

FINDING y GIVEN x

For any known value of x, the corresponding value of y can be found by **substituting** into the function.

Example 2 ◀)) **Self Tutor**

If $y = 2x^2 - 3x + 4$, find y when: **a** $x = 2$ **b** $x = -1$.

a When $x = 2$,
$y = 2(2)^2 - 3(2) + 4$
$= 8 - 6 + 4$
$= 6$

b When $x = -1$,
$y = 2(-1)^2 - 3(-1) + 4$
$= 2 \times 1 + 3 + 4$
$= 2 + 3 + 4$
$= 9$

Given the coordinates of a point, we can test if the point lies on the graph of a function by **substitution**.

We substitute the x-coordinate of the point into the function and see if the resulting value matches the y-coordinate of the point.

Example 3 ◀)) **Self Tutor**

Do A(2, -8) and B(-3, 4) satisfy the quadratic function $y = -x^2 - 3x + 2$?

When $x = 2$,
$y = -(2)^2 - 3(2) + 2$
$= -4 - 6 + 2$
$= -8$

When $x = -3$,
$y = -(-3)^2 - 3(-3) + 2$
$= -9 + 9 + 2$
$= 2$ which is $\neq 4$

∴ A satisfies the function, so A lies on its graph.

∴ B does not satisfy the function, so B does not lie on its graph.

FINDING x GIVEN y

When we substitute a value for y, we are left with a quadratic equation which we need to solve for x. Since the equation is quadratic, there may be 0, 1 or 2 possible values for x for any one value of y.

Example 4   ◀) **Self Tutor**

If $y = x^2 + 4x - 5$ find the value(s) of x when: **a** $y = 0$ **b** $y = -9$

a When $y = 0$,
$$x^2 + 4x - 5 = 0$$
$$\therefore \quad (x+5)(x-1) = 0$$
$$\therefore \quad x = -5 \quad \text{or} \quad x = 1$$
So, there are 2 solutions.

b When $y = -9$,
$$x^2 + 4x - 5 = -9$$
$$\therefore \quad x^2 + 4x + 4 = 0$$
$$\therefore \quad (x+2)^2 = 0$$
$$\therefore \quad x = -2$$
So, there is only one solution.

EXERCISE 19A

1 Are the following quadratic functions?
Give values for a, b, c if they are, and a reason if they are not.

  a $y = 2x^2 + x + 4$
  b $y = 3x + 8$
  c $y = -x^2 - x - 1$
  d $y = x^2$
  e $y = \frac{1}{2}x^2 - 6$
  f $y = -2x + 5x^3$
  g $y = -3x^2$
  h $y = -x^{-2} - 4x$
  i $2y + 4x^2 + 5 = 0$

2 If $y = x^2 + 2x - 5$,   find y when:

  a $x = 3$
  b $x = 0$
  c $x = -2$

3 If $y = 3x^2 - x + 4$,   find y when:

  a $x = 0$
  b $x = 2$
  c $x = -3$

4 If $y = -2x^2 - 4x + 7$,   find y when:

  a $x = 1$
  b $x = -3$
  c $x = \frac{1}{2}$

5 **a** Do A(3, 10) and B(−2, 13) lie on $y = 2x^2 - 3x - 1$?
  b Do P(0, 2), Q(2, 7) and R(−1, −4) lie on $y = -x^2 + 5x + 2$?

6 **a** The point C(2, k) lies on $y = x^2 + 3x - 7$.   Find k.
  b The point D(−3, d) lies on $y = 4x - 3x^2$.   Find d.
  c The point E(m, 7) lies on $y = x^2 + 3x - 3$.   Find m.
  d The point F(n, 29) lies on $y = 2x^2 + x + 8$.   Find n.

7 If $y = x^2 + 2x + 3$,   find x when:

  a $y = 0$
  b $y = 2$

8 If $y = x^2 - 4x + 6$,   find x when:

  a $y = 2$
  b $y = 0$
  c $y = 11$

9 **a** If $y = x^2 + 6x + 7$,   find x when $y = 7$.

    b If $y = x^2 - 4x - 1$,   find x when $y = 11$.

    c If $y = 3x^2 - 5x$,   find x when $y = -2$.

10 A cannonball is fired into the air. Its height is given by $H = 60t - 5t^2$ metres   where t is the time in seconds after its release.

    a Find the height of the cannonball when:

        i $t = 0$ **ii** $t = 1$ **iii** $t = 3$ **iv** $t = 8$

    b Find the time at which the height:

        i $H = 0$ **ii** $H = 100$ **iii** $H = 160$

    c Graph $H = 60t - 5t^2$ using the information from **a** and **b** only. Connect the points using a smooth curve.

11 A stone is thrown into the air. Its height above the ground is given by the function $H = -5t^2 + 30t + 2$ metres   where t is the time in seconds from when the stone is released.

    a How high above the ground was the stone released?

    b How high above the ground is the stone at time $t = 3$ seconds?

    c At what times was the stone's height above the ground: **i** 27 m **ii** 42 m?

    d Draw the graph of H against t using **a**, **b** and **c** only.

12 Bjorn makes wooden display cabinets. He finds that the profit in euros for making x cabinets per day is given by $P = 320x - 20x^2 - 320$ euro.

    a Calculate Bjorn's profit if he makes:

        i 0   **ii** 4   **iii** 10 cabinets per day.

    b How many cabinets per day must Bjorn make to receive a €460 profit?

B   GRAPHS OF QUADRATIC FUNCTIONS

The simplest quadratic function is $y = x^2$.
Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

The **vertex** is the point where the graph is at its maximum or minimum.

Notice that:

- The curve is a **parabola** and it opens upwards.

- There are no negative y values,   i.e., the curve does not go below the x-axis.

- The curve is **symmetrical** about the y-axis because, for example, when $x = -3$, $y = (-3)^2 = 9$ and when $x = 3$, $y = 3^2 = 9$.

- The curve has a **turning point** or **vertex** at $(0, 0)$.
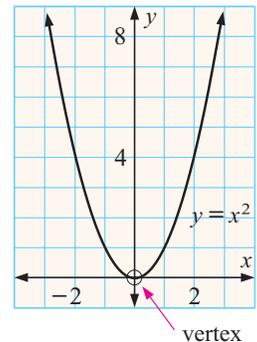
Example 5   ◀) **Self Tutor**

Draw the graph of $y = x^2 - 2x - 1$ from a table of values from $x = -3$ to $x = 3$.

When $x = -3$, $y = (-3)^2 - 2(-3) - 1$ When $x = 1$, $y = 1^2 - 2(1) - 1$
$= 9 + 6 - 1$ $= 1 - 2 - 1$
$= 14$ $= -2$

When $x = -2$, $y = (-2)^2 - 2(-2) - 1$ When $x = 2$, $y = 2^2 - 2(2) - 1$
$= 4 + 4 - 1$ $= 4 - 4 - 1$
$= 7$ $= -1$

When $x = -1$, $y = (-1)^2 - 2(-1) - 1$ When $x = 3$, $y = 3^2 - 2(3) - 1$
$= 1 + 2 - 1$ $= 9 - 6 - 1$
$= 2$ $= 2$

When $x = 0$, $y = (0)^2 - 2(0) - 1$
$= -1$

The tabled values are:

x	-3	-2	-1	0	1	2	3
y	14	7	2	-1	-2	-1	2
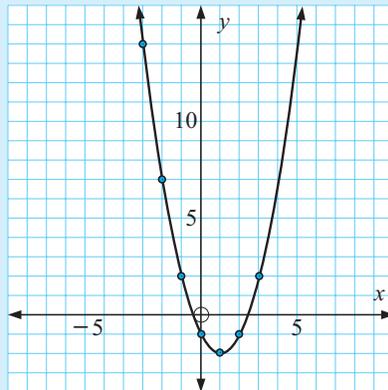
EXERCISE 19B

1 Use a table of values from $x = -3$ to $x = 3$ to draw the graph of:

    a $y = x^2$ **b** $y = x^2 - 2x$ **c** $y = x^2 - 2x - 2$

    d $y = 2x^2$ **e** $y = 2x^2 + 1$ **f** $y = \frac{1}{2}x^2 - 2$

    g $y = -x^2$ **h** $y = -x^2 + x$ **i** $y = -x^2 + 2x + 1$

2 Use the graphing package to check your answers to question **1**.   **GRAPHING PACKAGE**

3 What part of $y = ax^2 + bx + c$ affects:

    a whether the parabola opens upwards ∪ or downwards ∩

    b whether the parabola is thinner ∪ or wider ∪
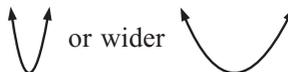
    c where the parabola cuts the y-axis?

INVESTIGATION 1   GRAPHS OF QUADRATIC FUNCTIONS

In this investigation we consider different forms of quadratic functions, and how the form of the quadratic affects its graph.

GRAPHING PACKAGE

Part 1:   Graphs of the form $y = x^2 + k$ where k is a constant

What to do:

1 Using a **graphing package** or **graphics calculator**:

    i graph the two functions on the same set of axes

    ii state the coordinates of the vertex of each function.

    a $y = x^2$ and $y = x^2 + 1$ **b** $y = x^2$ and $y = x^2 - 1$

    c $y = x^2$ and $y = x^2 + 3$ **d** $y = x^2$ and $y = x^2 - 3$

2 What effect does the value of k have on:

    a the position of the graph   **b** the shape of the graph?

3 What transformation is needed to graph $y = x^2 + k$ from $y = x^2$?

Part 2:   Graphs of the form $y = ax^2$, $a \neq 0$

What to do:

1 Using a **graphing package** or **graphics calculator**:

    i graph the two functions on the same set of axes

    ii state the coordinates of the vertex of each function.

    a $y = x^2$ and $y = 2x^2$ **b** $y = x^2$ and $y = 3x^2$

    c $y = x^2$ and $y = -x^2$ **d** $y = x^2$ and $y = \frac{1}{2}x^2$

    e $y = x^2$ and $y = -2x^2$ **f** $y = x^2$ and $y = -3x^2$

2 These functions are all members of the family $y = ax^2$ where a is the coefficient of the x^2 term. What effect does a have on:

    a the position of the graph   **b** the shape of the graph

    c the direction in which the graph opens?

Part 3:   Graphs of the form $y = (x - h)^2$

What to do:

1 Using a **graphing package** or **graphics calculator**:

    i graph the two functions on the same set of axes

    ii state the coordinates of the vertex of each function.

    a $y = x^2$ and $y = (x - 1)^2$ **b** $y = x^2$ and $y = (x + 1)^2$

    c $y = x^2$ and $y = (x - 3)^2$ **d** $y = x^2$ and $y = (x + 3)^2$

2 What effect does the value of h have on:

    a the position of the graph   **b** the shape of the graph?

3 What transformation is needed to graph $y = (x - h)^2$ from $y = x^2$?

Part 4: Graphs of the form $y = (x - h)^2 + k$

What to do:

1 *Without the assistance of technology*, sketch the graph of $y = (x - 1)^2 + 2$. State the coordinates of the vertex and comment on the shape of the graph.

2 Use a **graphing package** or **graphics calculator** to draw, on the same set of axes, the graphs of $y = x^2$ and $y = (x - 1)^2 + 2$.

3 Repeat steps **1** and **2** for $y = (x + 2)^2 - 3$.

4 Copy and complete:

- The graph of $y = (x - h)^2 + k$ is the same shape as the graph of ....
- The graph of $y = (x - h)^2 + k$ is an .... of the graph of $y = x^2$ through a translation of ....

Part 5: Graphs of the form $y = a(x - h)^2 + k, \quad a \neq 0$

What to do:

1 *Without the assistance of technology*, sketch the graphs of $y = 2x^2$ and $y = 2(x - 3)^2 + 1$ on the same set of axes. State the coordinates of the vertices and comment on the shape of the two graphs.

2 Use a **graphing package** or **graphics calculator** to check your graphs in step **1**.

3 Repeat steps **1** and **2** for:

    a $y = -x^2$ and $y = -(x + 1)^2 + 2$

    b $y = \frac{1}{2}x^2$ and $y = \frac{1}{2}(x - 3)^2 + 1$

4 Copy and complete:

- The graph of $y = a(x - h)^2 + k$ has the same shape and opens in the same direction as the graph of ....
- The graph of $y = a(x - h)^2 + k$ is an .... of the graph of $y = ax^2$ through a translation of ....

From the investigation the following important facts should have been discovered.

- Graphs of the form $y = x^2 + k$ have exactly the same shape as the graph of $y = x^2$. In fact, k is the **vertical translation** factor.

  Every point on the graph of $y = x^2$ is translated $\begin{pmatrix} 0 \\ k \end{pmatrix}$ to give the graph of $y = x^2 + k$.

- Graphs of the form $y = (x - h)^2$ have exactly the same shape as the graph of $y = x^2$. In fact, h is the **horizontal translation** factor.

  Every point on the graph of $y = x^2$ is translated $\begin{pmatrix} h \\ 0 \end{pmatrix}$ to give the graph of $y = (x - h)^2$.

- Graphs of the form $y = (x - h)^2 + k$ have the same shape as the graph of $y = x^2$ and can be obtained from $y = x^2$ by using a **horizontal shift** of h units and a **vertical shift** of k units. This is a **translation** of $\begin{pmatrix} h \\ k \end{pmatrix}$. The **vertex** is at (h, k).

- If $a > 0$, $y = ax^2$ opens upwards   i.e., $\smile$

  If $a < 0$, $y = ax^2$ opens downwards   i.e., $\frown$

  If $a < -1$ or $a > 1$ then $y = ax^2$ is 'thinner' than $y = x^2$.
  If $-1 < a < 1$, $a \neq 0$ then $y = ax^2$ is 'wider' than $y = x^2$.

-
$a > 0$ $\smile$	vertical shift of k units:
$a < 0$ $\frown$	if $k > 0$ it goes up
	if $k < 0$ it goes down

  $$y = a(x - h)^2 + k$$

$a < -1$ or $a > 1$, thinner than $y = x^2$	horizontal shift of h units:
$-1 < a < 1$, $a \neq 0$, wider than $y = x^2$	if $h > 0$ it goes right
	if $h < 0$ it goes left

C   USING TRANSFORMATIONS TO SKETCH QUADRATICS

In this section we will graph quadratic functions without finding a table of values.

To do this we use the findings from **Investigation 1** only. When we sketch graphs, they do not have to be perfectly to scale. However, we must show the main features of the graph using labels.

Example 6   ◀)) Self Tutor

Sketch the graphs of the following using transformations:

a $y = x^2 + 2$ b $y = -2x^2$

a $y = x^2 + 2$ comes from $y = x^2$ under the translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
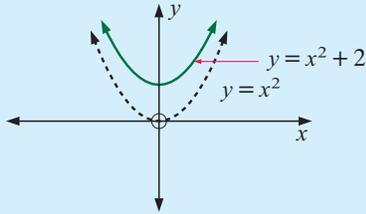
b $y = -2x^2$ comes from $y = x^2$ by making it thinner and reflecting it in the x-axis.
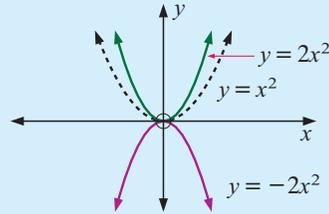
Example 7 ◀) Self Tutor

Using a translation, sketch the graphs of:

a $y = (x - 2)^2 + 3$

b $y = -(x + 1)^2 - 2$

State the coordinates of each vertex.

a $y = (x - 2)^2 + 3$ is a translation of $y = x^2$ under $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
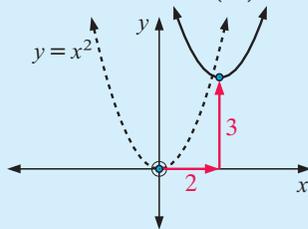
The vertex is at $(2, 3)$.

b $y = -(x + 1)^2 - 2$ is a translation of $y = -x^2$ under $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$
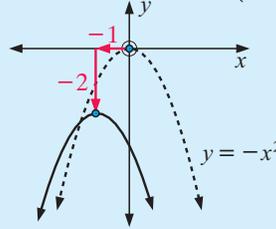
The vertex is at $(-1, -2)$.

EXERCISE 19C

1 On separate axes, sketch the graphs of the following using transformations:

a $y = x^2 + 1$ **b** $y = x^2 + 3$ **c** $y = x^2 - 1$

d $y = 2x^2$ **e** $y = -3x^2$ **f** $y = \frac{1}{2}x^2$

g $y = -x^2 + 2$ **h** $y = -x^2 - 1$ **i** $y = -4x^2$

Check your answers using the **graphing package** or a **graphics calculator**.

GRAPHING
PACKAGE

2 On separate axes, sketch the graphs and find the vertices of:

a $y = (x - 1)^2 + 2$ **b** $y = (x + 1)^2 + 3$ **c** $y = (x - 2)^2 + 1$

d $y = (x + 2)^2 - 1$ **e** $y = (x + 3)^2$ **f** $y = (x - 4)^2$

g $y = -(x - 1)^2 + 3$ **h** $y = -(x + 1)^2 - 1$ **i** $y = -(x - 2)^2 + 1$

j $y = -(x + 4)^2 + 2$ **k** $y = -(x - 3)^2 - 2$ **l** $y = -(x + 4)^2 + 10$

Check your answers using the **graphing package** or a **graphics calculator**.

D	# GRAPHING BY COMPLETING THE SQUARE

In **Chapter 16** we used the process of completing the square to assist us in solving quadratic equations which did not factorise.

This same process can be used here to convert quadratic functions into the form $y = (x - h)^2 + k$.

Consider $y = x^2 - 4x + 1$

$\therefore \quad y = x^2 - 4x + 2^2 + 1 - 2^2 \qquad$ {keeping the equation balanced}

$\therefore \quad y = x^2 - 4x + 2^2 - 3$

$\therefore \quad y = (x - 2)^2 - 3$

So, $y = x^2 - 4x + 1$ is really $y = (x - 2)^2 - 3$.

The graph of $y = x^2 - 4x + 1$ is the image of the graph of $y = x^2$ after it has been translated 2 units to the right and 3 units down,

i.e., through $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
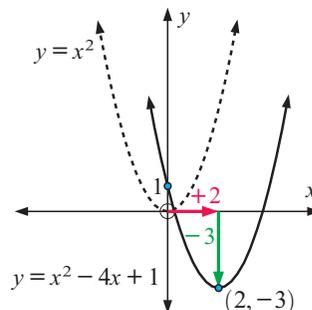
Example 8 ◀) **Self Tutor**

Write $y = x^2 + 4x + 1$ in the form $y = (x - h)^2 + k$ by completing the square. Hence, state the coordinates of its vertex and sketch the graph of the function.

$y = x^2 + 4x + 1$

$\therefore \quad y = x^2 + 4x \qquad + 1$

$\therefore \quad y = x^2 + 4x + 2^2 + 1 - 2^2$

$\therefore \quad y = (x + 2)^2 + 1 - 4$

$\therefore \quad y = (x + 2)^2 - 3$

The graph is a translation of $y = x^2$ through $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\therefore$ we shift it 2 units to the left and 3 units down.

The vertex is $(-2, -3)$.
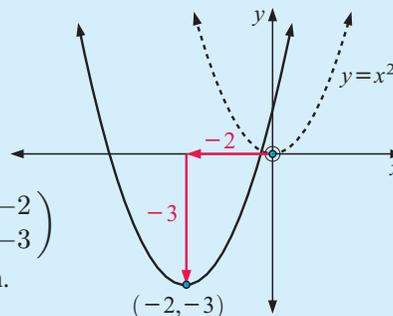
EXERCISE 19D

1 Write the following quadratics in the form $y = (x - h)^2 + k$ by completing the square. Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 + 2x + 2$ **b** $y = x^2 - 4x + 2$ **c** $y = x^2 + 2x - 3$

d $y = x^2 - 6x + 1$ **e** $y = x^2 + 2x$ **f** $y = x^2 - 3x$

g $y = x^2 + x - 1$ **h** $y = x^2 - 3x + 2$ **i** $y = x^2 + 5x - 5$

GRAPHING PACKAGE

Check your answers using a **graphing package** or **graphics calculator**.

Example 9   ◀)) **Self Tutor**

Write $y = -x^2 - 6x - 5$ in the form $y = -(x-h)^2 + k$ by completing the square. Hence:

a state the coordinates of its vertex   **b** sketch the graph of the function.

a $y = -x^2 - 6x - 5$
$\therefore \quad -y = x^2 + 6x + 5$
$\therefore \quad -y = x^2 + 6x + 3^2 + 5 - 3^2$
$\therefore \quad -y = (x+3)^2 - 4$
$\therefore \quad \ y = -(x+3)^2 + 4$

The graph is a translation of $y = -x^2$

through $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

$\therefore$ we shift it 3 units to the left and
4 units up.
The vertex is $(-3,\, 4)$.

b
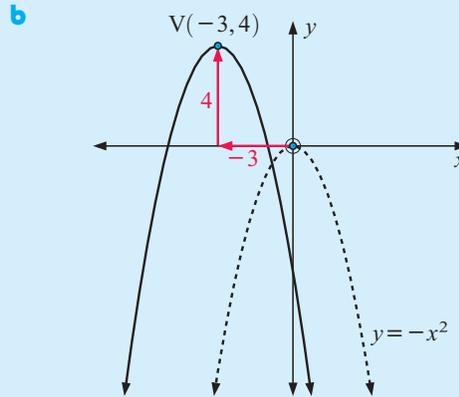
2 For each of the following functions:

  i   write the function in the form $y = -(x-h)^2 + k$ by completing the square
  ii state the coordinates of its vertex
  iii sketch the graph of the function.

  a $y = -x^2 + 2x + 2$ **b** $y = -x^2 - 4x - 2$ **c** $y = -x^2 + 6x - 10$

  d $y = -x^2 - 8x - 12$ **e** $y = -x^2 + 10x - 28$ **f** $y = -x^2 + 2$

  g $y = -x^2 + 3x - 2$ **h** $y = -x^2 - x - 1$ **i** $y = -x^2 - 3x - 2$

  j $y = -x^2 + x$ **k** $y = -x^2 - 5x - 7$ **l** $y = -x^2 + 7x - 13$

E   AXES INTERCEPTS

Given the equation of any curve:

An **x-intercept** is a value of x where the graph meets the x-axis,

A **y-intercept** is a value of y where the graph meets the y-axis.

x-intercepts are found by letting y be 0 in the equation of the curve.

y-intercepts are found by letting x be 0 in the equation of the curve.
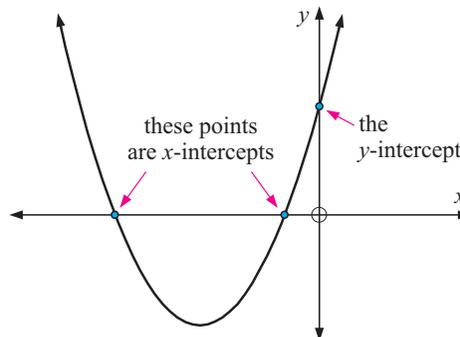
INVESTIGATION 2   AXES INTERCEPTS

What to do:

1 For the following quadratic functions, use a graphing package or graphics calculator to:

GRAPHING PACKAGE

       i draw the graph   **ii** find the y-intercept

      iii find any x-intercepts

  a $y = x^2 - 3x + 2$ **b** $y = -x^2 + 4x - 3$ **c** $y = x^2 - 3x$

  d $y = -x^2 + 4x$ **e** $y = (x-2)(x-4)$ **f** $y = -(x+3)(x-1)$

  g $y = (x+1)(x+4)$ **h** $y = (x-2)^2$ **i** $y = -(x+1)^2$

2 From your observations in question **1**:

  a State the y-intercept of a quadratic function in the form $y = x^2 + bx + c$ or $y = -x^2 + bx + c$.

  b State the x-intercepts of quadratic function in the form $y = (x - \alpha)(x - \beta)$ or $y = -(x - \alpha)(x - \beta)$.

  c What do you notice about the x-intercepts of quadratic functions in the form $y = (x - \alpha)^2$ or $y = -(x - \alpha)^2$?

THE y-INTERCEPT

You will have noticed that for a quadratic function of the form $y = ax^2 + bx + c$, the y-intercept is the constant term c. This is because any curve cuts the y-axis when $x = 0$.

For example,   if $y = x^2 + 3x - 4$ and we let $x = 0$

                then $y = 0^2 + 3(0) - 4$

              $\therefore$ $y = -4$, which is the constant term.

THE x-INTERCEPTS

You should have noticed that for a quadratic function of the form $y = (x - \alpha)(x - \beta)$ or $y = -(x - \alpha)(x - \beta)$, the x-intercepts are α and β.

This is in fact true for all quadratic functions of the form $y = a(x - \alpha)(x - \beta)$, since any curve cuts the x-axis when $y = 0$.

If we substitute $y = 0$ into the function we get $a(x - \alpha)(x - \beta) = 0$

            $\therefore$ $x = \alpha$ or β {by the Null Factor law}

This suggests that x-intercepts are easy to find when the quadratic is in **factorised** form.

Example 10   ◀) Self Tutor

Find the x-intercepts of:

a $y = (x+1)(x-2)$ **b** $y = -(x-2)^2$

> If a quadratic function has only one x-intercept then its graph must **touch** the x-axis.

a When $y = 0$,
$$(x+1)(x-2) = 0$$
$\therefore$ $x = -1$ or $x = 2$
$\therefore$ the x-intercepts are -1 and 2.

b When $y = 0$,
$$-(x-2)^2 = 0$$
$\therefore$ $x = 2$
$\therefore$ the x-intercept is 2.

FACTORISING TO FIND x-INTERCEPTS

> For any quadratic function of the form $y = ax^2 + bx + c$,   the x-intercepts can be found by solving the equation $ax^2 + bx + c = 0$.

You will recall from **Chapter 16** that quadratic equations may have *two solutions*, *one solution* or *no solutions*.

These solutions correspond to the *two x-intercepts*, *one x-intercept*, or *no x-intercepts* found when the graphs of the quadratic functions are drawn.

Example 11   ◀) Self Tutor

Find the x-intercept(s) of the quadratic functions:

a $y = x^2 + 4x + 4$ **b** $y = -x^2 + 2x + 3$

a When $y = 0$,
$$x^2 + 4x + 4 = 0$$
$\therefore$ $(x+2)^2 = 0$
$\therefore$ $x = -2$
$\therefore$ x-intercept is -2.

b When $y = 0$,
$$-x^2 + 2x + 3 = 0$$
$\therefore$ $x^2 - 2x - 3 = 0$
$\therefore$ $(x-3)(x+1) = 0$
$\therefore$ $x = 3$ or -1
$\therefore$ x-intercepts are -1 and 3.

EXERCISE 19E

1 For the following functions, state the y-intercept:

a $y = x^2 + x + 1$ **b** $y = x^2 - 2x + 3$ **c** $y = x^2 + 4x - 2$

d $y = -x^2 - x + 4$ **e** $y = -x^2 + x + 6$ **f** $y = -x^2 - 2x + 1$

g $y = 4 + x^2$ **h** $y = 3 + x - x^2$ **i** $y = 2x - x^2 + 7$

2 For the following functions, find the y-intercept:

a $y = (x+1)(x-2)$ **b** $y = (x-1)(x+3)$ **c** $y = -(x+2)(x-1)$

d $y = x(x-3)$ **e** $y = -(x-4)(x+1)$ **f** $y = -(x-1)(x-2)$

3 For the following functions, find the x-intercepts:

    a $y = (x - 1)(x + 3)$ **b** $y = -(x + 2)(x - 3)$ **c** $y = (x - 3)(x + 7)$

    d $y = -(x - 2)(x - 5)$ **e** $y = (x + 5)^2$ **f** $y = -(x - 7)^2$

4 For the following functions, find the x-intercepts:

    a $y = x^2 - 4$ **b** $y = x^2 - 9$ **c** $y = x^2 - 7x + 12$

    d $y = x^2 - x - 6$ **e** $y = 3x - x^2$ **f** $y = -x^2 + 7x - 10$

    g $y = -x^2 - 2x - 1$ **h** $y = x^2 - 6x + 9$ **i** $y = -x^2 - 5x + 6$

    j $y = x^2 + 4x - 5$ **k** $y = -x^2 + 6x - 16$ **l** $y = -x^2 - 4x + 21$

5 Check your answers to **4** using technology.

F   QUADRATIC GRAPHS

Consider $y = (x - 2)(x - 4)$.

The graph of this quadratic has x-intercepts   2 when the factor $x - 2 = 0$,

                             and   4 when the factor $x - 4 = 0$.

All quadratic graphs are symmetric about a vertical line drawn through the vertex. The axis of symmetry must be midway between the x-intercepts. The average of 2 and 4 is $\frac{2+4}{2} = 3$ so the axis of symmetry for $y = (x - 2)(x - 4)$ has equation $x = 3$.

When $x = 3$, $y = (3 - 2)(3 - 4)$
$$= (1)(-1)$$
$$= -1$$

So, the vertex is at $(3, -1)$.

Also, when $x = 0$, $y = (-2)(-4) = 8$
So, the y-intercept is 8.

Notice that from the factored form we are able to find:

- the x-intercepts
- the equation of the axis of symmetry
- the coordinates of the vertex
- the y-intercept.
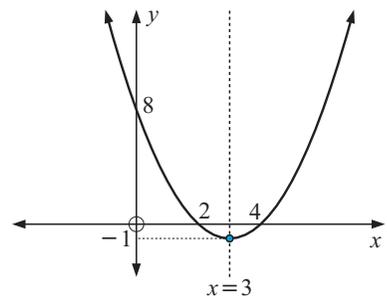
We can use these features to sketch a graph of the function.

Example 12   ◀⬩ **Self Tutor**

For $y = -(x + 1)(x - 3)$ find:

    a the x-intercepts   **b** the equation of the axis of symmetry

    c the coordinates of the vertex   **d** the y-intercept.

Hence, sketch the function.

$y = -(x+1)(x-3)$

a The x-intercepts are -1 and 3.

b The axis of symmetry is $x = \dfrac{-1+3}{2}$

i.e., $x = 1$

c When $x = 1$, $y = -(2)(-2) = 4$

$\therefore$ the vertex is at $(1, 4)$.

d When $x = 0$, $y = -(1)(-3) = 3$

$\therefore$ the y-intercept is 3.
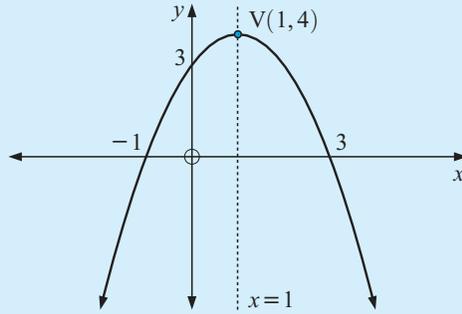
Example 13

◀》 **Self Tutor**

For $y = (x-2)^2$ find:

a the x-intercepts

b the equation of the axis of symmetry

c the coordinates of the vertex

d the y-intercept.

Hence, sketch the function.

$y = (x-2)^2$

a The x-intercept is 2.

b The axis of symmetry is $x = 2$.

c When $x = 2$, $y = 0^2 = 0$

$\therefore$ the vertex is at $(2, 0)$.

d When $x = 0$, $y = (-2)^2 = 4$

$\therefore$ the y-intercept is 4.
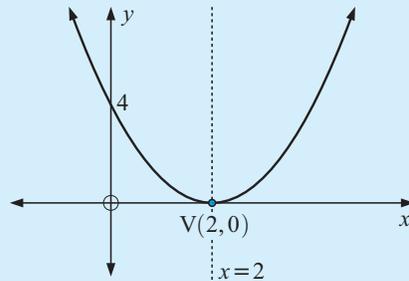
EXERCISE 19F

1 Find the equation of the axis of symmetry of:
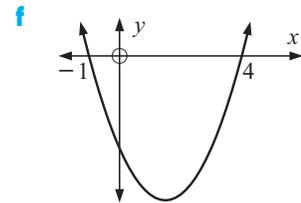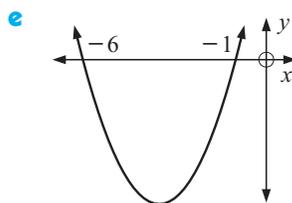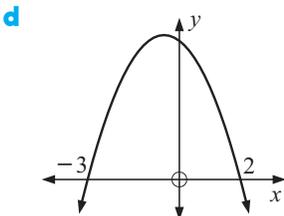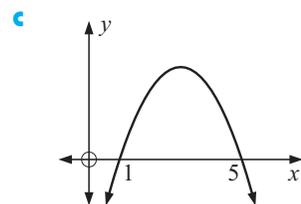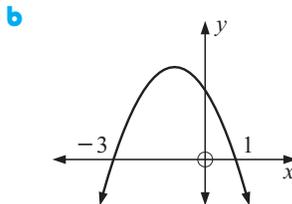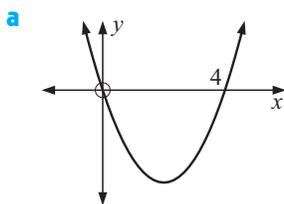
2 From the following quadratics, find:

  i the x-intercepts   **ii** the equation of the axis of symmetry
  iii the coordinates of the vertex   **iv** the y-intercept.

Hence sketch graphs of the functions.

  a $y = (x-1)(x-3)$ **b** $y = -(x-1)(x-5)$ **c** $y = x(x-4)$
  d $y = -x(x+2)$ **e** $y = (x+2)(x-4)$ **f** $y = -(x-1)^2$
  g $y = (x+4)(x-2)$ **h** $y = -(x+4)(x-2)$ **i** $y = (x+3)^2$
  j $y = -(x+1)(x-2)$ **k** $y = x^2 + 2x$ **l** $y = -x^2 - 4x - 4$
  m $y = x^2 + 3x$ **n** $y = x^2 - 3x - 10$ **o** $y = -x^2 - 7x - 6$

Hint: In **k** to **o** you should start by factorising the quadratic.

3 What are the limitations of the method used in **Examples 12** and **13** for graphing the general functions $y = x^2 + bx + c$ or $y = -x^2 + bc + c$?

4 Find all x-intercepts of the graph of a quadratic function which:

  a cuts the x-axis at 2 and has axis of symmetry $x = 4$
  b cuts the x-axis at -1 and has axis of symmetry $x = -3$
  c touches the x-axis at 3.

G   MAXIMUM AND MINIMUM VALUES OF QUADRATICS

We have already seen that the graphs of quadratic functions may open upwards or downwards.

For example:

The graph of $y = (x-2)^2 + 3$ opens upwards. Its vertex is $(2, 3)$ and at this point the function is a **minimum**.
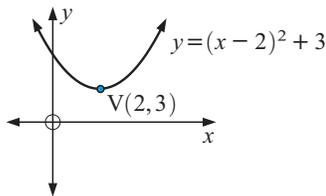
The graph of $y = -(x-3)^2 + 5$ opens downwards. Its vertex is $(3, 5)$ and at this point the function is a **maximum**.
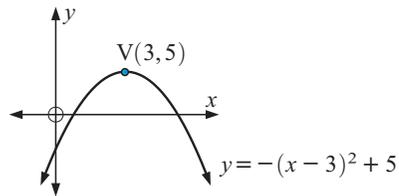
In general,

- for $y = (x-h)^2 + k$, the **minimum value** of y is k when $x = h$
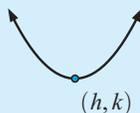
- for $y = -(x-h)^2 + k$, the **maximum value** of y is k when $x = h$.
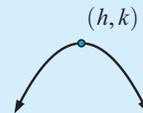
Example 14

◀) **Self Tutor**

Find the maximum or minimum value of the following and the value of x at which they occur:

a $y = x^2 - 3x + 1$

b $y = -x^2 - 4x + 1$

a $y = x^2 - 3x + 1$
$= x^2 - 3x + \left(\frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2$
$= \left(x - \frac{3}{2}\right)^2 + 1 - \frac{9}{4}$
$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

$\left(\frac{3}{2}, -\frac{5}{4}\right)$
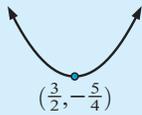
∴   the minimum value of y is $-\frac{5}{4}$
when $x = \frac{3}{2}$.

b $y = -x^2 - 4x + 1$
∴ $-y = x^2 + 4x - 1$
∴ $-y = x^2 + 4x + 2^2 - 1 - 2^2$
∴ $-y = (x + 2)^2 - 5$
∴ $y = -(x + 2)^2 + 5$

$(-2, 5)$
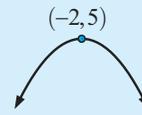
∴   the maximum value of y is 5
when $x = -2$.

Example 15

◀) **Self Tutor**

The square of a number is added to 8 times the number.
Find the minimum possible value of this sum.

If the number is x, then its square is x^2 and 8 times the number is $8x$.

So, we consider $y = x^2 + 8x$
$= x^2 + 8x + 4^2 - 4^2$
$= (x + 4)^2 - 16$

Thus, the minimum value is -16
and this occurs when the number is -4.

$(-4, -16)$
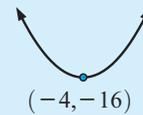
EXERCISE 19G

1 Find the maximum or minimum value of each of the following and state the value of x at which it occurs:

a $y = x^2 - 2x + 7$ **b** $y = -x^2 - 2x + 7$ **c** $y = x^2 + 2x + 3$

d $y = x^2 - 2x - 5$ **e** $y = -x^2 + 2x + 1$ **f** $y = -x^2 - 2x - 4$

g $y = x^2 + 4x + 1$ **h** $y = -x^2 + 4x + 1$ **i** $y = -x^2 - 4x + 1$

j $y = x^2 + 3x + 6$ **k** $y = -x^2 - 3x - 2$ **l** $y = x^2 + 8x$

m $y = -x^2 + 4x$ **n** $y = x^2 + 5x$ **o** $y = -x^2 - 7x$

2   **a**   The square of a number is added to 6 times the number.
Find the minimum possible value of this sum.

b   The square of a number is subtracted from 3 times the number.
Find the maximum possible value of this difference.

3 A billiard table manufacturer finds that the cost per table of making x tables per month is given by $C = x^2 - 16x + 67$ hundred dollars.

    a How many tables should be made per month to minimise the production cost of each?

    b What is the minimum cost per table?

    c What is the cost per table if 5 tables are made in a month?

4 A car driving along a city street has speed given by the function
$s = -t^2 + 6t + 40 \text{ km h}^{-1}$ where $0 \leqslant t \leqslant 10$ seconds.

    a How fast was the car travelling at time $t = 0$ seconds?

    b After how many seconds did the speed of the car first reach 45 km h^{-1}?

    c At what time did the speed reach 45 km h^{-1} again?

    d What was the maximum speed of the car and at what time did it occur?

5 A company supplies stalls which sell drinks at a festival. The profit obtained from operating n stalls is given by $P = 160n - 1520 - n^2$ dollars.

    a What number of stalls gives the maximum profit?

    b What is the maximum profit?

    c How much money is lost if there is a thunderstorm and the festival is cancelled at the last minute?

MAXIMISING AREAS OF ENCLOSURES

LINKS
click here

Areas of interaction:
Human ingenuity/The environment

REVIEW SET 19A

1 Find a if the point $(a, 3)$ lies on the graph of the function $y = x^2 + 3x + 5$.

2 Using a table of values from $x = -3$ to $x = 3$, draw the graph of
$y = 2 - x - x^2$.

3 Using a translation, sketch the graph of $y = (x - 1)^2 + 3$.
State the coordinates of the vertex.

4   **a** Write $y = x^2 - 4x + 3$ in the form $y = (x - h)^2 + k$ by completing the square.

    b Hence, state the coordinates of the vertex of $y = x^2 - 4x + 3$.

    c Sketch the graphs of $y = x^2$ and $y = x^2 - 4x + 3$ on the same set of axes.

5   **a** Write $y = -x^2 - 2x + 1$ in the form $y = -(x - h)^2 + k$ by completing the square.

    b Hence, state the coordinates of the vertex of $y = -x^2 - 2x + 1$.

    c Sketch the graphs of $y = -x^2$ and $y = -x^2 - 2x + 1$ on the same set of axes.

6 Find the x-intercepts of:

    a $y = (x+2)(x-3)$ **b** $y = -x^2 + x + 12$

7 For the graph of the quadratic function $y = (x+3)(x-2)$, find:

    a the x-intercepts   **b** the equation of the axis of symmetry

    c the coordinates of the vertex   **d** the y-intercept.

    e Use **a** to **d** to sketch the graph of the function.

8 Find the maximum or minimum value of y for the function:

    a $y = -x^2 + 2x - 3$ **b** $y = x^2 - 5x + 2$

9 4 times a number is subtracted from the square of the number.
What is the minimum value that could result?

REVIEW SET 19B

1 If $(t, -2)$ lies on graph of the function $y = x^2 - 3x - 2$, find t.

2 Using a table of values from $x = -3$ to $x = 3$, draw the graph of $y = x^2 - 2x + 3$.

3 Using a translation, sketch the graph of $y = -(x+1)^2 + 3$.
State the coordinates of the vertex.

4   **a** Write $y = x^2 + 2x + 1$ in the form $y = (x-h)^2 + k$ by completing the square.

    b Hence, state the coordinates of the vertex of $y = x^2 + 2x + 1$.

    c Sketch the graphs of $y = x^2$ and $y = x^2 + 2x + 1$ on the same set of axes.

5   **a** Write $y = -x^2 + 4x$ in the form $y = -(x-h)^2 + k$ by completing the square.

    b Hence, state the coordinates of the vertex of $y = -x^2 + 4x$.

    c Sketch the graphs of $y = -x^2$ and $y = -x^2 + 4x$ on the same set of axes.

6 Find the x-intercepts of: **a** $y = -(x+2)(x-5)$ **b** $y = (x+2)^2$

7 For the graph of the quadratic function $y = -(x-2)(x+4)$, find:

    a the x-intercepts   **b** the equation of the axis of symmetry

    c the coordinates of the vertex   **d** the y-intercept.

    e Use **a** to **d** to sketch the graph of the function.

8 Find the maximum or minimum value of y for the function:

    a $y = x^2 - 6x + 9$ **b** $y = -x^2 - 3x + 1$

9 A rectangle has one side x m long. Its perimeter is 40 m.

    a Find the length of the other sides of the rectangle in terms of x.

    b Write the area of the rectangle A in terms of x.

    c Find the maximum possible area of the rectangle. Comment on the shape of the rectangle in this case.
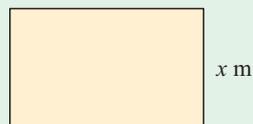
Chapter 20

Tree diagrams and binomial probabilities

Contents:

A   SAMPLE SPACES USING TREE DIAGRAMS

In **Chapter 10** we displayed sample spaces by:

- **listing** sets of single outcomes
- using **2-dimensional grids**.

Another way to represent a sample space is to use a **tree diagram**.

Suppose a three sided spinner has outcomes 1, 2 and 3.

If this spinner is spun twice, the sample space can be listed as shown opposite:

First spin	Second spin
1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

However, instead of repeating the numbers in the first column, we could list the outcomes using a diagram.

The lines indicate how the second outcome is related to the first.

The outcomes of the first spin are connected by a line coming from a single point.

We call this a **tree diagram**.
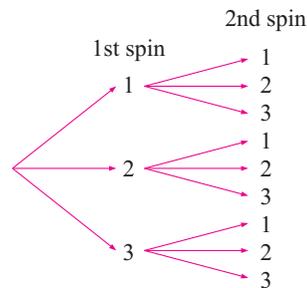
Example 1   🔊 Self Tutor

Use a tree diagram to illustrate the sample space when tossing a coin twice.
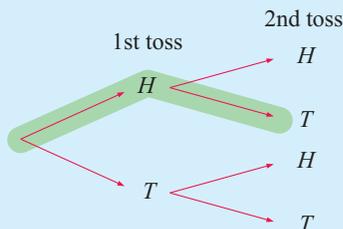
Each of the four branches represents a different possible outcome in the sample space.

The branch highlighted represents getting a head with the first toss and a tail with the second toss.

EXERCISE 20A

1 Use a tree diagram to illustrate the sample space for:

a tossing a 10-cent coin and a 20-cent coin simultaneously

b tossing a 6 sided die and a coin simultaneously

c the genders of a 3-child family

d the orderings in which 3 girls Sarah, Tracy and Beth, may sit in a row of 3 chairs.

Example 2

John plays Peter at tennis. The first to win two sets wins the match.
Illustrate the sample space using a tree diagram.

If J means "John wins the set"
and P means "Peter wins the set"
then the tree diagram is:

We could write the sample space
in set notation as
$S = \{JJ, JPJ, JPP, PJJ, PJP, PP\}$.

1st set   2nd set   3rd set
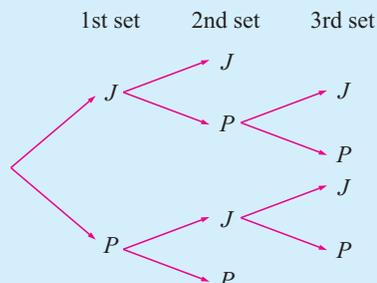
2   Use a tree diagram to illustrate the sample space for the following:

   a   The genders of a 4-child family.

   b   Bag A contains red and white marbles and bag B contains blue and yellow marbles.
A bag is selected and one marble is taken from it.

   c   Hats A, B and C each contain pink and purple tickets. A hat is selected and then
two tickets are taken from it.

   d   Two teams, X and Y, play football. The first team to kick 3 goals wins the match.

   e   Jody and Petria play tennis. The first to win two sets in a row or a total of 3 sets,
wins the match.

3   A bag contains five marbles. One is blue, one is red, and three are green. Two marbles
are selected from the bag without replacement. Use a tree diagram to show all possible
outcomes.

B   PROBABILITIES FROM TREE DIAGRAMS

Tree diagrams can be used to illustrate sample spaces providing there are not too many
different alternatives. Once the sample space has been illustrated, the tree diagram can be
used to determine probabilities.

INDEPENDENT EVENTS

Suppose Charles has probability $\frac{4}{5}$ of passing Physics and $\frac{7}{10}$ of passing Chemistry.

We assume that these events are
independent, which means the
probability of Charles passing
Physics is not related in any way to
him passing Chemistry.

A tree diagram showing all possible
outcomes is shown alongside with P
being a pass and F being a fail.

Physics   Chemistry   outcomes   probabilities

$$PP \qquad \tfrac{4}{5} \times \tfrac{7}{10} = \tfrac{28}{50}$$
$$PF \qquad \tfrac{4}{5} \times \tfrac{3}{10} = \tfrac{12}{50}$$
$$FP \qquad \tfrac{1}{5} \times \tfrac{7}{10} = \tfrac{7}{50}$$
$$FF \qquad \tfrac{1}{5} \times \tfrac{3}{10} = \tfrac{3}{50}$$
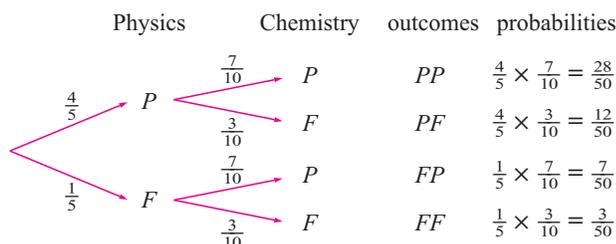
The probabilities for passing and failing each subject are marked along the branches.

When using tree diagrams to solve probability questions:

- the probability for each branch is calculated by **multiplying** the probabilities along that branch
- if two or more branches meet the description of the compound event, the probability of each branch is found and then they are **added**.

Example 3   ◄》 Self Tutor

Jessica has probability $\frac{4}{5}$ of getting an A in Mathematics and probability $\frac{2}{5}$ of getting an A for English.

a Display this information on a tree diagram.

b What is the probability Jessica gets one A in the two subjects?

a A is the event of getting an A. N is the event of not getting an A.

| Mathematics | English | outcomes | probabilities |

$$\frac{4}{5} \times \frac{3}{5} = \frac{12}{25}$$

$$\frac{1}{5} \times \frac{2}{5} = \frac{2}{25}$$
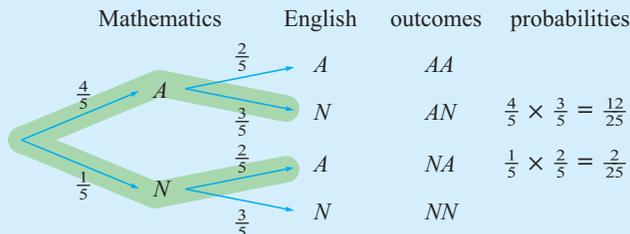
b Jessica can get one A either by getting an A in Mathematics but not in English, or by getting an A in English but not in Mathematics.

So, the total probability is $\frac{4}{5} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5} = \frac{14}{25}$

These events are highlighted on the tree diagram.

EXERCISE 20B.1

1 A spinner has probability $\frac{1}{3}$ of finishing on blue and probability $\frac{2}{3}$ of finishing on green. Xin spins the spinner two times.

  a Display this information on a tree diagram.

  b Find the probability that the spinner finishes on blue once and on green once.

2 Andrew has probability $\frac{1}{5}$ of winning the 100 metre sprint and probability $\frac{1}{3}$ of winning the 200 metre race.

  a Draw a tree diagram showing all of Andrew's chances.

  b What is the probability Andrew wins both races?

  c What is the probability that Andrew wins exactly one of the two races?

  d What is the probability Andrew loses both races?

  e Find the sum of the probabilities in **b**, **c** and **d**. Explain why this answer is to be expected.

DEPENDENT EVENTS

Suppose a hat contains 5 red and 3 blue tickets. One ticket is randomly chosen, its colour is noted, and it is then thrown away. A second ticket is randomly selected. What is the chance that it is red?
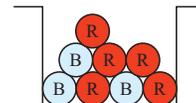
If the first ticket was red, $P(\text{second is red}) = \frac{4}{7}$ ← 4 reds remaining ← 7 to choose from

If the first ticket was blue, $P(\text{second is red}) = \frac{5}{7}$ ← 5 reds remaining ← 7 to choose from

So, the event of the second ticket being red **depends** on what colour the first ticket was. In such a case we say we have **dependent events**.

> Two or more events are **dependent** if they are **not independent**.
>
> **Dependent** events are events where the occurrence of one of the events *does affect* the occurrence of the other event.
>
> If A and B are dependent events then
>
> $$\mathbf{P(A \text{ then } B) = P(A) \times P(B \text{ given that } A \text{ has occurred}).}$$

A typical example of dependent events is when we sample two objects *without replacement*. This means that the first object is not replaced before the second is selected. It therefore cannot be selected twice.

Example 4   ◀) Self Tutor

A box contains 4 red and 2 yellow tickets. Two tickets are randomly selected one after the other from the box, *without replacement*.

a Display this information on a tree diagram.

b What is the probability that both are red?

c What is the probability that one is red and the other is yellow?

a Let R be the event that a red ticket is drawn and Y be the event that a yellow ticket is drawn.

Note that the outcome of the second event depends on the first.

1st selection   2nd selection

R — $\frac{3}{5}$ → R $\frac{4}{6} \times \frac{3}{5}$
     — $\frac{2}{5}$ → Y $\frac{4}{6} \times \frac{2}{5}$

$\frac{4}{6}$ to R, $\frac{2}{6}$ to Y

Y — $\frac{4}{5}$ → R $\frac{2}{6} \times \frac{4}{5}$
     — $\frac{1}{5}$ → Y $\frac{2}{6} \times \frac{1}{5}$
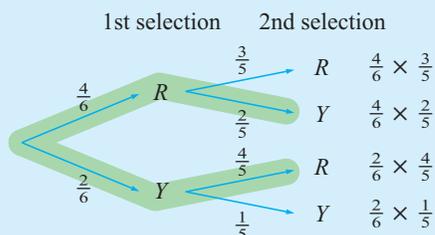
b $P(R \text{ and then } R) = P(R) \times P(R \text{ given that } R \text{ has occurred})$

$$= \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$$

c $P(\text{one is } R \text{ and the other is } Y)$

$= P(R \text{ and then } Y \text{ or } Y \text{ and then } R)$ {highlighted branches}

$= \frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{4}{5}$

$= \frac{16}{30}$ or $\frac{8}{15}$

EXERCISE 20B.2

1 A box contains 7 red and 3 green balls. Two balls are randomly selected from the box one after the other. The first is *not* replaced in the box before the second is selected. Determine the probability that:

    a both are red

    b the first is green and the second is red

    c a green and a red are obtained.

2 A hat contains the names of the 7 players in a tennis squad including the captain and the vice captain. A team of 3 players is chosen at random by drawing the names from the hat.

> Selecting items simultaneously means there is no replacement.

    a Display this information on a tree diagram. Distinguish between the captain, the vice captain, and the other players.

    b Find the probability that the team:

        i does not contain the captain

        ii contains neither the captain nor the vice captain

        iii contains either the captain or the vice captain, but not both.

3 Amelie has a bag containing two different varieties of apples. They are approximately the same size and shape, but one variety is red and the other is green. There are 4 red apples and 6 green ones. She selects one apple at random, eats it, and then takes another, also at random. Determine the probability that:

    a both apples were red

    b both apples were green

    c the first was red and the second was green

    d the first was green and the second was red

    e she ate one red and one green apple.

4 Marjut has a carton containing 10 cans of soup. 4 cans are tomato and the rest are pumpkin. She selects 2 cans at random without looking at the labels.

    a Let T represent tomato and P represent pumpkin. Draw a tree diagram to illustrate this sampling process.

    b What is the probability that both cans were tomato soup?

    c What is the probability that one can was tomato and the other can was pumpkin soup?

LARGE SAMPLE SPACES

Sometimes the number of possible outcomes is sufficiently large that it is a waste of time to draw a tree diagram. We can still use the same principles we used with the tree diagrams to calculate the probabilities.

Example 5   ◀୬) **Self Tutor**

A hat contains 20 tickets with the numbers 1, 2, 3, ..., 19 and 20 printed on them.
If 3 tickets are drawn from the hat without replacement, determine the probability that
all are prime numbers.

There are 20 numbers of which 8 are primes: $\{2, 3, 5, 7, 11, 13, 17, 19\}$

$\therefore$ P(3 primes)

$= $ P(1st drawn is prime *and* 2nd is prime *and* 3rd is prime)

$= \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18}$
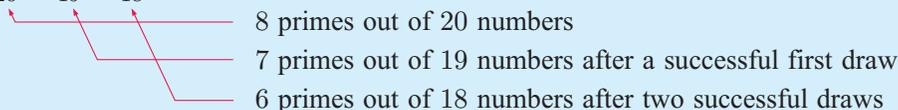
8 primes out of 20 numbers

7 primes out of 19 numbers after a successful first draw

6 primes out of 18 numbers after two successful draws

≈ 0.0491

EXERCISE 20B.3

1 A lottery has 100 tickets which are placed in a barrel. Two tickets are drawn at random from the barrel to decide 2 prizes. If John has 2 tickets in the lottery, determine his probability of winning:

    a first prize   **b** first and second prize

    c second prize but not first prize.   **d** none of the prizes.

2 A bin contains 12 identically shaped chocolates of which 8 are strawberry creams. If 3 chocolates are selected at random from the bin, determine the probability that:

    a they are all strawberry creams   **b** none of them are strawberry creams.

3 A bag contains two white and five red marbles. Three marbles are selected simultaneously. Determine the probability that:

    a all are red   **b** only two are red   **c** at least two are red.

SELECTION WITH AND WITHOUT REPLACEMENT

You may have noticed that when we work with tree diagrams, the probabilities of independent and dependent events are calculated using the same method. The following example compares two such events.

Consider a box containing 3 red, 2 blue and 1 green marble. Suppose we wish to sample two marbles:

- **with replacement** of the first before the second is drawn
- **without replacement** of the first before the second is drawn.

Examine how the tree diagrams differ:

With replacement   **Without replacement**
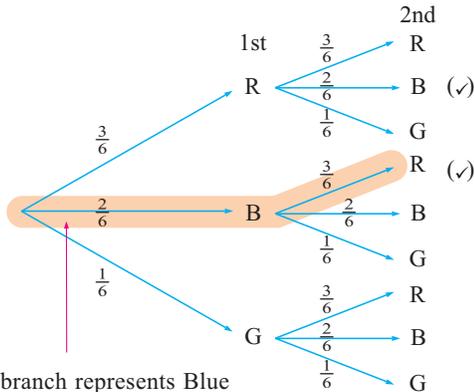
This branch represents Blue with the first draw and Red with the second draw. This is written as BR.
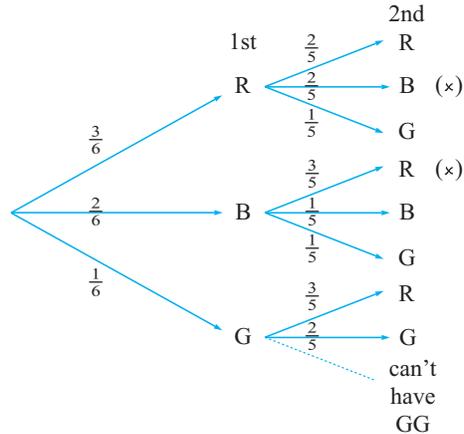
Notice that:
- with replacement $P(\text{two reds}) = \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$
- without replacement $P(\text{two reds}) = \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$

We can thus see why replacement is important.

Example 6   ◀)) **Self Tutor**

For the example above of the box containing 3 red, 2 blue and 1 green marble, find the probability of getting a red and a blue:

a with replacement   **b** without replacement.

a   P(a red and a blue)

= P(RB or BR)   {ticked ones}

$= \frac{3}{6} \times \frac{2}{6} + \frac{2}{6} \times \frac{3}{6}$

$= \frac{12}{36}$

$= \frac{1}{3}$

b   P(a red and a blue)

= P(RB or BR)   {crossed ones}

$= \frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{3}{5}$

$= \frac{12}{30}$

$= \frac{2}{5}$

EXERCISE 20B.4

Use tree diagrams to help answer the following questions:

1   Jar A contains 4 blue and 2 red marbles. Jar B contains 1 blue and 5 red marbles. A jar is randomly selected and one marble is taken from it. Determine the probability that the marble is blue.

2   Two marbles are drawn in succession from a box containing 2 purple and 5 green marbles. Determine the probability that the two marbles are different colours if:

a   the first is replaced   **b**   the first is *not* replaced.

3   5 tickets numbered 1, 2, 3, 4 and 5 are placed in a bag. Two are taken from the bag without replacement. Determine the probability that:

a   both are odd   **b**   both are even   **c**   one is odd and the other is even.

4   Jar A contains 3 red and 2 green tickets. Jar B contains 3 red and 7 green tickets. A die has 4 faces marked A and 2 faces marked B. The die is rolled and the result is used to select either jar A or jar B. When a jar has been selected, two tickets are randomly selected from it without replacement. Determine the probability that:

    a   both are green   **b**   they are different in colour.
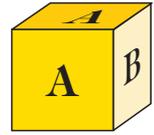
Example 7   ◀)) **Self Tutor**

A bag contains 5 red and 3 blue marbles. Two marbles are drawn simultaneously from the bag. Determine the probability that at least one is red.
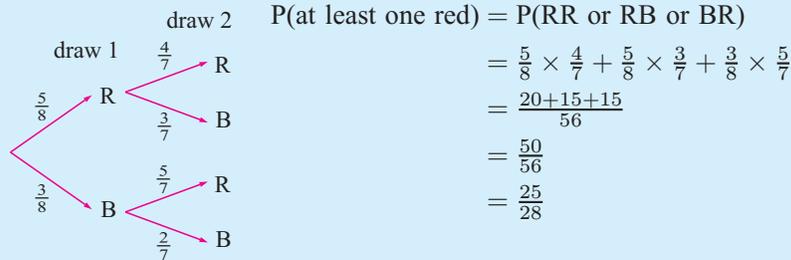
$$P(\text{at least one red}) = P(RR \text{ or } RB \text{ or } BR)$$
$$= \tfrac{5}{8} \times \tfrac{4}{7} + \tfrac{5}{8} \times \tfrac{3}{7} + \tfrac{3}{8} \times \tfrac{5}{7}$$
$$= \tfrac{20+15+15}{56}$$
$$= \tfrac{50}{56}$$
$$= \tfrac{25}{28}$$

Alternatively, $P(\text{at least one red}) = 1 - P(\text{no reds})$ {complementary events}
$$= 1 - P(BB) \quad \text{and so on.}$$

---

5   A bag contains four red and two blue marbles. Three marbles are selected simultaneously. Determine the probability that:

    a   all are red   **b**   only two are red   **c**   at least two are red.

6   Box A contains 3 red and 4 green marbles. Box B contains 5 red and 2 green marbles. One marble is randomly selected from box A and its colour noted. If it is red, 2 reds are added to box B. If it is green, 2 greens are added to box B.

A marble is then selected from box B.

Find the probability that the marble selected from box B is green.
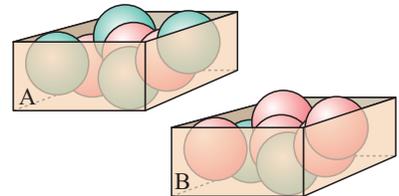
C   BINOMIAL PROBABILITIES

In many situations there are only two possible outcomes. For example:

- a tennis player either wins or loses a game
- you either catch a bus or you don't
- you either make a free throw or you miss.

> **Binomial experiments** are concerned with the repetition of several *independent trials* where there are only two possible outcomes, *success* and *failure*. Each trial has the *same* probability of success.

Since a coin has two outcomes when tossed, either heads or tails, we often use the tossing of a coin to model binomial experiments. In this case we regard a head as success and a tail as failure.

EXERCISE 20C.1

1 Which of the following is a binomial experiment?

a A coin is tossed 100 times. The outcome is the number of heads.

b A box contains 12 hard centred and 18 soft centred chocolates. I take out three chocolates and eat them. The outcome is the number of hard centred chocolates I eat.

c Ten winning tickets are drawn from a hat containing 100 tickets. Each ticket is replaced after it is drawn, so a single ticket may win more than one prize.

d Ten winning tickets are drawn from a hat containing 100 tickets. Tickets are not replaced, so each ticket can win at most one prize.

e A researcher tries the same drug on 32 volunteer migraine patients. He records a success if the pain eases after 3 hours. As far as the researcher is concerned, each volunteer is regarded the same.

f A sample of 100 vehicles passing a school are classified into motor cycles, cars, and trucks.

g A student sits for 7 examinations. The result of each exam is either a pass or a fail.

INVESTIGATION 1   SAMPLING SIMULATION
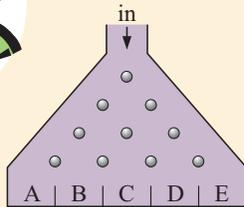
When balls enter the 'sorting' chamber they hit a series of metal rods. Each time they hit a rod, they may go to the left or right with *equal chance*. The balls finally come to rest in collection chambers at the bottom of the sorter.

This sorter looks very much like a tree diagram rotated through $90°$.

SIMULATION

Click on the icon to open the simulation. Notice the sliding bar which we can use to alter the probabilities of balls going to the left or right at each rod.

What to do:

1 To simulate the results of tossing *two coins*, set the bar to 50% and the sorter to show 3 rods as shown.
Run the simulation 200 times and repeat this four more times. Record each set of results.

2 A bag contains 7 blue and 3 red marbles. Two marbles randomly selected from it, the first being *replaced* before the second is drawn. You should set the bar to 70%, since P(blue) $= \frac{7}{10} = 0.7 = 70\%$.
Run the simulation a large number of times. Use the results to estimate the probabilities of getting: **a** two blues   **b** one blue   **c** no blues.
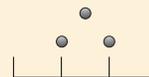
3 The tree diagram for the marble selection in **2** is:
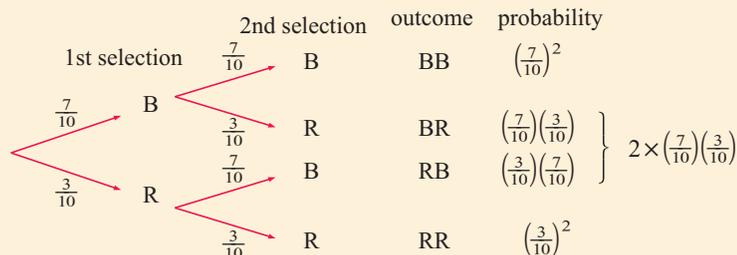
   a The tree diagram gives us the expected or theoretical probabilities for the different outcomes. Do they agree with the experimental results from **2**?

   b Write down the algebraic expansion of $(a + b)^2$.

   c Substitute $a = \frac{7}{10}$ and $b = \frac{3}{10}$ into the $(a + b)^2$ expansion. What do you notice?

4 Consider again the bag with 7 blue and 3 red marbles. This time, *three* marbles are randomly selected with replacement. Set the sorter as shown alongside and the bar to 70%.
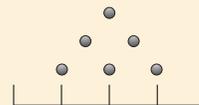
Run the simulation a large number of times. Obtain experimental estimates of the probabilities of getting:

   a three blues   **b** two blues   **c** one blue   **d** no blues.

5   **a** Use a tree diagram showing a 1st selection, 2nd selection and 3rd selection to find theoretical probabilities of getting the results in **4**.

   b Show that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Use this expansion with $a = \frac{7}{10}$ and $b = \frac{3}{10}$ to check the results of **4** and **5 a**.

From the investigation you should have discovered that where there is a repetition of several independent trials, there is a direct link between:

- probabilities obtained from a tree diagram   **(theoretical probabilities)**
- proportions obtained from a sampling simulator **(experimental probabilities)**
- the binomial expansion of $(a + b)^n$ for $n = 2, 3, \ .....$

Example 8   ◀) **Self Tutor**

Jacki paints 4 faces of a die red and the other two blue.

   a With one roll of the die, what is the probability the result will be:

      i red   **ii** blue?

   b If the die is rolled *twice*, what is the probability of rolling:

      i two reds   **ii** one red   **iii** no reds?

   c Use the binomial expansion $(a + b)^2 = a^2 + 2ab + b^2$ to expand $(\frac{2}{3} + \frac{1}{3})^2$.

   d What do you notice from **b** and **c**?

a   **i** $P(\text{red}) = \frac{4}{6} = \frac{2}{3}$ **ii** $P(\text{blue}) = \frac{2}{6} = \frac{1}{3}$

b Tree diagram:
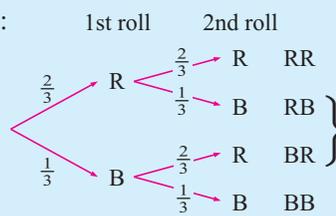
i $P(\text{two reds}) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

ii $P(\text{one red}) = 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{4}{9}$

iii $P(\text{no reds}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

c $\left(\frac{2}{3} + \frac{1}{3}\right)^2 = \left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2$

d The expansion of $\left(\frac{2}{3} + \frac{1}{3}\right)^2$ generates the same probabilities as found from the tree diagram.

EXERCISE 20C.2

1 Aliki paints 5 faces of a die purple and the other one green.

   a With one roll of the die, what is the probability the result will be:

      i purple   **ii** green?

   b If the die is rolled *twice*, use a tree diagram to find the probability of rolling:

      i two purples   **ii** one purple   **iii** no purples.

   c Use the binomial expansion $(a + b)^2 = a^2 + 2ab + b^2$ to expand $\left(\frac{5}{6} + \frac{1}{6}\right)^2$.

   d What do you notice from **b** and **c**?

2 Consider the genders of three child families.

   a List the eight possible families.

     One of them may be written MFF for male, female, female.

     Determine the probability that a randomly chosen three child family consists of:

      i 3 boys   **ii** 2 boys   **iii** 1 boy   **iv** 0 boys.

   b Use a tree diagram showing the first child, second child and third child to display the sample space and probabilities. Check that this gives the same results as in **a**.

   c Use the binomial expansion $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ to expand $\left(\frac{1}{2} + \frac{1}{2}\right)^3$.

   d What do you notice?

3 In **Example 8**, Jacki has a die with 4 red faces and 2 blue faces. Suppose she now rolls it *three* times.

   a Use a tree diagram to find the probability of rolling:

      i three reds   **ii** two reds   **iii** one red   **iv** no reds.

   b Use the binomial expansion $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ to expand $\left(\frac{2}{3} + \frac{1}{3}\right)^3$.

   c What do you notice from **a** and **b**?

4 In sampling *with replacement* we put back each drawn item before the next one is randomly selected. If there are two possible outcomes, this results in a *binomial experiment*.

It is known that 3% of a large batch of electric light globes are defective. Four of them are randomly selected with replacement and tested.

a To find the probability distribution of this data, explain why using the expansion of $(0.03 + 0.97)^4$ instead of drawing a tree diagram would be an advantage.

b Use the binomial expansion $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand $(0.03 + 0.97)^4$.

c From **b**, what is the probability of getting:

   i 4 defectives   **ii** 3 defectives   **iii** 2 defectives

   iv 1 defective   **v** 0 defectives?

SIMULATION

d Use the sampling demonstraton to check that your theoretical calculations are matched reasonably closely by those obtained by experiment.

5 An archer has a 90% chance of hitting the target with each arrow. She fires 4 arrows at the target. Use the binomial expansion for $(a + b)^4$ in question **4b** to determine the chance of hitting the target once only.

BINOMIAL PROBABILITIES USING A GRAPHICS CALCULATOR

Suppose we conduct a binomial experiment with $n = 5$ trials. For each trial there is a probability $p = 0.7$ of getting a success. We want to find probability of getting $x = 2$ successes. The answer should be 0.1323.

For a **TI-83**	For a **Casio**
Press 2nd DISTR and scroll down to binompdf.	From the menu select STAT.
Press ENTER .	Press F5 to select DIST.
For the trial example, press	Press F5 to select BINM.
5 , 0.7 , 2) (n, p, x)	Press F1 to select Bpd.
and then ENTER .	Press F2 to select VAR.
	Now fill in the information on the screen.
	For x put 2, for Numtrial put 5, for P put 0.7. Now press EXE .

INVESTIGATION 2 THE MEAN RESULT OF A BINOMIAL EXPERIMENT

What to do:

1 Click on the icon to open the binomial sorting simulator shown alongside.

SIMULATION

The simulator drops balls onto 4 rows of bars. When it hits the first bar the ball moves to the right with probability p and to the left with probability $1 - p$.

The ball continues in this manner until it arrives at the bottom in one of 5 positions. This is a binomial experiment with number of trials $n = 4$ and probability of success p.
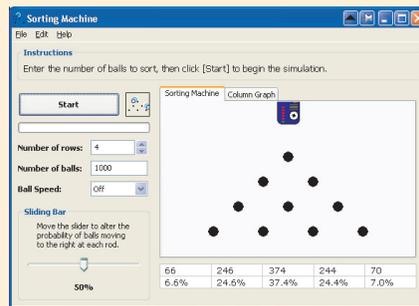
2 Click the "Start" button to run the experiment 1000 times with $p = \frac{1}{2}$. For the results shown in the screenshot, the data can be summarised in the frequency table:

Number of heads	0	1	2	3	4
Frequency	66	246	374	244	70

Construct a frequency table for your results.

3 **a** Find the mean of the data from the frequency table in the usual way.
   b Compare this with the value of np. What do you notice?

4 Obtain experimental binomial distribution results for 1000 repetitions with:
   a $n = 4$ and $p = 0.5$ **b** $n = 5$ and $p = 0.6$ **c** $n = 6$ and $p = 0.75$

5 For each of the distributions obtained in **4**, find the mean.

6 Compare your experimental values in **5** with the corresponding values of np. Comment on your results.

WHY CASINOS ALWAYS WIN

LINKS
click here

Areas of interaction:
Health and social education

REVIEW SET 20A

1 Use a tree diagram to illustrate the following sample space:
Lleyton and Pat play tennis. The first to win three sets wins the match.

2 In a darts game, Alex has a 20% chance of hitting the bullseye, and Irena has a 10% chance of hitting the bullseye. If they each have one throw at the dartboard, determine the probability that:
   a both hit the bullseye
   b neither hits the bullseye
   c at least one hits the bullseye
   d only Alex hits the bullseye.

3 Bag X contains three white and one red marble. Bag Y contains one white and four red marbles. A bag is randomly chosen and two marbles are drawn from it. Illustrate the given information on a tree diagram and hence determine the probability of drawing two marbles of the same colour.

4 Decide which of the following is a binomial experiment:

   a A student guesses every answer in a true-false test that has 20 questions.

   b An athlete enters 5 different competitions. The outcome of each is either a win or a loss for the athlete.

5 A basketball player has probability 0.3 of sinking a basket each time he shoots. What is the probability the player:

   a sinks no baskets in 4 attempts   **b** sinks 2 baskets in 4 attempts?

REVIEW SET 20B

1 Use a tree diagram to illustrate the sample space for the following:
Bags A, B and C contain green and yellow tickets. A bag is selected at random and one ticket is taken from it.

2 Jar X contains 5 red and 3 green marbles. Jar Y contains 4 red and 2 green marbles. A jar is selected at random and then two marbles are selected from it without replacement. Determine the probability that:

   a both marbles are red   **b** two green marbles are picked from Jar X.

3 Michelle has probability 0.9 of passing Mathematics and 0.7 of passing Biology.

   a Construct a tree diagram showing all possible outcomes for the two subjects.

   b What is the probability that Michelle will pass one subject only?

4 Which of the following is a binomial experiment?

   a A box contains 23 soft centred and 28 hard centred chocolates. June draws out 5 chocolates and eats them. The outcome is the number of hard centred chocolates that June eats.

   b A darts player has probability 0.6 of hitting the bullseye with each throw. The darts player tries 14 times to hit the bullseye.

5 In a test there are 3 multiple choice questions. Each question has 4 choices. Suppose a student randomly guesses the answers to each of the questions.

   a Draw a tree diagram to show all possible ways the student can answer the questions and the probability for each outcome.

   b What is the probability the student gets:

      i the first question right, but the second and third questions wrong

      ii exactly 1 of the 3 multiple choice questions right?

HISTORICAL NOTE   MARIA GAETANA AGNESI 1718 - 1799

Maria Gaetana Agnesi was born in 1718 in Italy. Her father, Pietro Agnesi, was a professor of mathematics at the University of Bologna. Maria was the eldest of 21 children and her family was both wealthy and cultured.

It became evident when she was still very young that her grasp of languages, in particular Latin, was outstanding. Her grasp of Latin was important as it was the language spoken by scholars all over Europe.

Encouraged by her parents, Maria engaged in discussions on abstract mathematical and philosophical topics with other intellectuals from all over Europe who came to her home to listen and speak to her. Maria was able to discuss these topics in many different languages but, being shy, she asked to be excused from these meetings at the age of 21, using the death of her mother as an excuse.

She took over the management of the household and at the same time concentrated on her own mathematical development. She became fascinated by the development of calculus due to Leibniz and Newton but as these works appeared in a variety of languages and papers she decided to produce a book of clarity covering the topic.

The result of ten years work, at the age of 30, her two volume "Analytical Institutions" brought fame not only to Maria but to the cause of women by demonstrating that women could compete with men in abstract reasoning. She was highly honoured for this work and praised by the French Academy but could not be admitted as a member because she was female.

She was offered the professorship at the University of Bologna in 1752 when her father died but she declined and retired from academic pursuits. The last 40 years of her life were spent assisting the sick and the poor. There is some dispute as to whether Maria actually became a nun but she lived and worked the simple life of a nun and eventually became director of a hospital for the poor in Milan. There is no doubt she was an exceptional mathematician and one wonders what she may have achieved if she had continued her mathematical pursuits after the death of her father.

Chapter 21

Algebraic fractions

Contents:

Algebraic fractions are fractions which contain at least one variable or unknown.

The variable may be in the numerator, the denominator, or both the numerator and denominator.

For example, $\dfrac{x}{7}$, $\dfrac{-2}{5-y}$ and $\dfrac{x+2y}{1-y}$ are all algebraic fractions.

Algebraic fractions are sometimes called **rational expressions**.

A   EVALUATING ALGEBRAIC FRACTIONS

To **evaluate** an algebraic expression we replace the variables with their known values. We simplify the expression so we can give our answer in simplest form.

Example 1   ◀) **Self Tutor**

If $a = 2$, $b = -3$ and $c = -5$, evaluate: **a** $\dfrac{a-b}{c}$ **b** $\dfrac{a-c-b}{b-a}$

a $\dfrac{a-b}{c}$

$= \dfrac{2-(-3)}{(-5)}$

$= \dfrac{2+3}{-5}$

$= \dfrac{5}{-5}$

$= -1$

b $\dfrac{a-c-b}{b-a}$

$= \dfrac{2-(-5)-(-3)}{(-3)-2}$

$= \dfrac{2+5+3}{-3-2}$

$= \dfrac{10}{-5}$

$= -2$

EXERCISE 21A

1 If $a = 3$, $b = 2$, $c = 6$, evaluate:

a $\dfrac{c}{2}$ **b** $\dfrac{c}{a}$ **c** $\dfrac{a}{c}$ **d** $\dfrac{c}{b-a}$ **e** $\dfrac{a+c}{b}$

f $\dfrac{ab}{c}$ **g** $\dfrac{a^2}{b}$ **h** $\dfrac{c^2}{a}$ **i** $\dfrac{ab^2}{c}$ **j** $\dfrac{(ab)^2}{c}$

2 If $a = 2$, $b = -3$ and $c = -4$, evaluate:

a $\dfrac{c}{a}$ **b** $\dfrac{a}{c}$ **c** $\dfrac{-1}{b}$ **d** $\dfrac{c^2}{a}$ **e** $\dfrac{c}{a+b}$

f $\dfrac{a-c}{2b}$ **g** $\dfrac{b}{c-a}$ **h** $\dfrac{a-c}{a+c}$ **i** $\dfrac{c-a}{b^2}$ **j** $\dfrac{a^2}{c-b}$

B   SIMPLIFYING ALGEBRAIC FRACTIONS

CANCELLATION

We have observed previously that number fractions can be simplified by cancelling common factors.

For example, $\frac{15}{35} = \frac{3 \times \cancel{5}^{1}}{7 \times \cancel{5}_{1}} = \frac{3}{7}$ where the common factor 5 was cancelled.

The same principle can be applied to algebraic fractions:

> If the numerator and denominator of an algebraic fraction are both written in factored form and common factors are found, we can simplify by **cancelling the common factors**.

For example,
$$\frac{6bc}{3c} = \frac{2 \times \cancel{3}^{1} \times b \times \cancel{c}^{1}}{_{1}\cancel{3} \times \cancel{c}_{1}} \qquad \text{\{fully factorised\}}$$
$$= \frac{2b}{1} \qquad\qquad \text{\{after cancellation\}}$$
$$= 2b$$

Fractions such as $\dfrac{3xy}{7z}$ cannot be simplified since the numerator and denominator do not have any common factors.

So, to simplify algebraic expressions:

> When cancelling in algebraic fractions, only factors can be cancelled, not terms.

- make sure that the numerator and denominator are fully factorised
- cancel any common factors
- simplify the result.

ILLEGAL CANCELLATION

Take care with fractions such as $\dfrac{x+4}{2}$.

The expression in the numerator, $x + 4$, **cannot be** written as the product of factors other than $1 \times (x + 4)$. x and 4 are *terms* of the expression, not factors.

A typical **error** is the **illegal cancellation** $\dfrac{x + \cancel{4}^{2}}{\cancel{2}_{1}} = \dfrac{x+2}{1} = x + 2.$

You can check that this cancellation of terms is incorrect by substituting a value for x.

For example, if $x = 2$, $\dfrac{x+4}{2} = \dfrac{2+4}{2} = 3$

whereas $x + 2 = 4$.

Example 2   ◀) **Self Tutor**

Simplify: **a** $\dfrac{a^2}{2a}$ **b** $\dfrac{6a^2b}{3b}$ **c** $\dfrac{a+b}{a}$

a $\dfrac{a^2}{2a}$

$= \dfrac{a \times \overset{1}{\cancel{a}}}{2 \times \cancel{a}^1}$

$= \dfrac{a}{2}$

b $\dfrac{6a^2b}{3b}$

$= \dfrac{\overset{2}{\cancel{6}} \times a \times a \times \overset{1}{\cancel{b}}}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{b}}}$

$= \dfrac{2 \times a \times a}{1}$

$= 2a^2$

c $\dfrac{a+b}{a}$ cannot be simplified as $a+b$ is a sum, not a product.

EXERCISE 21B.1

1 Joe Lin says that you should always check algebraic simplification by using substitution.

For example,   Wei Soong suggested that $\dfrac{2x+6}{3} = 2x + 2$.

When $x = 3$, $\dfrac{2x+6}{3} = \dfrac{6+6}{3} = \dfrac{12}{3} = 4$,

but $2x + 2 = 6 + 2 = 8 \neq 4$.

This one example shows that $\dfrac{2x+6}{3} \neq 2x + 2$.

Check the following simplifications using the given substitutions:

a $\dfrac{3x+8}{4} = 3x + 2$ (substituting $x = 4$)

b $\dfrac{4x+2}{2} = 2x + 1$ (substituting $x = 1$, 2 and 3)

c $\dfrac{4x+3}{2} = 2x + 3$ (substituting $x = 3$)

d $\dfrac{2a+b+10}{5} = 2a + b + 2$ (substituting $a = 2$ and $b = 1$)

2 Simplify:

a $\dfrac{2a}{4}$ **b** $\dfrac{4m}{2}$ **c** $\dfrac{6a}{a}$ **d** $\dfrac{6a}{2a}$ **e** $\dfrac{2a^2}{a}$

f $\dfrac{2x^3}{2x}$ **g** $\dfrac{2x^3}{x^2}$ **h** $\dfrac{2x^3}{x^3}$ **i** $\dfrac{2a^2}{4a^3}$ **j** $\dfrac{8m^2}{4m}$

k $\dfrac{4a^2}{a^2}$ **l** $\dfrac{6t}{3t^2}$ **m** $\dfrac{4d^2}{2d}$ **n** $\dfrac{ab^2}{2ab}$ **o** $\dfrac{4ab^2}{6a^2b}$

3 Simplify if possible:

a $\dfrac{2t}{2}$ **b** $\dfrac{2+t}{2}$ **c** $\dfrac{xy}{x}$ **d** $\dfrac{x+y}{x}$ **e** $\dfrac{x+y}{y}$

f $\dfrac{ac}{bc}$ **g** $\dfrac{a+c}{b+c}$ **h** $\dfrac{2a^2}{4a}$ **i** $\dfrac{2+a^2}{4a}$ **j** $\dfrac{(2a)^2}{4a}$

4 Split these expressions into *two* parts and if possible simplify:

For example, $\dfrac{x+4}{2} = \dfrac{x}{2} + \dfrac{4}{2} = \dfrac{x}{2} + 2.$

a $\dfrac{a+4}{2}$ **b** $\dfrac{2a+4}{2}$ **c** $\dfrac{2x+3}{2}$ **d** $\dfrac{2x+8}{2}$ **e** $\dfrac{4x+3}{2}$

f $\dfrac{2x^2+4x}{2}$ **g** $\dfrac{2x^2+4x}{4}$ **h** $\dfrac{x^2-6x}{x^2}$ **i** $\dfrac{x^2-6x}{x}$ **j** $\dfrac{x^2-6x}{6}$

Example 3   🔊 **Self Tutor**

Simplify: **a** $\dfrac{(-4b)^2}{2b}$ **b** $\dfrac{3(x+4)^2}{x+4}$

a $\dfrac{(-4b)^2}{2b}$

$= \dfrac{(-4b) \times (-4b)}{2 \times b}$

$= \dfrac{^{8}\cancel{16} \times b \times \cancel{b}^{\,1}}{_{1}\cancel{2} \times \cancel{b}_{\,1}}$

$= 8b$

b $\dfrac{3(x+4)^2}{x+4}$

$= \dfrac{3(x+4)\cancel{(x+4)}^{\,1}}{\cancel{x+4}_{\;1}}$

$= 3(x+4)$

---

5 Simplify:

a $\dfrac{(2a)^2}{a^2}$ **b** $\dfrac{(4n)^2}{8n}$ **c** $\dfrac{(-a)^2}{a}$ **d** $\dfrac{a^2}{(-a)^2}$

e $\dfrac{(-2a)^2}{4}$ **f** $\dfrac{(-3n)^2}{6n}$ **g** $\dfrac{(x+y)^2}{x+y}$ **h** $\dfrac{2(x+2)}{(x+2)^2}$

Example 4   🔊 **Self Tutor**

Simplify: **a** $\dfrac{(2x+3)(x+4)}{5(2x+3)}$ **b** $\dfrac{4(x+2)(x-1)}{2(x-1)}$

a $\dfrac{^{1}\cancel{(2x+3)}(x+4)}{5\cancel{(2x+3)}^{\,1}}$

$= \dfrac{(x+4)}{5}$

$= \dfrac{x+4}{5}$

b $\dfrac{^{2}\cancel{4}(x+2)\cancel{(x-1)}^{\,1}}{_{1}\cancel{2}\cancel{(x-1)}^{\,1}}$

$= \dfrac{2(x+2)}{1}$

$= 2(x+2)$

6 Simplify:

a $\dfrac{2(a+3)}{2}$ **b** $\dfrac{4(x+2)}{2}$ **c** $\dfrac{6(c+3)}{3}$

d $\dfrac{2(d-3)}{6}$ **e** $\dfrac{4}{2(x+1)}$ **f** $\dfrac{12}{4(2-x)}$

g $\dfrac{3(x+4)}{9(x+4)}$ **h** $\dfrac{12(a-3)}{6(a-3)}$ **i** $\dfrac{(x+y)(x-y)}{3(x-y)}$

j $\dfrac{2xy(x-y)}{6x(x-y)}$ **k** $\dfrac{5(y+2)(y-3)}{15(y+2)}$ **l** $\dfrac{x(x+1)(x+2)}{3x(x+2)}$

Example 5   ◄⑴ **Self Tutor**

Simplify if possible: **a** $\dfrac{4(x+8)}{2}$ **b** $\dfrac{x+8}{2}$

a $\dfrac{4(x+8)}{2}$

$= \dfrac{2 \times \overset{1}{\cancel{2}} \times (x+8)}{\underset{1}{\cancel{2}}}$

$= 2(x+8)$

b $\dfrac{x+8}{2}$ cannot be simplified because the coefficient of x is not a multiple of 2, and so we cannot cancel the denominator.

---

7 Simplify if possible:

a $\dfrac{3(x+2)}{6}$ **b** $\dfrac{6(x+2)}{3}$ **c** $\dfrac{x+6}{3}$ **d** $\dfrac{x-4}{2}$

e $\dfrac{2(x-1)}{4(x+1)}$ **f** $\dfrac{2x+5}{10}$ **g** $\dfrac{2x+6}{10}$ **h** $\dfrac{3}{3a+9}$

FACTORISATION AND SIMPLIFICATION

It is often necessary to **factorise** either the numerator or denominator before simplification can be done. To do this we use the rules for factorisation that we have seen previously.

Example 6   ◄⑴ **Self Tutor**

Simplify: **a** $\dfrac{3a+9}{3}$ **b** $\dfrac{4a+12}{8}$

a $\dfrac{3a+9}{3}$

$= \dfrac{\overset{1}{\cancel{3}}(a+3)}{\underset{1}{\cancel{3}}}$

$= a+3$

b $\dfrac{4a+12}{8}$

$= \dfrac{\overset{1}{\cancel{4}}(a+3)}{\underset{2}{\cancel{8}}}$

$= \dfrac{a+3}{2}$

Example 7   ◀) **Self Tutor**

Simplify by factorising: **a** $\dfrac{ab+ac}{b+c}$ **b** $\dfrac{6x^2-6xy}{3x-3y}$

a $\dfrac{ab+ac}{b+c} = \dfrac{a(b+c)}{b+c}$ {remove HCF a from numerator}

$\qquad = \dfrac{a\cancel{(b+c)}^{\,1}}{\cancel{(b+c)}_{\,1}}$

$\qquad = a$

b $\dfrac{6x^2-6xy}{3x-3y} = \dfrac{6x(x-y)}{3(x-y)}$ ⟵ HCF is $6x$
$\qquad\qquad\qquad\qquad\quad$ ⟵ HCF is 3

$\qquad = \dfrac{\overset{2}{\cancel{6}} \times x \times \cancel{(x-y)}^{\,1}}{\underset{1}{\cancel{3}} \times \cancel{(x-y)}_{\,1}}$

$\qquad = 2x$

$\boxed{b - a = -1(a - b)}$ is a useful rule for converting $b - a$ into $a - b$. It can sometimes allow us to cancel common factors.

Example 8   ◀) **Self Tutor**

Simplify, if possible: **a** $\dfrac{6a-6b}{b-a}$ **b** $\dfrac{xy^2-xy}{1-y}$

a $\dfrac{6a-6b}{b-a}$

$= \dfrac{6\cancel{(a-b)}^{\,1}}{-1\cancel{(a-b)}_{\,1}}$

$= -6$

b $\dfrac{xy^2-xy}{1-y}$

$= \dfrac{xy\cancel{(y-1)}^{\,1}}{-1\cancel{(y-1)}_{\,1}}$

$= -xy$

Example 9   ◀) **Self Tutor**

Simplify: $\dfrac{x^2-1}{x^2+3x+2}$

$\dfrac{x^2-1}{x^2+3x+2}$

$= \dfrac{(x-1)\cancel{(x+1)}^{\,1}}{(x+2)\cancel{(x+1)}_{\,1}}$

$= \dfrac{x-1}{x+2}$

> Don't forget to expand your factorisations to check them.

EXERCISE 21B.2

1 Simplifying by removing common factors:

a $\dfrac{4}{2(x+1)}$ **b** $\dfrac{12}{4(2-x)}$ **c** $\dfrac{2x+4}{2}$ **d** $\dfrac{3x+6}{3}$

e $\dfrac{3x+6}{6}$ **f** $\dfrac{4x+20}{8}$ **g** $\dfrac{4y+12}{12}$ **h** $\dfrac{ax+bx}{x}$

i $\dfrac{ax+bx}{cx+dx}$ **j** $\dfrac{(a+2)^2}{2(a+2)}$ **k** $\dfrac{3(b-4)}{6(b-4)^2}$ **l** $\dfrac{8(p+q)^2}{12(p+q)}$

2 Simplify by factorising:

a $\dfrac{3x+6}{4x+8}$ **b** $\dfrac{ax+bx}{2x}$ **c** $\dfrac{ax+bx}{a+b}$ **d** $\dfrac{x}{ax+bx}$

e $\dfrac{a+b}{ay+by}$ **f** $\dfrac{ax+bx}{ay+by}$ **g** $\dfrac{4x^2+8x}{x+2}$ **h** $\dfrac{3x^2+9x}{x+3}$

3 Simplify, if possible:

a $\dfrac{2x-2y}{y-x}$ **b** $\dfrac{3x-3y}{2y-2x}$ **c** $\dfrac{m+n}{n-m}$ **d** $\dfrac{m-n}{n-m}$

e $\dfrac{r-2s}{4s-2r}$ **f** $\dfrac{3r-6s}{2s-r}$ **g** $\dfrac{2x-2}{x-x^2}$ **h** $\dfrac{ab^2-ab}{2-2b}$

i $\dfrac{4x^2-4x}{2-2x}$ **j** $\dfrac{4x+6}{2}$ **k** $\dfrac{4x+6}{3}$ **l** $\dfrac{4x+6}{4}$

m $\dfrac{4x+6}{5}$ **n** $\dfrac{4x+6}{6}$ **o** $\dfrac{6a+1}{2}$ **p** $\dfrac{6a+1}{3}$

q $\dfrac{6a+2}{4}$ **r** $\dfrac{3b+9}{2}$ **s** $\dfrac{3b+9}{6}$ **t** $\dfrac{4x-2}{2-x}$

4 Simplify:

a $\dfrac{x^2-1}{x-1}$ **b** $\dfrac{x^2-1}{x+1}$ **c** $\dfrac{x^2-1}{1-x}$ **d** $\dfrac{x+2}{x^2-4}$

e $\dfrac{a^2-b^2}{a+b}$ **f** $\dfrac{a^2-b^2}{b-a}$ **g** $\dfrac{2x+2}{x^2-1}$ **h** $\dfrac{9-x^2}{3x-x^2}$

i $\dfrac{3x^2-3y^2}{2xy-2y^2}$ **j** $\dfrac{2b^2-2a^2}{a^2-ab}$ **k** $\dfrac{4xy-y^2}{16x^2-y^2}$ **l** $\dfrac{4x(x-4)}{16-x^2}$

5 Simplify by factorising and cancelling common factors:

a $\dfrac{x^2-x-2}{x-2}$ **b** $\dfrac{x+3}{x^2-2x-15}$ **c** $\dfrac{2x^2+2x}{x^2-4x-5}$

d $\dfrac{x^2-4}{x^2+4x+4}$ **e** $\dfrac{x^2-x-12}{x^2-5x+4}$ **f** $\dfrac{x^2+2x+1}{1-x^2}$

g $\dfrac{x^2-x-20}{x^2+7x+12}$ **h** $\dfrac{2x^2+5x+2}{2x^2+7x+3}$ **i** $\dfrac{3x^2+7x+2}{6x^2-x-1}$

j $\dfrac{8x^2+2x-1}{4x^2-5x+1}$ **k** $\dfrac{12x^2-5x-3}{6x^2+5x+1}$ **l** $\dfrac{15x^2+17x-4}{5x^2+9x-2}$

C   MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS

The rules for multiplying and dividing algebraic fractions are identical to those used with numerical fractions. These are:

To **multiply** two or more fractions, we multiply the numerators to form the new numerator, and we multiply the denominators to form the new denominator.

To **divide** by a fraction we multiply by its reciprocal.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

MULTIPLICATION

> *Step 1:*   Multiply numerators and multiply denominators.
>
> *Step 2:*   Separate the factors.
>
> *Step 3:*   Cancel any common factors.
>
> *Step 4:*   Write in simplest form.

For example, $\dfrac{a^2}{2} \times \dfrac{4}{a} = \dfrac{a^2 \times 4}{2 \times a}$ {*Step 1*}

$$= \frac{a \times \cancel{a} \times 2 \times \cancel{2}^{\,1}}{_1\cancel{2} \times \cancel{a}\ _1}$$ {*Steps 2 and 3*}

$$= \frac{2a}{1}$$

$$= 2a$$ {*Step 4*}

Example 10   ◀)) **Self Tutor**

Simplify: **a** $\dfrac{3}{m} \times \dfrac{m}{6}$ **b** $\dfrac{3}{m} \times m^2$

a $\dfrac{3}{m} \times \dfrac{m}{6}$

$= \dfrac{^1\cancel{3} \times \cancel{m}^{\,1}}{_1\cancel{m} \times \cancel{6}\ _2}$

$= \dfrac{1}{2}$

b $\dfrac{3}{m} \times m^2$

$= \dfrac{3}{m} \times \dfrac{m^2}{1}$

$= \dfrac{3m^2}{m}$

$= \dfrac{3 \times m \times \cancel{m}^{\,1}}{\cancel{m}\ _1}$

$= \dfrac{3m}{1}$

$= 3m$

DIVISION

> *Step 1:*   To divide by a fraction, multiply by its reciprocal.
>
> *Step 2:*   Multiply numerators and multiply denominators.
>
> *Step 3:*   Cancel any common factors.
>
> *Step 4:*   Write in simplest form.

For example,

$$\frac{a}{2} \div \frac{b}{4} = \frac{a}{2} \times \frac{4}{b} \quad \{Step\ 1\}$$

$$= \frac{a \times 4}{2 \times b} \quad \{Step\ 2\}$$

$$= \frac{a \times \overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}} \times b} \quad \{Step\ 3\}$$

$$= \frac{2a}{b} \quad \{Step\ 4\}$$

Example 11   ◀) **Self Tutor**

Simplify: **a** $\dfrac{4}{n} \div \dfrac{2}{n^2}$ **b** $\dfrac{3}{a} \div 2$

a

$$\frac{4}{n} \div \frac{2}{n^2}$$

$$= \frac{4}{n} \times \frac{n^2}{2}$$

$$= \frac{4 \times n^2}{n \times 2}$$

$$= \frac{\overset{1}{\cancel{2}} \times 2 \times \cancel{n} \times \overset{1}{n}}{\underset{1}{\cancel{n}} \times \cancel{2}_{\ 1}}$$

$$= 2n$$

b

$$\frac{3}{a} \div 2$$

$$= \frac{3}{a} \times \frac{1}{2}$$

$$= \frac{3 \times 1}{a \times 2}$$

$$= \frac{3}{2a}$$

EXERCISE 21C

1 Simplify:

a $\dfrac{x}{2} \times \dfrac{y}{5}$ **b** $\dfrac{a}{2} \times \dfrac{3}{a}$ **c** $\dfrac{a}{2} \times a$ **d** $\dfrac{a}{4} \times \dfrac{2}{3a}$

e $\dfrac{c}{5} \times \dfrac{1}{c}$ **f** $\dfrac{c}{5} \times \dfrac{c}{2}$ **g** $\dfrac{a}{b} \times \dfrac{c}{d}$ **h** $\dfrac{a}{b} \times \dfrac{b}{a}$

i $\dfrac{1}{m^2} \times \dfrac{m}{2}$ **j** $\dfrac{m}{2} \times \dfrac{4}{m}$ **k** $\dfrac{a}{x} \times \dfrac{x}{b}$ **l** $m \times \dfrac{4}{m}$

m $\dfrac{3}{m^2} \times m$ **n** $\left(\dfrac{a}{b}\right)^2$ **o** $\left(\dfrac{2}{x}\right)^2$ **p** $\dfrac{1}{a} \times \dfrac{a}{b} \times \dfrac{b}{c}$

2 Simplify:

a $\dfrac{a}{2} \div \dfrac{a}{3}$ **b** $\dfrac{2}{a} \div \dfrac{2}{3}$ **c** $\dfrac{3}{4} \div \dfrac{4}{x}$ **d** $\dfrac{3}{x} \div \dfrac{4}{x}$

e $\dfrac{2}{n} \div \dfrac{1}{n}$ **f** $\dfrac{c}{5} \div 5$ **g** $\dfrac{c}{5} \div c$ **h** $m \div \dfrac{2}{m}$

i $m \div \dfrac{m}{2}$ **j** $1 \div \dfrac{m}{n}$ **k** $\dfrac{3}{g} \div 4$ **l** $\dfrac{3}{g} \div \dfrac{9}{g^2}$

D   ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

Variables are used in algebraic fractions to represent unknown numbers. We can treat algebraic fractions in the same way that we treat numerical fractions since they are in fact *representing* numerical fractions.

The rules for addition and subtraction of algebraic fractions are identical to those used with numerical fractions.

To **add** two or more fractions we obtain the *lowest common denominator* and then add the resulting numerators.

Since $\frac{2}{5} + \frac{4}{5} = \frac{2+4}{5}$,

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

To **subtract** two or more fractions we obtain the *lowest common denominator* and then subtract the resulting numerators.

Since $\frac{7}{9} - \frac{5}{9} = \frac{7-5}{9}$,

$$\frac{a}{c} - \frac{d}{c} = \frac{a-d}{c}$$

To find the lowest common denominator, we look for the **lowest common multiple of the denominators**.

For example,   when adding $\frac{3}{2} + \frac{5}{3}$, the lowest common denominator is 6,

when adding $\frac{5}{6} + \frac{4}{9}$, the lowest common denominator is 18.

The same method is used when there are variables in the denominator.

For example,   when adding $\dfrac{2}{a} + \dfrac{3}{b}$,   the lowest common denominator is ab,

when adding $\dfrac{2}{x} + \dfrac{4}{5x}$,   the lowest common denominator is $5x$,

when adding $\dfrac{5}{6x} + \dfrac{4}{9y}$,   the lowest common denominator is $18xy$.

To find $\dfrac{x}{3} + \dfrac{5x}{4}$ we find the LCD and then proceed in the same manner as for ordinary fractions.

The LCM of 3 and 4 is 12, so the LCD is 12.

$$\therefore \quad \frac{x}{3} + \frac{5x}{4} = \frac{x \times 4}{3 \times 4} + \frac{5x \times 3}{5 \times 3}$$

$$= \frac{4x}{12} + \frac{15x}{12}$$

$$= \frac{19x}{12}$$

Example 12
◀) **Self Tutor**

Simplify: **a** $\dfrac{x}{2} + \dfrac{3x}{4}$ **b** $\dfrac{a}{3} - \dfrac{2a}{5}$

a $\quad \dfrac{x}{2} + \dfrac{3x}{4} \qquad \{\text{LCD} = 4\}$

$\quad = \dfrac{x \times 2}{2 \times 2} + \dfrac{3x}{4}$

$\quad = \dfrac{2x}{4} + \dfrac{3x}{4}$

$\quad = \dfrac{2x + 3x}{4}$

$\quad = \dfrac{5x}{4}$

b $\quad \dfrac{a}{3} - \dfrac{2a}{5} \qquad \{\text{LCD} = 15\}$

$\quad = \dfrac{a \times 5}{3 \times 5} - \dfrac{2a \times 3}{5 \times 3}$

$\quad = \dfrac{5a}{15} - \dfrac{6a}{15}$

$\quad = \dfrac{-a}{15} \quad \text{or} \quad -\dfrac{a}{15}$

Example 13
◀) **Self Tutor**

Simplify: **a** $\dfrac{4}{a} + \dfrac{3}{b}$ **b** $\dfrac{5}{x} - \dfrac{4}{3x}$

a $\quad \dfrac{4}{a} + \dfrac{3}{b} \qquad \{\text{LCD} = ab\}$

$\quad = \dfrac{4 \times b}{a \times b} + \dfrac{3 \times a}{b \times a}$

$\quad = \dfrac{4b}{ab} + \dfrac{3a}{ab}$

$\quad = \dfrac{4b + 3a}{ab}$

b $\quad \dfrac{5}{x} - \dfrac{4}{3x} \qquad \{\text{LCD} = 3x\}$

$\quad = \dfrac{5 \times 3}{x \times 3} - \dfrac{4}{3x}$

$\quad = \dfrac{15}{3x} - \dfrac{4}{3x}$

$\quad = \dfrac{11}{3x}$

Example 14 ◄》 **Self Tutor**

Simplify: **a** $\dfrac{b}{3} + 1$ **b** $\dfrac{a}{4} - a$

a $\dfrac{b}{3} + 1$

$= \dfrac{b}{3} + \dfrac{3}{3}$

$= \dfrac{b+3}{3}$

b $\dfrac{a}{4} - a$

$= \dfrac{a}{4} - \dfrac{a \times 4}{1 \times 4}$

$= \dfrac{a}{4} - \dfrac{4a}{4}$

$= \dfrac{-3a}{4}$ or $-\dfrac{3a}{4}$

EXERCISE 21D

1 Simplify by writing as a single fraction:

a $\dfrac{a}{2} + \dfrac{a}{3}$ **b** $\dfrac{b}{5} - \dfrac{b}{10}$ **c** $\dfrac{c}{4} + \dfrac{3c}{2}$ **d** $\dfrac{x}{7} - \dfrac{x}{2}$

e $\dfrac{a}{3} + \dfrac{b}{4}$ **f** $\dfrac{t}{3} - \dfrac{5t}{9}$ **g** $\dfrac{m}{7} + \dfrac{2m}{21}$ **h** $\dfrac{5d}{6} - \dfrac{d}{3}$

i $\dfrac{3p}{5} - \dfrac{2p}{7}$ **j** $\dfrac{m}{2} + \dfrac{m}{3} + \dfrac{m}{6}$ **k** $\dfrac{a}{2} - \dfrac{a}{3} + \dfrac{a}{4}$ **l** $\dfrac{x}{4} - \dfrac{x}{3} + \dfrac{x}{6}$

2 Simplify:

a $\dfrac{7}{a} + \dfrac{3}{b}$ **b** $\dfrac{3}{a} + \dfrac{2}{c}$ **c** $\dfrac{4}{a} + \dfrac{5}{d}$ **d** $\dfrac{2a}{m} - \dfrac{a}{m}$

e $\dfrac{a}{x} + \dfrac{b}{2x}$ **f** $\dfrac{3}{a} - \dfrac{1}{2a}$ **g** $\dfrac{4}{x} - \dfrac{1}{xy}$ **h** $\dfrac{a}{b} + \dfrac{c}{d}$

i $\dfrac{a}{b} - \dfrac{x}{y}$ **j** $\dfrac{2}{3} + \dfrac{a}{2}$ **k** $\dfrac{x}{3} + \dfrac{3}{4}$ **l** $\dfrac{x}{y} + \dfrac{2}{3}$

3 Simplify:

a $\dfrac{x}{2} + 1$ **b** $\dfrac{y}{3} - 1$ **c** $\dfrac{a}{2} + a$ **d** $\dfrac{b}{4} - 3$

e $\dfrac{x}{2} - 4$ **f** $2 + \dfrac{a}{3}$ **g** $3 - \dfrac{x}{5}$ **h** $2 + \dfrac{1}{x}$

i $5 - \dfrac{2}{x}$ **j** $a + \dfrac{2}{a}$ **k** $\dfrac{3}{b} + b$ **l** $\dfrac{x}{3} - 2x$

4 Simplify:

a $\dfrac{x}{2} + \dfrac{2x}{5}$ **b** $\dfrac{4x}{5} - \dfrac{3x}{2}$ **c** $\dfrac{3}{a} + \dfrac{2}{3a}$ **d** $\dfrac{4}{y} - \dfrac{3}{2y}$

e $\dfrac{5}{a} + \dfrac{3}{b}$ **f** $\dfrac{4}{3a} - \dfrac{5}{b}$ **g** $\dfrac{x}{7} + 2$ **h** $3 - \dfrac{x}{4}$

E   MORE COMPLICATED FRACTIONS

Addition and subtraction of more complicated algebraic fractions can be made relatively straightforward if we adopt a consistent approach.

For example: $\dfrac{2x+1}{4} - \dfrac{x+2}{7} = \dfrac{7}{7}\left(\dfrac{2x+1}{4}\right) - \dfrac{4}{4}\left(\dfrac{x+2}{7}\right)$ {achieves LCD of 28}

$$= \dfrac{7(2x+1)}{28} - \dfrac{4(x+2)}{28} \qquad \text{\{simplify each fraction\}}$$

We can then write the expression as a single fraction and simplify the numerator.

Example 15   ◀ᴺ **Self Tutor**

Write as a single fraction: **a** $\dfrac{x}{6} + \dfrac{x-2}{3}$ **b** $\dfrac{x+1}{2} - \dfrac{x-2}{3}$

a $\dfrac{x}{6} + \dfrac{x-2}{3}$

$= \dfrac{x}{6} + \dfrac{2}{2}\left(\dfrac{x-2}{3}\right)$

$= \dfrac{x + 2(x-2)}{6}$

$= \dfrac{x + 2x - 4}{6}$

$= \dfrac{3x - 4}{6}$

b $\dfrac{x+1}{2} - \dfrac{x-2}{3}$

$= \dfrac{3}{3}\left(\dfrac{x+1}{2}\right) - \dfrac{2}{2}\left(\dfrac{x-2}{3}\right)$

$= \dfrac{3(x+1)}{6} - \dfrac{2(x-2)}{6}$

$= \dfrac{3(x+1) - 2(x-2)}{6}$

$= \dfrac{3x + 3 - 2x + 4}{6}$

$= \dfrac{x + 7}{6}$

Example 16   ◀ᴺ **Self Tutor**

Write as a single fraction: **a** $\dfrac{1}{x} + \dfrac{2}{x-1}$ **b** $\dfrac{2}{x-1} - \dfrac{3}{x+1}$

a $\dfrac{1}{x} + \dfrac{2}{x-1}$

$= \dfrac{1}{x}\left(\dfrac{x-1}{x-1}\right) + \left(\dfrac{2}{x-1}\right)\dfrac{x}{x}$ {LCD $= x(x-1)$}

$= \dfrac{1(x-1) + 2x}{x(x-1)}$

$= \dfrac{x - 1 + 2x}{x(x-1)}$

$= \dfrac{3x - 1}{x(x-1)}$

b $\dfrac{2}{x-1} - \dfrac{3}{x+1}$

$= \left(\dfrac{2}{x-1}\right)\left(\dfrac{x+1}{x+1}\right) - \left(\dfrac{3}{x+1}\right)\left(\dfrac{x-1}{x-1}\right)$ $\{\text{LCD} = (x-1)(x+1)\}$

$= \dfrac{2(x+1) - 3(x-1)}{(x-1)(x+1)}$

$= \dfrac{2x+2-3x+3}{(x-1)(x+1)}$

$= \dfrac{-x+5}{(x-1)(x+1)}$ or $\dfrac{5-x}{(x-1)(x+1)}$

EXERCISE 21E

1 Write as a single fraction:

   a $\dfrac{x}{2} + \dfrac{x+1}{3}$
   b $\dfrac{x-1}{4} - \dfrac{x}{2}$
   c $\dfrac{2x}{3} + \dfrac{x+3}{4}$

   d $\dfrac{x+1}{2} + \dfrac{x-1}{3}$
   e $\dfrac{x-1}{3} + \dfrac{1-2x}{4}$
   f $\dfrac{2x+3}{2} + \dfrac{2x-3}{3}$

2 Simplify:

   a $\dfrac{x}{3} + \dfrac{x+1}{4}$
   b $\dfrac{3x+2}{4} + \dfrac{x}{2}$
   c $\dfrac{x}{6} + \dfrac{3x-1}{5}$

   d $\dfrac{a+b}{3} + \dfrac{b-a}{2}$
   e $\dfrac{x+1}{5} + \dfrac{2x-1}{4}$
   f $\dfrac{x+1}{7} + \dfrac{3-x}{2}$

   g $\dfrac{x}{6} - \dfrac{2-x}{5}$
   h $\dfrac{2x-1}{5} - \dfrac{x}{4}$
   i $\dfrac{x}{8} - \dfrac{1-x}{4}$

   j $\dfrac{x}{5} - \dfrac{2-x}{10}$
   k $\dfrac{x-1}{5} - \dfrac{2x-7}{3}$
   l $\dfrac{1-3x}{4} - \dfrac{2x+1}{3}$

   m $\dfrac{x}{30} - \dfrac{4x-1}{10}$
   n $\dfrac{3}{x} + \dfrac{4}{x+1}$
   o $\dfrac{5}{x+2} - \dfrac{3}{x}$

   p $\dfrac{4}{x+1} - \dfrac{3}{x-1}$
   q $3 + \dfrac{1}{x+2}$
   r $\dfrac{1}{x} + \dfrac{4}{x-4}$

   s $\dfrac{2}{x+3} - 4$
   t $\dfrac{x+1}{x-1} + \dfrac{x}{x+1}$
   u $\dfrac{5}{x} + \dfrac{6}{x-2}$

REVIEW SET 21A

1 If $p = 5$, $q = -3$ and $r = 6$, evaluate:

   a $\dfrac{r}{q}$
   b $\dfrac{p-q}{p+q}$
   c $\dfrac{\sqrt{p^2-16}}{r-q}$
   d $\dfrac{p+2q-2r}{r^2-p^2}$

2 Simplify:

a $\dfrac{(2t)^2}{6t}$ **b** $\dfrac{16a + 8b}{6a + 3b}$ **c** $\dfrac{x(x-4)}{3(x-4)}$ **d** $\dfrac{8}{4x+8}$

3 Simplify:

a $\dfrac{2a - 2b}{b - a}$ **b** $\dfrac{2x + 6}{x^2 - 9}$ **c** $\dfrac{x^2 + 4x + 4}{x^2 + 2x}$ **d** $\dfrac{3x^2 - 6x}{3x^2 - 5x - 2}$

4 Simplify:

a $\dfrac{a}{b} \times \dfrac{b}{3}$ **b** $\dfrac{a}{b} \div \dfrac{b}{3}$ **c** $\dfrac{a}{b} + \dfrac{b}{3}$ **d** $\dfrac{a}{b} - \dfrac{b}{3}$

5 Write as a single fraction:

a $\dfrac{2x}{3} + \dfrac{x}{4}$ **b** $2 + \dfrac{x}{7}$ **c** $\dfrac{x}{4} - 1$ **d** $\dfrac{x}{2} + \dfrac{x}{4} - \dfrac{x}{3}$

6 Simplify:

a $\dfrac{x}{3} + \dfrac{x-1}{4}$ **b** $\dfrac{x+2}{3} - \dfrac{2-x}{6}$ **c** $\dfrac{1}{x+1} + \dfrac{2}{x-2}$

d $\dfrac{2x+1}{5} - \dfrac{x-1}{10}$ **e** $\dfrac{5}{x-1} - \dfrac{4}{x+1}$ **f** $\dfrac{1}{x^2} + \dfrac{1}{x+1}$

REVIEW SET 21B

1 If $m = -4$, $n = 3$ and $p = 6$, evaluate:

a $\dfrac{p}{m+n}$ **b** $\dfrac{p - m}{\sqrt{m^2 + n^2}}$ **c** $\dfrac{p - 2n}{m + n}$ **d** $\dfrac{2p + n}{p - 2n}$

2 Simplify:

a $\dfrac{(3x)^2}{6x^3}$ **b** $\dfrac{3a + 6b}{3}$ **c** $\dfrac{(x+2)^2}{x^2 + 2x}$ **d** $\dfrac{9}{3x + 6y}$

3 Simplify:

a $\dfrac{a + b}{3b + 3a}$ **b** $\dfrac{2x^2 - 8}{x + 2}$ **c** $\dfrac{x^2 - 6x + 9}{4x - 12}$ **d** $\dfrac{2x^2 + 3x - 2}{3x^2 + 7x + 2}$

4 Simplify:

a $\dfrac{m}{n} \times \dfrac{2}{n}$ **b** $\dfrac{m}{n} \div \dfrac{2}{n}$ **c** $\dfrac{3}{x} + \dfrac{5}{2x}$ **d** $\dfrac{6}{y} - \dfrac{a}{b}$

5 Simplify:

a $\dfrac{3x}{7} - \dfrac{x}{14}$ **b** $5 + \dfrac{x}{2}$ **c** $3 - \dfrac{y}{x}$ **d** $1 + \dfrac{x}{2} + \dfrac{y}{3}$

6 Simplify:

a $\dfrac{x}{4} - \dfrac{2-x}{8}$ **b** $\dfrac{x+5}{2} + \dfrac{2x+1}{5}$ **c** $\dfrac{2}{x-1} - \dfrac{3}{x+2}$

d $\dfrac{3x-1}{2} - \dfrac{1-x}{4}$ **e** $\dfrac{2}{x-3} - \dfrac{3}{x+3}$ **f** $\dfrac{1}{x-1} - \dfrac{2}{x^2}$

Chapter

22

Other functions: Their graphs and uses

Contents:

A   EXPONENTIAL FUNCTIONS

Consider a population of 100 mice which is growing under plague conditions.

If the mouse population doubles each week, we can construct a **table** to show the population number P after t weeks.

t (weeks)	0	1	2	3	4	......
P	100	200	400	800	1600	......
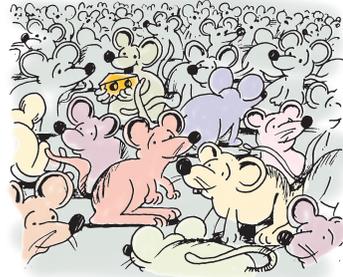
We can also represent this information on a graph as:
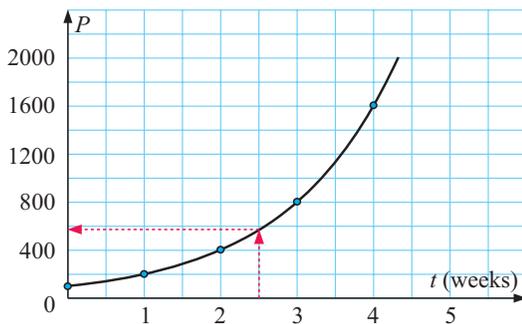
> If we use a smooth curve to join the points, we can predict the mouse population when $t = 2.5$ weeks!

We can find a relationship between P and t using another table:

t	P values
0	$100 = 100 \times 2^0$
1	$200 = 100 \times 2^1$
2	$400 = 100 \times 2^2$
3	$800 = 100 \times 2^3$
4	$1600 = 100 \times 2^4$

So, the relationship which connects P and t is
$P = 100 \times 2^t$.

This is an **exponential relationship** and the graph is an **exponential graph**.

We can use the function to find P for any value of $t \geqslant 0$.

For example, when $t = 2.5$, $\quad P = 100 \times 2^{2.5}$
$$\approx 566 \text{ mice}$$

Example 1   ◀)) **Self Tutor**

For the function $y = 2^x - 3$, find y when: **a** $x = 0$ **b** $x = 2$ **c** $x = -2$.

a When $x = 0$,
$y = 2^0 - 3$
$= 1 - 3$
$= -2$

b When $x = 2$,
$y = 2^2 - 3$
$= 4 - 3$
$= 1$

c When $x = -2$,
$y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

EXERCISE 22A

1 If $y = 2^x + 3$ find the value of y when:

    a $x = 0$ **b** $x = 1$ **c** $x = 2$ **d** $x = -1$ **e** $x = -2$

2 If $y = 5 \times 2^x$ find the value of y when:

    a $x = 0$ **b** $x = 1$ **c** $x = 2$ **d** $x = -1$ **e** $x = -2$

3 If $g = 2^{x+1}$ find the value of g when:

    a $x = 0$ **b** $x = 1$ **c** $x = 3$ **d** $x = -1$ **e** $x = -3$

4 If $h = 2^{-x}$ find the value of h when:

    a $x = 0$ **b** $x = 2$ **c** $x = 4$ **d** $x = -2$ **e** $x = -4$

5 If $P = 6 \times 3^{-x}$ find the value of P when:

    a $x = 0$ **b** $x = 1$ **c** $x = -1$ **d** $x = 2$ **e** $x = -2$

B   GRAPHING SIMPLE EXPONENTIAL FUNCTIONS

One of the simplest exponential functions is $y = 2^x$. A **table of values** for this function is:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

If $x = -3$,	If $x = 0$,	If $x = 3$,
$y = 2^{-3} = \frac{1}{8}$	$y = 2^0 = 1$	$y = 2^3 = 8$

Using these values, we can generate the graph of $y = 2^x$:
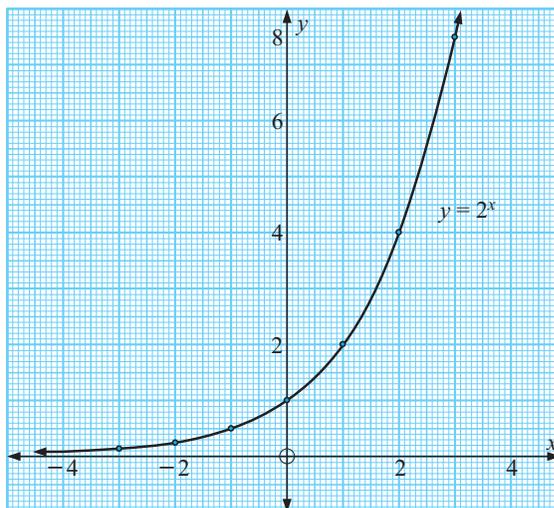
To the left of the y-axis the graph gets closer to the x-axis but always lies above it.

To the right of the y-axis the graph becomes very steep as x values increase.

We say that the graph of $y = 2^x$ is **asymptotic** to the x-axis, or the x-axis is a **horizontal asymptote** for the graph of $y = 2^x$.

We can use the graphs of exponential functions to find the value of numbers raised to **decimal powers**. We can also use them to help solve **exponential equations**.

EXERCISE 22B

1 Alongside is the graph of $y = 2^x$.

   a Use the graph to estimate, to one decimal place, the value of:

      i $2^{0.5}$ **ii** $2^{1.6}$ **iii** $2^{2.1}$

   b Check your estimations using the ⌃ key on your calculator.
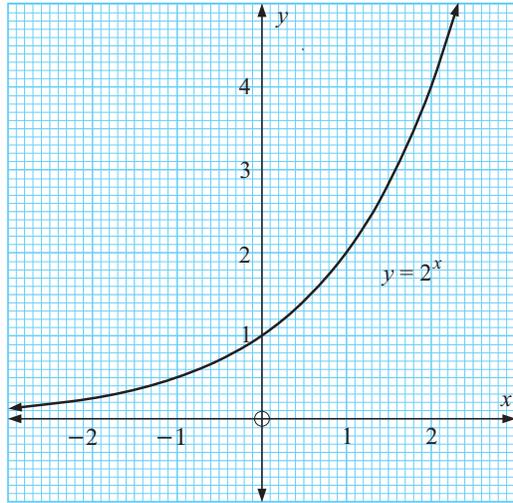
2 Alongside is the graph of $y = 3^x$.

   a Use the graph to estimate, to one decimal place, the value of:

      i 3^0 **ii** 3^1

      iii $3^{0.5}$ **iv** $3^{1.2}$

   b Check your estimations using the ⌃ key on your calculator.
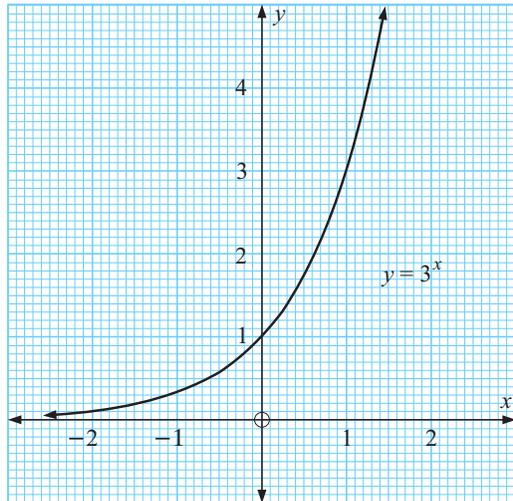
3   **a** Use the graph of $y = 2^x$ in question **1** to solve correct to one decimal place:

      i $2^x = 4$ **ii** $2^x = 3$ **iii** $2^x = 1$ **iv** $2^x = 0.8$

    Hint: In **i** find the x-value of a point on the graph with a y-value of 4.

  b Check your estimations using the ⌃ key on your calculator.

4   **a** Use the graph of $y = 3^x$ in question **2** to solve correct to one decimal place:

      i $3^x = 3$ **ii** $3^x = 4$ **iii** $3^x = 1$ **iv** $3^x = 0.4$

  b Check your estimations using your calculator.

5 Consider the function $y = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
y							

    a Copy and complete the table of values:

    b Use the table in **a** to sketch the graph of $y = 2^{-x}$.

    c Is the x-axis a horizontal asymptote of $y = 2^{-x}$?

6 Alongside is the graph of $y = 3^{-x}$.

    a Use the graph to estimate, to one decimal place, the value of:

        i $3^{-0.5}$ **ii** $3^{1.4}$

        iii $3^{-2.2}$ **iv** 3^{0}

    b Check your estimations using the $\boxed{\wedge}$ key on your calculator.
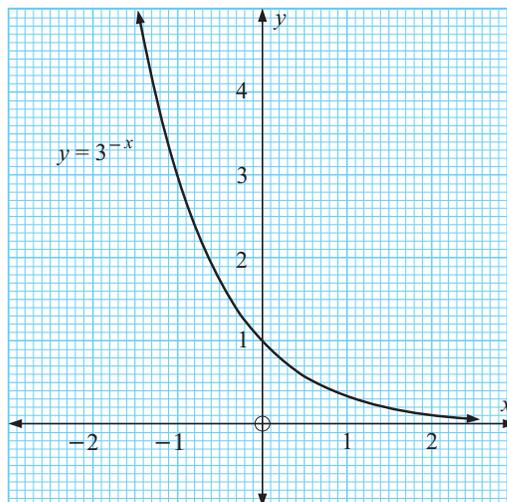
INVESTIGATION 1   SOLVING EXPONENTIAL EQUATIONS

Consider the exponential equation $2^x = 10$.

Since $2^3 = 8$ and $2^4 = 16$, the solution for x must lie between 3 and 4.

GRAPHING PACKAGE

A **graphics calculator** can be used to solve this equation by drawing the graphs of $y = 2^x$ and $y = 10$ and finding the **point of intersection**. You may need to consult the graphics calculator instructions at the start of the book. Alternatively, you could use the graphing package.

What to do:

1 Draw the graph of $Y_1 = 2^{\wedge}X$.

2 Use **trace** to estimate x when $y = 10$.

3 Draw the graph of $Y_2 = 10$ on the same set of axes as $Y_1 = 2^{\wedge}X$.

4 Check the estimation in **2** by finding the coordinates of the **point of intersection** of the graphs.

5 Use the method above to solve for x, correct to 3 decimal places:

    a $2^x = 3$ **b** $2^x = 7$ **c** $2^x = 34$

    d $2^x = 100$ **e** $3^x = 12$ **f** $3^x = 41$

    You may have to change the *viewing window scales*.

6 Use the method above to solve for x, correct to 3 decimal places:

    a $2^{-x} = 5$ **b** $2^{-x} = \frac{1}{15}$ **c** $3^{-x} = 2$

C   GROWTH PROBLEMS

At the start of the chapter, we used the example of a mouse plague. The population P of mice was 100 at time zero, and doubled each week thereafter. We saw the population **grows exponentially** according to the relationship $P = 100 \times 2^t$ after t weeks.

We can use this relationship to answer questions about the mouse population.

- What is the size of the population after $6\frac{1}{2}$ weeks?

  When $t = 6.5$, $P = 100 \times 2^{6.5}$

  $\qquad\qquad\qquad\qquad \approx 9051$ mice. $\{100 \boxed{\times} 2 \boxed{\wedge} 6.5 \boxed{\text{ENTER}}\,\}$

- How long will it take for the population to reach 6400 mice?

  We let $100 \times 2^t = 6400$

  $\qquad \therefore \quad 2^t = 64 \qquad$ {dividing by 100}

  $\qquad \therefore \quad 2^t = 2^6 \qquad$ {writing with base 2}

  $\qquad \therefore \quad t = 6 \qquad$ {equating indices}

  So, it will take 6 weeks.

Populations of people, animals and bacteria are examples of quantities which may show **exponential growth**. Many other quantities also grow exponentially.

Example 2   ◀) **Self Tutor**

The population of rabbits on a farm is given approximately by $R = 50 \times (1.07)^n$ where n is the number of weeks after the rabbit farm was established.

  a What was the original rabbit population?

  b How many rabbits were present after 15 weeks?

  c How many rabbits were present after 30 weeks?

  d Sketch the graph of R against n for $n \geqslant 0$.

  e How long will it take for the population to reach 500?
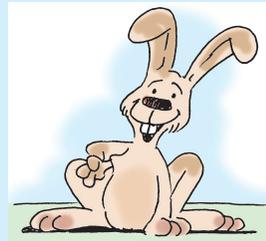
$R = 50 \times (1.07)^n$ where R is the population size, and

$\qquad\qquad\qquad\qquad\qquad\quad n$ is the number of weeks from the start.

  a When $n = 0$, $R = 50 \times (1.07)^0$

  $\qquad\qquad\qquad\quad = 50 \times 1$

  $\qquad\qquad\qquad\quad = 50 \qquad \therefore \quad$ there were 50 rabbits originally.

  b When $n = 15$, $R = 50 \times (1.07)^{15}$

  $\qquad\qquad\qquad\qquad \approx 137.95 \qquad \therefore \quad$ there were 138 rabbits.

  c When $n = 30$, $R = 50 \times (1.07)^{30}$

  $\qquad\qquad\qquad\qquad \approx 380.61 \qquad \therefore \quad$ there were 381 rabbits.

d
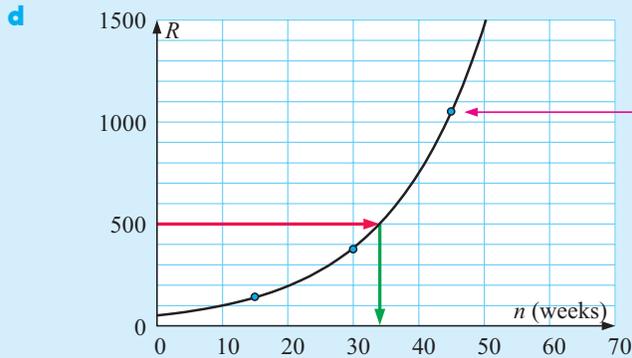
when $n = 45$
$R = 50 \times (1.07)^{45}$
≈ 1050

e From the graph, the approximate number of weeks to reach 500 rabbits is 34.
This solution can be refined finding the **point of intersection** of $Y_1 = 50 \times 1.07^\wedge X$ and $Y_2 = 500$ on a graphics calculator. The solution is ≈ 34.03 weeks.
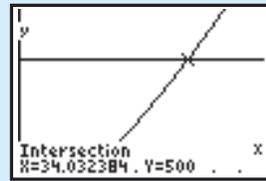
EXERCISE 22C

You are encouraged to use technology to help answer the following questions.

1 The population of a nest of ants n weeks after it is established is given by $P = 500 \times (1.12)^n$.

  a How many ants were originally in the nest?

  b How many ants were in the nest after: **i** 10 weeks   **ii** 20 weeks?

  c Sketch the graph of P against n for $n \geqslant 0$.

  d Use your graph or technology to find how many weeks it takes for the ant population to reach 2000.

2 The weight of bacteria in a culture t hours after it has been established is given by the formula $W = 20 \times (1.007)^t$ grams.

  a Find the original weight of bacteria in the culture.

  b Find the weight of the bacteria after 24 hours.

  c Sketch the graph of W against t for $t \geqslant 0$.

  d Use your graph or technology to find how long it takes for the weight to reach 100 grams.

3 The population of wasps in a nest n days after it was discovered is given by $P = 250 \times (1.06)^n$.

  a How many wasps were in the nest originally?

  b Find the number of wasps after: **i** 25 days   **ii** 8 weeks.

  c Sketch the graph of P against n for $n \geqslant 0$.

  d How long will it take for the population to double?
  Hint: Use your graph or technology.

4 The population of a city was determined by census at 10 year intervals:

Year	1960	1970	1980	1990	2000
Population (thousands)	23.0	27.6	33.1	39.7	47.7

Suppose x is the number of years since 1960 and P is the population (in thousands).

a Draw the graph of P against x with P on the vertical axis.

b It is suspected that the law connecting P and x is exponential. This means it has the form $P = a \times b^x$ where a and b are constants. Find the value of:

   i a, using 1960 population data   **ii** b, using 2000 population data.

c For the values of a and b found in **b**, check if your exponential formula fits the 1970, 1980 and 1990 data.

d Use your formula to predict the city's population in the year: **i** 2010 **ii** 2050.

D   DECAY PROBLEMS

Decay problems occur when the size of a variable decreases over time. Many quantities decay exponentially, for example the mass of radioactive materials, the current in circuits, and the value of depreciating goods.

Example 3   ◄ᴺ Self Tutor

When a diesel-electric generator is switched off, the current dies away according to the formula $I = 24 \times (0.25)^t$ amps, where t is the time in seconds.

a Find I when $t = 0, 1, 2$ and 3.

b What current flowed in the generator at the instant when it was switched off?

c Plot the graph of I against t for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

---

a $I = 24 \times (0.25)^t$ amps

When $t = 0$,	When $t = 1$,	When $t = 2$,	When $t = 3$,
$I = 24 \times (0.25)^0$	$I = 24 \times (0.25)^1$	$I = 24 \times (0.25)^2$	$I = 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b When $t = 0$, $I = 24$ $\therefore$ 24 amps of current flowed.

c
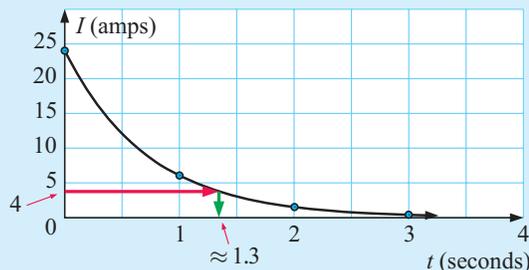
d From the graph above, the approximate time to reach 4 amps is 1.3 seconds.

or

By finding the **point of intersection** of $Y_1 = 24 \times (0.25)^{\wedge}X$ and $Y_2 = 4$ on a graphics calculator, the time ≈ 1.29 seconds.

```
Intersection
X=1.2924813  Y=4
```
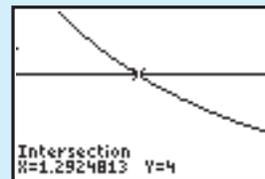
EXERCISE 22D

1 When a liquid in a container is placed in a refrigerator, its temperature in ^oC is given by $T = 100 \times (0.933)^t$, where t is the time in minutes. Find:

  a the initial temperature of the liquid

  b the temperature after:

    i 10 min   **ii** 20 min   **iii** 30 min.

  c Draw the graph of T against t for $t \geqslant 0$, using the information above.

  d Use your graph or a **graphics calculator** to estimate the number of minutes taken for the liquid to reach

    i 40^oC **ii** 10^oC.

2 The weight of a radioactive substance t years after being set aside is given by $W = 150 \times (0.997)^t$ grams.

  a How much radioactive substance was put aside?

  b Determine the weight of the substance after:

    i 100 years   **ii** 200 years   **iii** 400 years.

  c Sketch the graph of W against t for $t \geqslant 0$, using the above information.

  d Use your graph or **graphics calculator** to find how long it takes for the substance to decay to 25 grams.
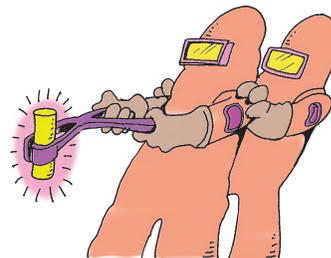
3 The marsupial *Eraticus* is endangered. There is only one colony remaining, and research into its population has shown its decline over the last 25 years:

Year	1980	1985	1990	1995	2000	2005
Number	255	204	163	131	104	84

Let n be the time since 1980 and P be the population size.

  a Graph P against n with P on the vertical axis.

  b It is suspected that the law connecting P and n is of the form $P = a \times b^n$ where a and b are constants. Find:

    i the value of a, using the 1980 data

    ii the value of b, using the 2005 population data.

  c Check to see if the data for 1985, 1990, 1995 and 2000 fit the law.

  d In what year do you expect the population size to reduce to 50?

E   SIMPLE RATIONAL FUNCTIONS

The simplest rational function has an equation of the form $y = \dfrac{k}{x}$ where k is a constant.

It has a graph which is called a **rectangular hyperbola**.

There are examples of hyperbolae in the world around us.
When an aeroplane flies faster than the speed of sound, which is around 1200 km h^{-1} , we
say it breaks the sound barrier. It sets up a shock wave in the shape of a **cone**, and when this
intersects the ground it does so in the shape of a **hyperbola**. The **sonic boom** formed hits
every point on the curve at the same time, so that people in different places along the curve

on the ground all hear it at the same time. No sound is heard
outside the curve but the boom eventually covers every place
inside it.
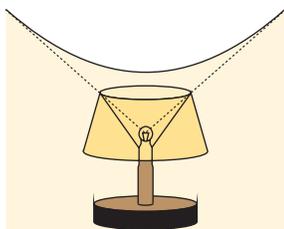
The hyperbolic shape is also noticed in the home when a lamp is
close to a wall. The light and shadow form part of a hyperbola on
the wall.

There are many situations in which two quantities
vary inversely. They form a **reciprocal relationship**
which can be described using a rational function.

For example, the pressure and volume of a gas at
room temperature vary inversely according to the
equation $P = \dfrac{77.4}{V}$.

If P is graphed against V, the curve is one branch of
a hyperbola.
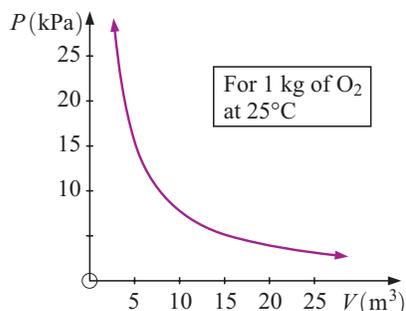
INVESTIGATION 2   THE FAMILY OF CURVES $y = \frac{k}{x}$

In this investigation, you should use a **graphing package** or
graphics calculator to draw curves of the form $y = \dfrac{k}{x}$
where $k \neq 0$.

GRAPHING PACKAGE

What to do:

1   On the same set of axes, draw the graphs of $y = \dfrac{1}{x}$, $y = \dfrac{4}{x}$ and $y = \dfrac{8}{x}$.

2   Describe the effect of the value of k on the graph for $k > 0$.

3   Repeat **1** for $y = \dfrac{-1}{x}$, $y = \dfrac{-4}{x}$ and $y = \dfrac{-8}{x}$.

4   Comment on the change in shape of the graph in **3**.

5   Explain why there is no point on the graph when $x = 0$.

6   Explain why there is no point on the graph when $y = 0$.

EXERCISE 22E.1

1 Consider the function $y = \dfrac{5}{x}$, which can be written as $xy = 5$.

  a Explain why both x and y can take all real values except 0.

  b What are the asymptotes of the function $y = \dfrac{5}{x}$?

  c Find y when: **i** $x = 500$ **ii** $x = -500$

  d Find x when: **i** $y = 500$ **ii** $y = -500$

  e By plotting points or by using technology, graph $y = \dfrac{5}{x}$.

  f Without calculating new values, sketch the graph of $y = \dfrac{-5}{x}$.

2 Sam has to type an assignment which is 400 words in length. How fast he can type will affect how long the job takes him.

Suppose Sam can type at n words per minute and the job takes him t minutes.

  a Complete the table of values:

  b Draw a graph of n versus t with n on the horizontal axis.

Words per minute (n)	10	20	30	....
Time taken (t)				

  c Is it reasonable to draw a smooth curve through the points plotted in **b**? What shape is the curve?

  d State the formula for the relationship between n and t.

3 Determine the equations of the following hyperbolae:

a   **b**   **c**
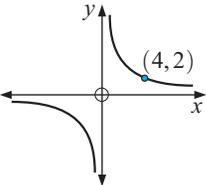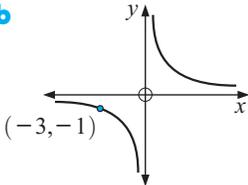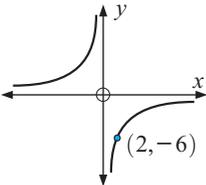
4 Find the axes of symmetry for functions of the form $y = \dfrac{k}{x}$.

APPLICATIONS OF MORE COMPLICATED RATIONAL FUNCTIONS

Example 4   ◀) **Self Tutor**

An experimental breeding colony of cuscus is set up and the size of the colony at time t years, $0 \leqslant t \leqslant 8$, is given by $N = 12 - \dfrac{200}{t - 10}$.

  a What was the original size of the colony?

  b What is the size of the colony after: **i** 2 years   **ii** 8 years?

  c How long would it take for the colony to reach a size of 60?

  d Draw the graph of N against t for $0 \leqslant t \leqslant 8$ using **a**, **b** and **c** only.

$$N = 12 - \frac{200}{t - 10}$$

a When $t = 0$, $N = 12 - \dfrac{200}{-10}$

$$= 12 + 20$$
$$= 32 \text{ cuscus}$$

b **i** When $t = 2$,

$$N = 12 - \left(\frac{200}{-8}\right)$$
$$= 37 \text{ cuscus}$$

ii When $t = 8$,

$$N = 12 - \left(\frac{200}{-2}\right)$$
$$= 112 \text{ cuscus}$$

c When $N = 60$,

$$60 = 12 - \frac{200}{t - 10}$$
$$\therefore \quad 48 = \frac{-200}{t - 10}$$
$$\therefore \quad t - 10 = \frac{-200}{48} \approx -4.17$$
$$\therefore \quad t = 5.83$$

d

So, it would take 5.83 years, which is 5 years and 10 months.
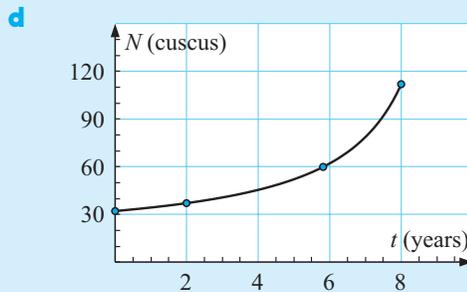
EXERCISE 22E.2

1 In order to remove noxious weeds from his property, a farmer sprays it with weedicide. The chemical is slow acting and the number of weeds per hectare remaining after t days is given by $N = 10 + \dfrac{100}{t + 4}$ weeds per hectare, $t \geqslant 0$.

 a How many weeds per hectare were alive before spraying the weedicide?
 b How many weeds were alive after 6 days?
 c How long did it take for the number of weeds still alive to reach 15 per hectare?
 d Sketch the graph of N against t, using **a**, **b** and **c** and your calculator.
 e Is the chemical spraying program going to eradicate all weeds? Explain your answer.

2 The current remaining in an electrical circuit t milliseconds after it is switched off, is given by $I = \dfrac{960}{t + 4}$ amps, $t \geqslant 0$.

 a What current flowed in the circuit at the instant it was switched off?
 b Find the current after: **i** 1 **ii** 6 **iii** 20 milliseconds.
 c How long would it take for the current to reach 10 amps?
 d Graph I against t using **a**, **b** and **c** only.
 e Use technology to graph the function.
    Use it to check your answers to **a**, **b** and **c**.

GRAPHING PACKAGE

3 A helicopter climbs vertically from its helipad so that at time t minutes, its height above the ground is given by $h = 4000 \left(1 - \dfrac{1}{t+1} \right)$ metres, $t \geqslant 0$.

GRAPHING PACKAGE

   a Check that $h = 0$ when $t = 0$.

   b Find the height reached after: **i** 9 minutes.   **ii** 19 minutes.

   c Use technology to help you sketch the graph of h against t for $t \geqslant 0$, and check your answers to **a** and **b**.

   d Is there a maximum altitude reached by the helicopter? Explain your answer.

4 On a wet day a cyclist was travelling at a constant speed along a bitumen road.

He braked suddenly and his speed afterwards was given by $S = \dfrac{24}{t+1} - 3$ m s^{-1}, $t \geqslant 0$.

   a How fast was he travelling at the instant when the brakes were applied?

   b How long did it take for him to reach a speed of 10 m s^{-1}?

   c How long did it take for him to become stationary?

   d Using technology, graph S against t for the braking interval.

   e The inequality $t \geqslant 0$ should be replaced with a more appropriate one. Which one?

F   OPTIMISATION WITH RATIONAL FUNCTIONS

In many problems we are given a lot of written information. From this we need to:

- identify what we are trying to achieve
- identify the variables in the problem
- formulate a model and use the model to solve the problem.

In this section we will use **technology** to assist us. However, we still need to be able to set up the model ourselves.

The problems we will solve involve finding the **maximum** or **minimum** value of one variable. This means we need to find a maximum or minimum **turning point**. We call this process **optimisation**.

GRAPHING PACKAGE

min turning point

max turning point

Example 5   ◀》 **Self Tutor**
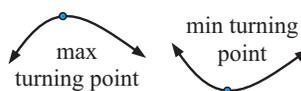
A chicken farmer wishes to build a rectangular enclosure to hold his chickens. Local government regulations indicate that for the number of chickens he wishes to enclose, an area of 60 m^2 is required. The farmer wishes to minimise the cost of fencing material he needs to buy, and uses an existing fence to form one side of the enclosure.

existing fence

60 m^2 x m
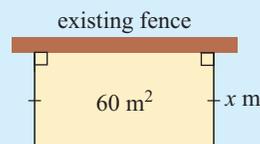
a If the two equal sides are x m long, find the length of the other side in terms of x.

b Find the total length of fencing needed, L m, in terms of x.

c Use technology to graph L against x.

GRAPHING PACKAGE

d Locate the coordinates of the optimum point.

e State the dimensions of the chicken pen of optimum size.

a
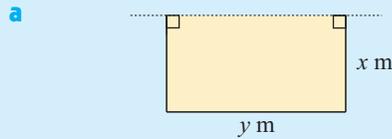
If the third side is y m long then

$$xy = 60 \quad \{\text{area is } 60 \text{ m}^2\}$$

$$\therefore \quad y = \frac{60}{x}$$

b $L = x + x + y \qquad \therefore \quad L = 2x + \dfrac{60}{x}$ metres

c
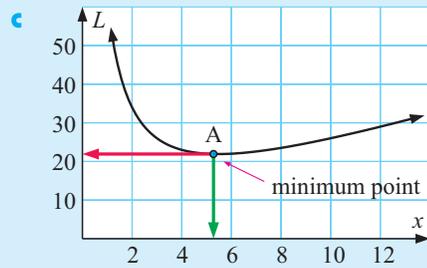
d The optimum point is at (5.477, 21.91)

e When $x = 5.48$, $y \approx \dfrac{60}{5.477} \approx 10.95$

So, the pen is 5.48 m by 10.95 m for a minimum length of 21.91 m.

EXERCISE 22F

1 Jason is allowed 500 m² for his rectangular market garden plot. He wishes to fence the plot, and wants to choose its shape so that the cost of fencing is minimised. He quickly realises that he must find the rectangular shape of least perimeter.

500 m² x m
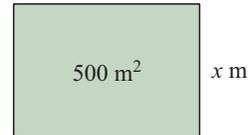
   a If one side is x m long, what is the length of the other side?

   b Find the total length of fencing, L m, in terms of x only.

   c Use technology to graph L against x for $x > 0$.

   d Locate the coordinates of the optimum point.

   e State the dimensions of the garden of optimum size.

2 Two sheds are to be built side by side with a common wall. They have exactly the same floor plan and each shed has a floor area of 600 m². The shed walls cost €85 per linear metre to build.

x m   A

B
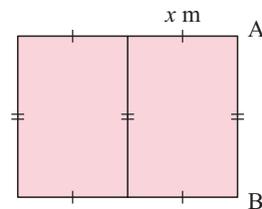
   a Let [AB] have length y m. Find y in terms of x.

   b Find a formula for the total cost of the walls, €C, in terms of x only.

   c Use technology to graph C against x for $x > 0$.

   d Locate the coordinates of the optimum point.

   e State the dimensions of the sheds of optimum size.

3 A box manufacturer has been asked to make gift boxes for a festival. Each is to have a square base and an open top, and the capacity of the box is to be $\frac{1}{4}$ m^3.

open top
h cm
x cm
x cm
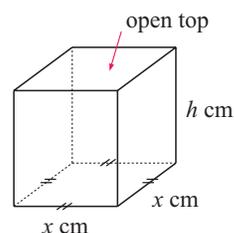
  a Explain why $x^2h = 250\,000$.

  b Suppose the inner surface area of the box is A cm^2. Explain why $A = x^2 + 4xh$.

  c Find a formula for A in terms of x only.

  d Use technology to graph A against x.

  e Locate the coordinates of the optimum point.

  f What shape should the box have so that the least amount of material is required to make it?

4 A cylindrical bin is to have a capacity of $\frac{1}{10}$ m^3 and an open top. Your task is to find the shape of the bin which requires the least amount of material to make. Use this method:

cylinder (no top)
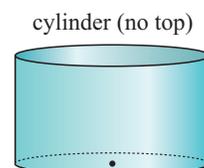
- Draw a fully labelled diagram with variables marked on it.
- Find one variable in terms of the other.
- Find a formula for the variable being optimised in terms of **one variable** only.
- Find the shape of the bin which uses the least material.

G   UNFAMILIAR FUNCTIONS

In this section we will use technology to help investigate functions which are unfamiliar to us. It is advisable to start with a larger viewing window so that we do not miss any key features of the graph.

GRAPHING PACKAGE

Example 6   ◀))) Self Tutor

Consider the function $y = \dfrac{2^x}{x-2} - 3$.

  a Find any turning points.

  b Find the equation of any vertical or horizontal asymptotes.

  c Find any axes intercepts.   **d** Sketch the graph of the function.

  a There is a turning point at $(3.44, 4.54)$ {to 3 s.f.}

  b Asymptotes appear to be:
    $x = 2$ (vertical)
    $y = -3$ (horizontal)

  c There is no x-intercept. The y-intercept is -3.5.

  d

$y = \dfrac{2^x}{x-2} - 3$
$(3.44, 4.54)$
-3.5
$y = -3$
$x = 2$
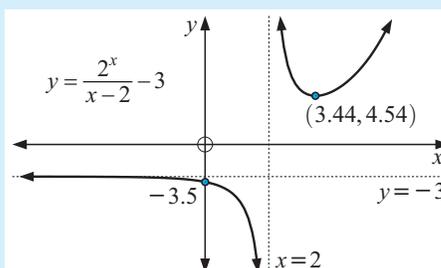
EXERCISE 22G

1 For the following functions:

    i find any turning points   **ii** find the equations of any vertical asymptotes

    iii find any axes intercepts   **iv** sketch the graph.

  a $y = \dfrac{3^x}{x}$ **b** $y = x^2 2^x$ **c** $y = x^2 + \dfrac{1}{x}$

  d $y = \dfrac{x^2}{2^x} - 4$ **e** $y = \dfrac{x - 2}{x^2 - 1}$ **f** $y = \dfrac{x^2 - 1}{x^2 - 4}$

  g $y = \dfrac{x + 1}{(x - 1)(x - 3)}$ **h** $y = \dfrac{2x^2}{(x + 2)(x - 3)}$

2 The effect of a pain killing injection t hours after it is given, is given by the *surge* function $E = 60t \times 2^{-0.212t}$.

  a Sketch a graph of the function.

  b Looking at the graph, describe the effect of the injection.

  c At what time is the effect at a maximum?

  d A surgeon can operate on a patient provided that the effectiveness is at least 100 units. Find the time interval in which the surgeon can operate.

3 From past experience, we know that everyone in a small community eventually hears a particular rumour. A small group inadvertently starts the rumour and the proportion of people who have heard the rumour t hours later is given by the *logistic model*

$$R = \dfrac{0.9877}{1 + 54.53 \times 2^{-1.202t}}.$$

  a Sketch a graph of the model.

  b Looking at the graph, describe the manner in which the rumour spreads.

  c At what time have half the people heard the rumour?

  d What proportion of the community started the rumour?

  e If the population of the community is 281, how many people started the rumour?

CARBON DATING

LINKS
click here

Areas of interaction:
The environment

REVIEW SET 22A

1 If $y = 3^x - 2$, find the value of y when:

  a $x = 0$ **b** $x = 2$ **c** $x = -2$

2 On the same set of axes, draw the graphs of: **a** $y = 2^x$ **b** $y = 2^x - 4$.
In each case state the y-intercept and the equation of the horizontal asymptote.

3 The temperature of a liquid t minutes after it was heated is given by $T = 80 \times (0.913)^t$ °C.   Find:

    a the initial temperature of the liquid

    b the temperature after: **i** $t = 12$ **ii** $t = 24$ **iii** $t = 36$ minutes.

    c Sketch the graph of T against t for $t \geqslant 0$, using the above or technology.

    d Hence, find the time taken for the temperature to reach $25°C$.

4 A population of ants t weeks after a new nest is founded is given by $P = 1500 \times (1.087)^t$.

    a Find the initial population.

    b Find the population after: **i** 3 weeks   **ii** 6 months, to 3 significant figures.

    c Sketch the graph of P against t using technology. On your graph mark details from **a** and **b**.

    d Find the time at which the population reaches 50 000.

5 A metal object is released at the top of an oil tank. After t seconds its speed of descent is given by $S = 3(1 - 2^{-3.71t})$ cm s^{-1}.

    a Find its initial speed of descent.

    b Find its speed after 2 seconds.

    c Find when its speed reaches 2.5 cm s^{-1}.

    d Sketch the graph of S against t for $t \geqslant 0$.

    e Describe the motion of the object as time goes by.

6 Use technology to graph the function $y = \dfrac{x^2 - 1}{x + 2} - 1$

    a Find the axes intercepts.

    b Find the coordinates of the turning points.

    c Find the equation of the vertical asymptote.

    d Sketch the graph showing the features from **a**, **b** and **c**.

REVIEW SET 22B

1 If $y = 2^{x-1}$,   find the value of y when:

    a $x = 0$ **b** $x = -2$ **c** $x = 3$

2 On the same set of axes, draw graphs of: **a** $y = 2^x$ **b** $y = 2^{-x}$
In each case, state the y-intercept and the equation of the horizontal asymptote.

3 A population P of zebra t years after an initial count is given by $P = 1000 \times (1.26)^t$.

    a Find the original number of zebras.

    b Find the number of zebras when: **i** $t = 6$ **ii** $t = 12$ **iii** $t = 18$ years.

    c Sketch the graph of P against t for $t \geqslant 0$, using the above or technology.

    d Hence, determine the time taken for the population to reach 5000 zebras.

4 The weight of a radioactive substance after t years is given by
$W = 1500 \times (0.993)^t$ grams.

  a Find the original amount of radioactive material.

  b Find the amount of radioactive material remaining after:

  i 400 years   **ii** 800 years.

  c Sketch the graph of W against t for $t \geqslant 0$, using the above or technology.

  d Hence, find the time taken for the weight to reduce to 100 grams.

5 A colony of marsupials has size given by $M = 6 - \dfrac{400}{t - 20}$ where t is the time in years, $0 \leqslant t \leqslant 19$.

  a What was the original size of the colony?

  b What was the size of the colony after 10 years?

  c How long did it take for the colony to reach a population of 60?

  d Sketch the graph of M against t.

6 Use technology to graph the function $y = \dfrac{1}{x^2 - x - 2}$.

  a Find any axes intercepts.

  b Find the coordinates of any turning points.

  c State the equation of the horizontal asymptote.

  d Sketch the curve showing the features from **a**, **b** and **c**.

Chapter 23

Vectors

Contents:

OPENING PROBLEM

We have all seen film or video of aeroplanes attempting to land on days when there is a high cross wind perpendicular to the landing strip.

Just before the plane touches down it is inclined at an angle to the strip. This helps to compensate for the wind and keep the aeroplane parallel to the runway.

Suppose an aeroplane is landing from the east. Its landing speed in still conditions is 150 km h^{-1}. If the aeroplane experiences a cross breeze of 30 km h^{-1} from the south, at what angle must it head and what is its actual speed?
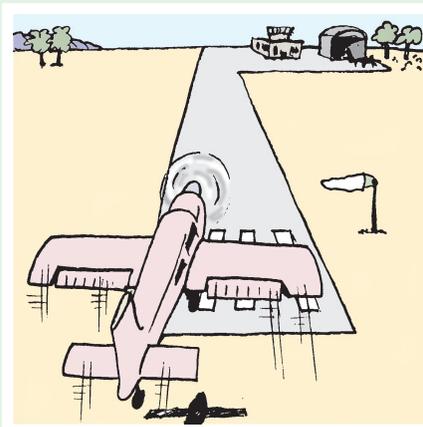
Throughout this course we have dealt with many quantities that have size. For example, we can measure length, area, volume, time, and speed.

While all of these quantities have size, they do not have a direction. We call these quantities **scalars**.

Some quantities such as displacement, acceleration, and momentum have size and also a direction.

> **Vectors** are quantities which have both magnitude or size, and direction.

One simple application of vectors is in transformation geometry.

For example,
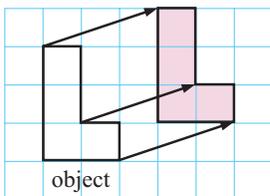
Every point on the object figure moves under a translation 3 units horizontally and then 1 unit vertically.

VIDEO CLIP

By Pythagoras' theorem, each point moves $\sqrt{10}$ units in the particular direction shown in the figure.

We saw in **Chapter 15** that this translation can be specified by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

where   3 represents the horizontal movement   and
         1 represents the vertical movement.

In general, a translation can be specified using a **column vector** $\begin{pmatrix} x \\ y \end{pmatrix}$.   The vector describes both the direction of motion and the distance travelled by each point on the object.

x tells us how far **horizontally** we move and in what direction.

- $x > 0$ means we move to the *right*.
- $x < 0$ means we move to the *left*.

y tells us how far **vertically** we move and in what direction.

- $y > 0$ means we move *upwards*.
- $y < 0$ means we move *downwards*.

x is often called the **horizontal component** and y the **vertical component** of the translation.

A   VECTOR REPRESENTATION

A vector quantity can be represented using a small arrow over a lower case letter.

However, in text books we use a bold lower case letter.

For example, in the diagram alongside, the illustrated vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is denoted **a** or $\overrightarrow{a}$.
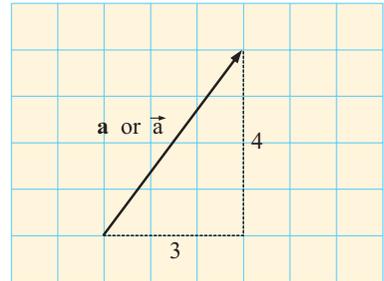
We write $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ or $\overrightarrow{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

An arrowhead is used to show the direction of the vector.

The non-arrow end is often called the **start** of the vector and the arrowhead end is the vector's **end**.

Another way to represent a vector is by referring to its end points.

If we label the end points A and B, then $\overrightarrow{AB}$ is the vector from point A to point B.

Example 1   ◀)) **Self Tutor**

a Represent $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ on squared paper.

b Write $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in component from given A, B and C in the diagram:

a

b To get from A to B we move 3 units right and 1 unit up.

$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

To get from B to C we move 2 units left and 3 units down.

$\therefore \quad \overrightarrow{BC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

EXERCISE 23A

1 On squared paper, draw the vectors:

    a $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
    b $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$
    c $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
    d $\mathbf{d} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

    e $\mathbf{e} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
    f $\mathbf{f} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$
    g $\mathbf{g} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$
    h $\mathbf{h} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

2 Write each of the following vectors in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:
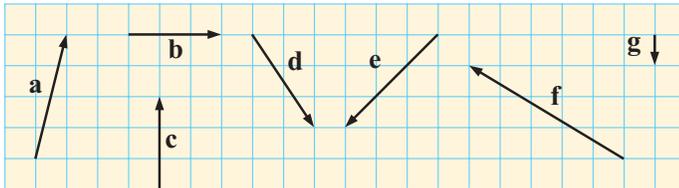
3 For the given figure, write the following as column vectors:

    a $\overrightarrow{AB}$
    b $\overrightarrow{BC}$
    c $\overrightarrow{CD}$
    d $\overrightarrow{DA}$

    e $\overrightarrow{BA}$
    f $\overrightarrow{AC}$
    g $\overrightarrow{BD}$
    h $\overrightarrow{AA}$
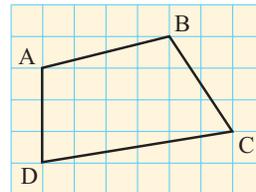
B   LENGTHS OF VECTORS

The length of a vector is the distance between its starting point and end point.

The length is also called the **magnitude** of the vector.

Notation: If **a** is a vector, its length is $|\mathbf{a}|$.

               If $\overrightarrow{a}$ is a vector, its length is $\left|\overrightarrow{a}\right|$.

               If $\overrightarrow{AB}$ is a vector, its length is $\left|\overrightarrow{AB}\right|$.

To find the length of a vector we can use **Pythagoras' theorem**.

Example 2   ◀ѕ) **Self Tutor**

Find the length or magnitude of the vector $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.
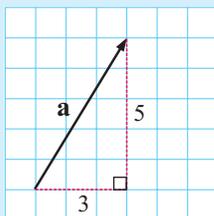
Now $|\mathbf{a}|^2 = 3^2 + 5^2$ {Pythagoras}

            $= 9 + 25$

            $= 34$

$\therefore$ $|\mathbf{a}| = \sqrt{34}$ {as $|\mathbf{a}| > 0$}

$\therefore$ the length is about 5.83 units.

We can use Pythagoras' theorem to prove that in general:

the vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ has length $|\mathbf{a}| = \sqrt{x^2 + y^2}$.

DISTANCE AND DISPLACEMENT

Consider a ship which starts at point A. It sails 3 km north, then 2 km east. It is now at point B.

The **displacement** of the ship is $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The **distance** travelled by the ship is $\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 3^2}$
$= \sqrt{13}$ units.
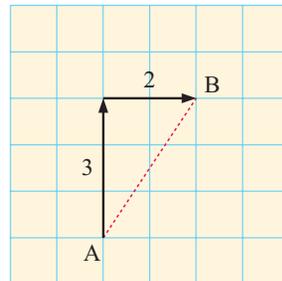
EXERCISE 23B

1 Use Pythagoras' theorem to find the length or magnitude of:

a $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

2 Use Pythagoras' theorem to find the length or magnitude of:

a $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

Comment on your answers.

3 Brigitta walks 4 km east and then turns and walks 6 km south.
a Write Brigitta's displacement in vector form.
b How far, in a straight line, is Brigitta from her starting point?

4 If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $|\mathbf{a}| = \sqrt{x^2 + y^2}$.

Use this formula to find the length of:

a $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ **d** $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$

e $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ **f** $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$ **g** $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ **h** $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$
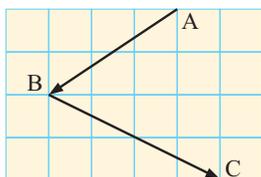
5
a Write $\overrightarrow{AB}$ in component form and hence find $\left|\overrightarrow{AB}\right|$.
b Write $\overrightarrow{BC}$ in component form and hence find $\left|\overrightarrow{BC}\right|$.

6 **a** What geometrical property do vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have?

  b Write in component form:

   i $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$

  c What are the lengths of $\overrightarrow{AB}$ and $\overrightarrow{CD}$?

7
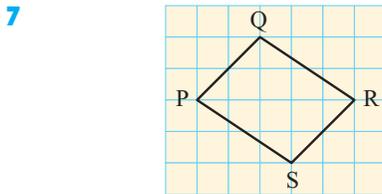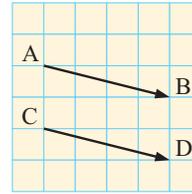
  a Find $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ and find their lengths.

  b Find $\overrightarrow{RQ}$ and $\overrightarrow{SP}$ and find their lengths.

  c Use **a** and **b** to classify quadrilateral PQRS.

8 $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ are called 'opposite vectors'.

  a Illustrate each vector on grid paper.

  b Explain why the vectors are called opposite vectors.

  c True or false?   "Opposite vectors have the same length."

C   EQUAL VECTORS

> Two vectors are **equal** if they have the same x and y-components.
>
> If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$,   then $a = c$ and $b = d$.
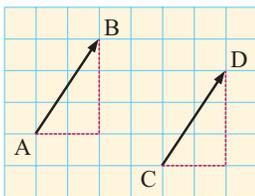
Notice that $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors and we write $\overrightarrow{AB} = \overrightarrow{CD}$.

Notice that equal vectors have the same length and direction. They are parallel.

EXERCISE 23C

1 **a** Explain why $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

  b What can be deduced if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$?

  c If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 \\ -b \end{pmatrix}$,   find a and b.

2
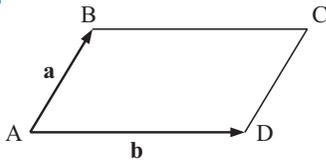
ABCD is a parallelogram with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

State, with reasons, vector expressions for $\overrightarrow{DC}$ and $\overrightarrow{BC}$.

3 ABD, BEC and BCD are equilateral triangles.

Suppose $\overrightarrow{AD} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{BD} = \mathbf{c}$.

a Find, in terms of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, vectors representing:

   i $\overrightarrow{BC}$ **ii** $\overrightarrow{BE}$ **iii** $\overrightarrow{DC}$ **iv** $\overrightarrow{EC}$

b Explain why $\mathbf{a} \neq \mathbf{b}$.

c Does $|\mathbf{a}| = |\mathbf{b}|$?
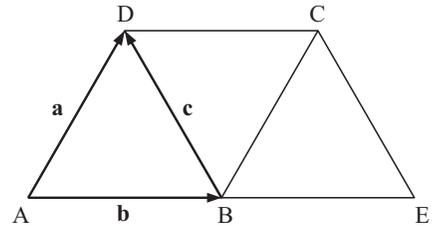
D   VECTOR ADDITION

Suppose a walker starts at point S and walks to point A with displacement vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The walker then walks from A to a final point F, this time with displacement vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

What is the walker's final displacement from S?

To get from S to F directly, the displacement vector is clearly $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$.
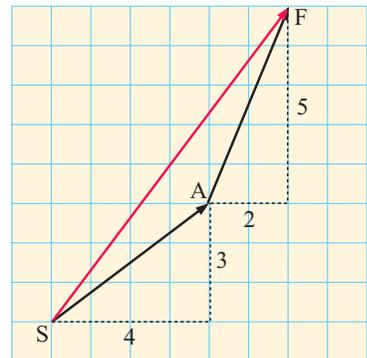
Notice also that $\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4+2 \\ 3+5 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

This example shows us how to **add vectors**.

If two vectors in **component form** are $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ then $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$.

If two vectors are given in **arrow** or **geometric form**, the method to add $\mathbf{a}$ and $\mathbf{b}$ is:

Step 1: Draw vector $\mathbf{a}$ accurately.

Step 2: At the arrow end of $\mathbf{a}$, draw vector $\mathbf{b}$.

Step 3: Draw a vector from the start of $\mathbf{a}$ to the end of $\mathbf{b}$.
The **resultant vector** is $\mathbf{a} + \mathbf{b}$.

DEMO
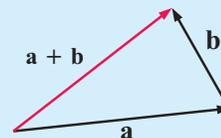
Example 3   ◄) **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{c}$ **c** $\mathbf{a} + \mathbf{b} + \mathbf{c}$

a $\mathbf{a} + \mathbf{b}$

$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 2+4 \\ 5+-1 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

b $\mathbf{b} + \mathbf{c}$

$= \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 4+-2 \\ -1+-3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

c $\mathbf{a} + \mathbf{b} + \mathbf{c}$

$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 2+4+-2 \\ 5+-1+-3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

The points K, L and M form the vertices of a triangle. Suppose we wish to go from K to M.
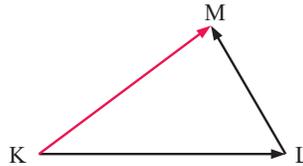
$\overrightarrow{KL} + \overrightarrow{LM} = \overrightarrow{KM}$

same letter

We could go from K to M directly along vector $\overrightarrow{KM}$.

Alternatively, we could go from K to L and then from L to M. In this case we add the two vectors $\overrightarrow{KL}$ and $\overrightarrow{LM}$, giving $\overrightarrow{KL} + \overrightarrow{LM}$.

The result is the same either way, so $\overrightarrow{KL} + \overrightarrow{LM} = \overrightarrow{KM}$.

We can use the same method to add any number of vectors.

Example 4   ◄) **Self Tutor**

Write a vector equation to connect the vectors in:
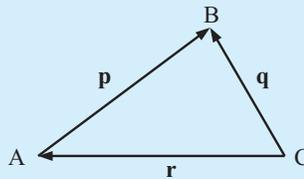
We can move from C to B directly along vector $\mathbf{q}$.

Alternatively, we can go from C to A and then from A to B. This is $\mathbf{r} + \mathbf{p}$.

So, $\mathbf{q} = \mathbf{r} + \mathbf{p}$.

Example 5   ◄) **Self Tutor**

Simplify: **a** $\overrightarrow{LM} + \overrightarrow{MN}$ **b** $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$

a $\overrightarrow{LM} + \overrightarrow{MN} = \overrightarrow{LN}$ **b** $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC} = \overrightarrow{AC}$

EXERCISE 23D

1 Find:

a $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

d $\begin{pmatrix} -6 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ **e** $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

f $\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

2 If $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{a}$

e $\mathbf{a} + \mathbf{a}$ **f** $\mathbf{b} + \mathbf{b}$ **g** $\mathbf{c} + \mathbf{c} + \mathbf{c}$ **h** $\mathbf{a} + \mathbf{b} + \mathbf{c}$

3 Write a vector equation to connect the vectors in:
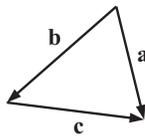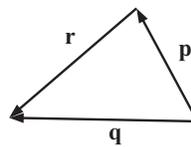
a   **b**   **c**
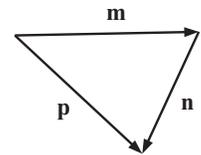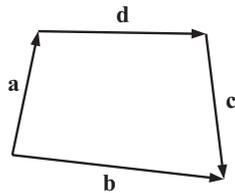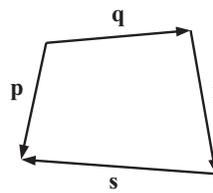
d   **e**   **f**
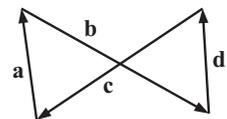
4 Find, in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{PS}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{QT}$ **d** $\overrightarrow{PT}$
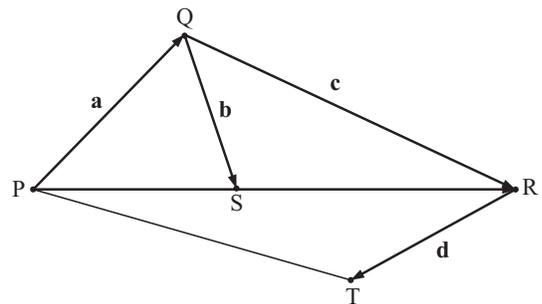
5 Simplify:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{d} + \mathbf{e}$

c $\mathbf{c} + \mathbf{f}$ **d** $\mathbf{a} + \mathbf{b} + \mathbf{f}$

e $\mathbf{c} + \mathbf{d} + \mathbf{e}$
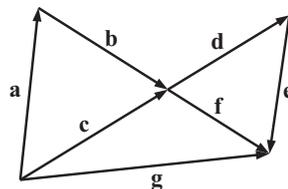
6 Simplify:

a $\overrightarrow{AB} + \overrightarrow{BE}$

b $\overrightarrow{BC} + \overrightarrow{CE}$

c $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$
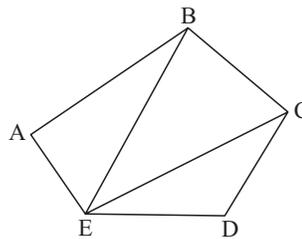
7 Simplify:

a $\overrightarrow{AP} + \overrightarrow{PB}$

b $\overrightarrow{PX} + \overrightarrow{XY} + \overrightarrow{YQ}$

c $\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{ND}$

d $\overrightarrow{SP} + \overrightarrow{PQ} + \overrightarrow{QN}$

e $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

f $\overrightarrow{CX} + \overrightarrow{XN} + \overrightarrow{ND} + \overrightarrow{DP}$

8 Alongside is a hole at Hackers Golf Club.

a Jack tees off from T and his ball finishes at A. Write a vector to describe the displacement of the ball from T to A.

b He plays his second stroke from A to B. Write a vector to describe the displacement with this shot.

c By great luck, Jack's next shot finishes in the hole H. Write a vector which describes this shot.

d Use vector lengths to find the distance from:

i T to H

ii T to A

iii A to B

iv B to H

e Find the sum of all three vectors for the ball travelling from T to A to B to H. What information does the sum give about the golf hole?
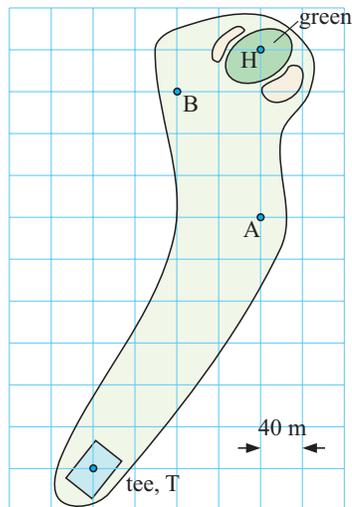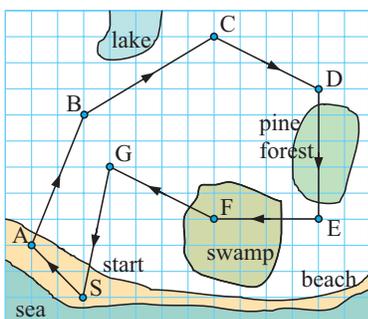
9 The diagram alongside shows an orienteering course run by Kahu.

a Write a column vector to describe each leg of the course.

b Find the sum of all of the vectors.

c What does the sum in b tell us?

ACTIVITY   **A VECTOR CIRCUIT**

Design an orienteering course like that in question **9** above. It must contain at least *six* legs and may cross over itself. It must finish where it started.

1 Write each leg in vector form.

2 Explain why sum of the vectors will always be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

E   MULTIPLYING VECTORS BY A NUMBER

If $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then $\mathbf{p} + \mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

If we define $\mathbf{p} + \mathbf{p}$ to be $2\mathbf{p}$,   we notice that

$$2\mathbf{p} = \begin{pmatrix} 2 \times 3 \\ 2 \times 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \quad \text{also.}$$
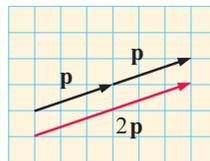
Examples like this one suggest that: if $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $k\mathbf{p} = \begin{pmatrix} ka \\ kb \end{pmatrix}$.

Multiplying a vector by a number in this way is called **scalar multiplication**. This is because the constant k is a scalar.

For example, if $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$,   then $4\mathbf{a} = \begin{pmatrix} 4 \times 3 \\ 4 \times 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$

and $-\mathbf{a} = \begin{pmatrix} -1 \times 3 \\ -1 \times 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ is called the **negative vector** of $\mathbf{a}$.

It can be obtained from $\mathbf{a}$ by reversing the signs of its components.

THE ZERO VECTOR

The **zero vector** is the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It can be obtained by performing scalar multiplication of any vector with the scalar zero or 0.

For example, $0 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \times 2 \\ 0 \times 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The **zero vector**, $\mathbf{0}$ has length 0.

It is the only vector with no direction.

Example 6   ◀) Self Tutor

If $\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ find: **a** $3\mathbf{m}$ **b** $-2\mathbf{m}$.   Illustrate your answers.

a $3\mathbf{m} = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \times 1 \\ 3 \times -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$
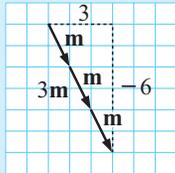
b $-2\mathbf{m} = \begin{pmatrix} -2 \times 1 \\ -2 \times -2 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 4 \end{pmatrix}$
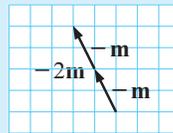
EXERCISE 23E

1 **a** If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ find $\mathbf{a} + \mathbf{a} + \mathbf{a}$.

   b Show using the rule above that $3\mathbf{a}$ gives the same result.

2 For $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ find:

   a $2\mathbf{a}$ **b** $4\mathbf{a}$ **c** $-\mathbf{a}$ **d** $3\mathbf{b}$ **e** $-2\mathbf{b}$

   f $\frac{1}{2}\mathbf{c}$ **g** $\frac{1}{3}\mathbf{d}$ **h** $-\frac{3}{2}\mathbf{c}$ **i** $-6\mathbf{d}$ **j** $-\frac{1}{2}\mathbf{d}$

3 For $\mathbf{m} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ draw, on grid paper, the vectors:

   a $\mathbf{m}$ **b** $2\mathbf{m}$ **c** $-2\mathbf{m}$ **d** $-3\mathbf{m}$ **e** $\frac{1}{2}\mathbf{m}$ **f** $-\frac{1}{3}\mathbf{m}$

4 If $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ find:

   a $3\mathbf{a}$ **b** $-\mathbf{a}$ **c** $\mathbf{a} + \mathbf{0}$ **d** $\mathbf{0} + \mathbf{a}$ **e** $-\mathbf{a} + \mathbf{a}$ **f** $\mathbf{a} + (-\mathbf{a})$

5 **a** On grid paper show these pairs of vectors:

   i $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **ii** $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ **iii** $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

   b Copy and complete:
   "Geometrically, $\mathbf{a}$ and $-\mathbf{a}$ are ...... and ...... in length, but ...... in direction."

   c Copy and complete: "If $\overrightarrow{AB}$ is   B   then $\overrightarrow{BA}$ is ......"
                                      A

DISCUSSION   MULTIPLYING VECTORS BY NUMBERS

Jason said that "multiplying a vector by a number does not change its direction, it just makes the vector longer or shorter."

Discuss the inaccuracy of Jason's statement by considering multiplication by 2, $\frac{1}{2}$, -2 and $-\frac{1}{2}$.   Correct Jason's statement.

F VECTOR SUBTRACTION

To **subtract** a vector we add the opposite or negative vector.

So, $\mathbf{a} - \mathbf{b} = \mathbf{a} + -\mathbf{b}$ and we use vector addition to find it.

Example 7   ◀ᴺ) **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ find: **a** $-\mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$

Illustrate how to find $\mathbf{a} - \mathbf{b}$ geometrically.

a $-\mathbf{b} = \begin{pmatrix} -1 \times 2 \\ -1 \times -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ **b** $\mathbf{a} - \mathbf{b} = \mathbf{a} + -\mathbf{b}$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Draw **a** and **b** first.
Draw **a** again and at its arrow end
draw $-\mathbf{b}$.
The resultant vector is $\mathbf{a} - \mathbf{b}$.
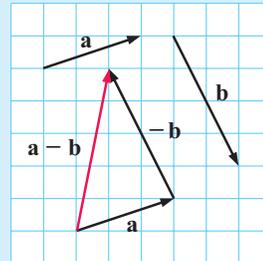
Example 8   ◀ᴺ) **Self Tutor**

Simplify: $\overrightarrow{PQ} - \overrightarrow{RQ}$.
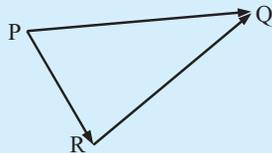
$\overrightarrow{PQ} - \overrightarrow{RQ}$
$= \overrightarrow{PQ} + -\overrightarrow{RQ}$
$= \overrightarrow{PQ} + \overrightarrow{QR}$ {adding the opposite vector}
$= \overrightarrow{PR}$

EXERCISE 23F

1 Write the opposite or negative vector of:

a $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$

2 Show by diagram how to find the resultant vector of:

   a $\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -6 \end{pmatrix}$

3 Without drawing a diagram find:

   a $\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

   d $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **f** $\begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

4 Simplify:

   a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{QP} - \overrightarrow{RP}$

   c $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$ **d** $\overrightarrow{PQ} - \overrightarrow{RQ} + \overrightarrow{RS} - \overrightarrow{TS} + \overrightarrow{TV}$

5 For the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ find:

   a $\mathbf{a} - \mathbf{b}$ **b** $\mathbf{b} - \mathbf{c}$ **c** $2\mathbf{a} - \mathbf{c}$ **d** $\mathbf{a} + \mathbf{b} - \mathbf{c}$

   e $\mathbf{b} + 2\mathbf{c} - \mathbf{a}$ **f** $\mathbf{a} - \frac{1}{2}\mathbf{b}$ **g** $\frac{1}{2}\mathbf{b} + \mathbf{c}$ **h** $2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$

6 ABCD is a parallelogram in which $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

   and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

   Find column vectors for:

   a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{BD}$
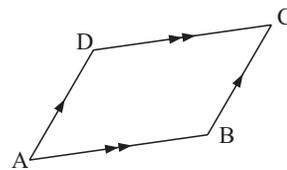
7 M is the midpoint of [PS].

   If $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$ and $\overrightarrow{RS} = \mathbf{c}$, find in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

   a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$ **d** $\overrightarrow{PM}$
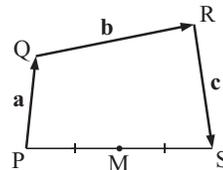
8 For   [diagram: vector **a**]   and   [diagram: vector **b**]   draw vector diagrams for:

   a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{a} - 2\mathbf{b}$

9 ABDE and ABCD are parallelograms. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, vector expressions for:

   a $\overrightarrow{ED}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{DB}$

   d $\overrightarrow{AD}$ **e** $\overrightarrow{BC}$ **f** $\overrightarrow{EC}$
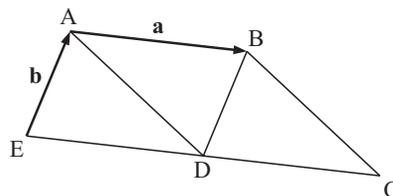
10 Use the diagram to simplify:

    a $\mathbf{a} + \mathbf{c}$ **b** $\mathbf{h} + \mathbf{f}$ **c** $\mathbf{j} - \mathbf{i}$

    d $\mathbf{d} - \mathbf{c}$ **e** $\mathbf{e} - \mathbf{b}$ **f** $-\mathbf{f} - \mathbf{h}$
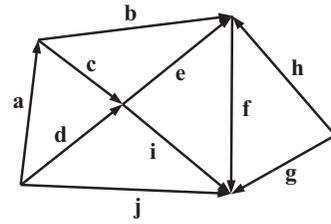
G   THE DIRECTION OF A VECTOR

In displacement problems the direction of a vector often needs to be found.

One way to measure the direction of travel is to use **true bearings**.

We saw in **Chapter 12** that a true bearing is a measurement of the **clockwise** angle from the **true north direction**.

Remember that the bearing of A from B and the bearing of B from A always differ by 180°.
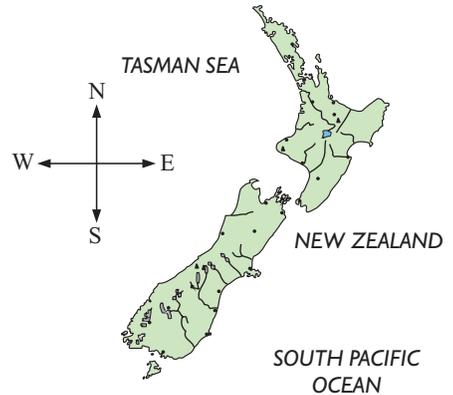
Example 9   ◀) **Self Tutor**

Find the angle that the given vector makes with true north:

  a $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

a

$\tan \theta = \frac{1}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{3}\right)$
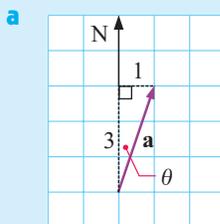
$\approx 19.4^o$ (to 3 s.f.)

$\therefore$ **a** has the bearing 019.4^o.

b

$\tan \theta = \frac{4}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\approx 53.1^o$

(to 3 s.f.)

and $\theta + 90^o \approx 143.1^o$
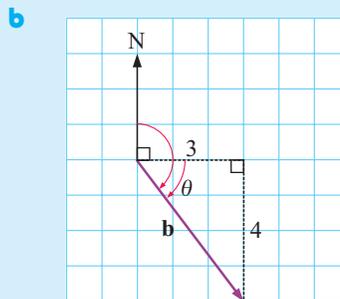
$\therefore$ **b** has the bearing 143.1^o.

SPEED AND VELOCITY

We have seen previously how **speed** describes how fast something is moving.

Velocity is a vector which describes speed in a given direction.

Example 10

◆》 **Self Tutor**

Henri is cycling at a constant speed of 40 km h^{-1} in the direction 222°.

a Draw an accurate scale diagram showing this information.

b Find Henri's velocity vector.

a
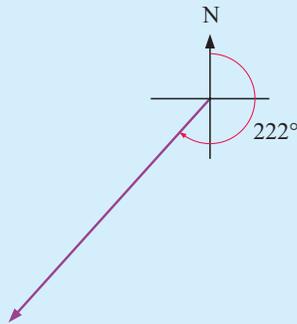
Scale: 1 mm $\equiv$ 1 km h^{-1}

b

$$\sin 42^o = \frac{a}{40} \quad \therefore \quad a = 40\sin 42^o$$

$$\cos 42^o = \frac{b}{40} \quad \therefore \quad b = 40\cos 42^o$$
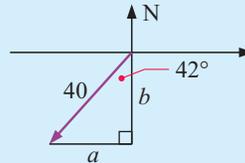
So, the velocity vector is

$$\begin{pmatrix} -40\sin 42^o \\ -40\cos 42^o \end{pmatrix} \approx \begin{pmatrix} -26.8 \\ -29.7 \end{pmatrix}$$

EXERCISE 23G

1 Using a diagram only, find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

2 Using a diagram only, find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

3 Use right angled triangle trigonometry to find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

4 Find the length and direction of:

a $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

5 Jasmin walks at a constant speed of 5 km h^{-1} in the direction 057°.

a Draw an accurate scale diagram showing this information.

b Find Jasmin's velocity vector.

6 Chai jogs at a constant speed of 12 km h^{-1} in the direction 146°.
   Find Chai's velocity vector.

7 Jiji walks for 8 km in the direction 303°.

a Draw an accurate scale diagram showing this information.

b Find Jiji's displacement vector.

H | PROBLEM SOLVING BY VECTOR ADDITION

In this section we use vector addition to help solve problems where there are different vector components.

Example 11   ◀)) **Self Tutor**

In windless conditions, Sergio runs at 10 m s^{-1}. One day he faces the east and tries to run at his usual speed. What will be his actual velocity if he encounters a wind of 1 m s^{-1} from:

   a the west   **b** the east   **c** the north?

Sergio's actual velocity will be the vector addition of his velocity in windless conditions, plus the wind.

Sergio's velocity in windless conditions is $\mathbf{s} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$.

If the wind has velocity vector $\mathbf{w}$, then Sergio's actual velocity is $\mathbf{v} = \mathbf{s} + \mathbf{w}$.

a $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$
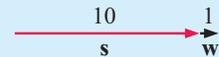
Sergio's velocity is 11 m s^{-1} to the east.

b $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$
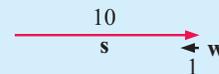
Sergio's velocity is 9 m s^{-1} to the east.

c $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$

$$|\mathbf{v}| = \sqrt{10^2 + (-1)^2} \qquad \tan\theta = \tfrac{1}{10}$$
$$= \sqrt{101} \qquad\qquad \therefore \quad \theta = \tan^{-1}\left(\tfrac{1}{10}\right)$$
$$\approx 10.05 \qquad\qquad\qquad \approx 5.7^o$$
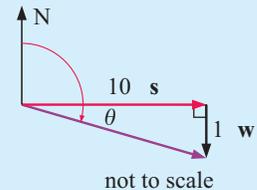
So, Sergio's velocity is 10.05 m s^{-1} on the bearing 095.7^o.

EXERCISE 23H

1 Quang runs for 10 km in a northerly direction and then for 5 km in a westerly direction.

   a Write each part of this run in component vector form.

   b Find Quang's displacement vector from his starting point.

   c Find Quang's distance from his starting point.

   d Find Quang's bearing from his starting point.

2 Aleksandra walks with displacement vector $\begin{pmatrix} 14 \\ 2 \end{pmatrix}$. She then changes direction and

walks with displacement vector $\begin{pmatrix} 3 \\ -11 \end{pmatrix}$. Units are in kilometres.

    a Illustrate Aleksandra's movement on grid paper.

    b Find Aleksandra's displacement vector from her starting point?

    c How far is Aleksandra from her starting point?

    d What is Aleksandra's bearing from her starting point?

3 The diagram shows a river running from north to south at 1 m s^{-1}. A swimmer swims directly out from the bank at 2 m s^{-1}.
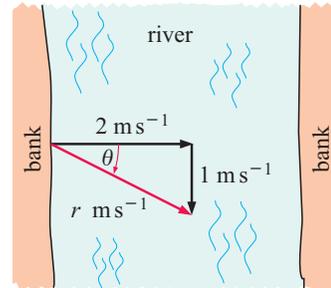
    a What is the actual speed of the swimmer?

    b Find θ.

    c On what bearing is the swimmer actually heading?

4 The Interislander ferry is steaming due east across Cook Straight at a speed of 20 km h^{-1}. Johanna is a passenger on the ferry. She walks from the bow of the ferry towards the stern at a speed of 5 km h^{-1}.

Find Johanna's resultant speed as she walks to the stern of the ferry.

Is it possible for Johanna to be moving faster than the ferry? Explain your answer.

5 A yacht is sailing at 14 km h^{-1} on a bearing of 045°. It is suddenly hit by the wake of a large ship, which pushes it at 4 km h^{-1} on the bearing 135°.

    a Draw a scale diagram of the yacht's original velocity vector.

    b Show that the yacht's original velocity vector was $\begin{pmatrix} 7\sqrt{2} \\ 7\sqrt{2} \end{pmatrix}$.

    c Draw a scale diagram of the wake's velocity vector.

    d Show that the wake's velocity vector is $\begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$.

    e Assuming that the yacht's captain makes no allowance for the wake, find the new speed and direction of the yacht.

6 An aeroplane takes off from Changi airport in Singapore. Its flight is affected by a 50 km h^{-1} wind on the bearing 305°. The aeroplane's *actual* velocity is 350 km h^{-1} on the bearing 035°.

    a Draw a vector diagram to represent the situation.

    b Calculate the velocity vector that the aeroplane would have if it were not affected by the wind.

REVIEW SET 23A

1 On grid paper draw the vectors:

    a $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:
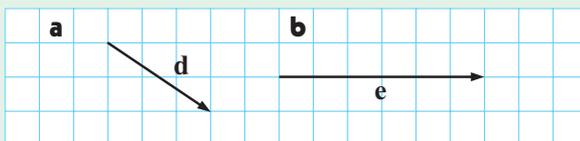
3 Suppose $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

    a Draw a diagram which shows how to find $\mathbf{a} + \mathbf{b}$.

    b Find: **i** $\mathbf{c} + \mathbf{b}$ **ii** $\mathbf{a} - \mathbf{b}$ **iii** $\mathbf{a} + \mathbf{b} - \mathbf{c}$

4 Consider $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

    a Sketch $3\mathbf{p}$.

    b Calculate: **i** $2\mathbf{q}$ **ii** $3\mathbf{p} + 2\mathbf{q}$ **iii** $\mathbf{p} - 2\mathbf{q}$

    c Draw a diagram which shows how to find $\mathbf{q} + 2\mathbf{p}$.

5 Consider the vector $\mathbf{m} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

    a Illustrate the vector on grid paper.

    b Find the vector's length.

    c Find the bearing of the vector.

6
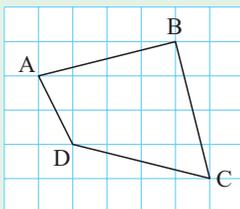
    a Find in component form:

        i $\overrightarrow{BC}$ **ii** $\overrightarrow{BD}$

    b Simplify $\overrightarrow{AD} + \overrightarrow{DC}$.

    c Find $\left| \overrightarrow{AC} \right|$.

7 Draw a scale diagram of a displacement vector of magnitude 4 km and bearing 235°.

8 A plane is flown on a bearing of 045° for 200 km. Its course is then changed to a bearing of 135° and it is flown for 500 km.

    a Draw a vector diagram of the plane's flight.

    b Calculate how far the plane is now from its starting point.

    c On what bearing would the plane need to fly to return directly to its starting point?

REVIEW SET 23B

1 On grid paper draw the vectors:

   a $p = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
   b $q = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$
   c $r = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:
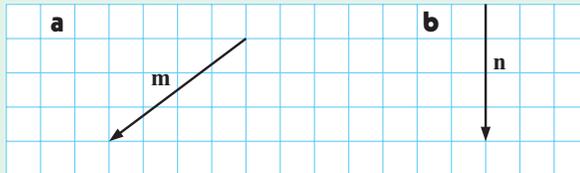
3 Suppose $d = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $e = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

   a Draw a vector diagram to illustrate $d - e$.
   b Find $d - e$ in component form.
   c Find: **i** $2e + 3d$ **ii** $4d - 3e$

4 What results when opposite vectors are added?

5
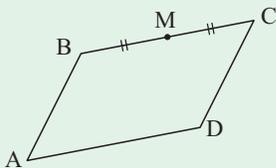
If $\overrightarrow{AB} = p$ and $\overrightarrow{BC} = q$ and ABCD is a parallelogram, find vector expressions for:

   a $\overrightarrow{CD}$ **b** $\overrightarrow{BM}$ **c** $\overrightarrow{MD}$ **d** $\overrightarrow{AD}$

6 Draw a scale diagram of a velocity vector of 5 km h^{-1} with a bearing of 315°.

7 P and Q are the midpoints of sides [AB] and [BC].
   Let $\overrightarrow{AP} = p$ and $\overrightarrow{BQ} = q$.

   a Find vector expressions for:

       i $\overrightarrow{PB}$ **ii** $\overrightarrow{QC}$ **iii** $\overrightarrow{PQ}$ **iv** $\overrightarrow{AC}$

   b How are $\overrightarrow{PQ}$ and $\overrightarrow{AC}$ related?
   c Copy and complete:
      "the line joining the midpoints of two sides of a triangle is .... to the third side and .... its length".
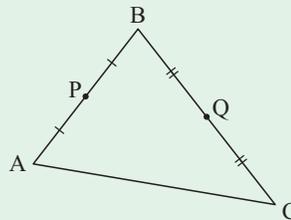
8 A man starts rowing his boat due east towards an island 850 m away. Since he is facing the wrong way, he does not realise a northerly current is pushing him off course. The man can row at 1.5 m s^{-1} in still water, and the current pushes him at 0.3 m s^{-1}.

   a Draw a vector diagram illustrating the man's actual velocity.
   b Find the man's actual speed and bearing.
   c If there was no current, how long would it take for the man to reach the island?
   d By how far will the man miss the point on the island he was trying to land at?

Chapter 24

Deductive geometry

Contents:

The geometry of triangles, quadrilaterals and circles has been used for at least 3000 years in art, design and architecture. Simple geometrical figures often have very interesting and useful properties.

In **deductive geometry** we use logical reasoning to prove that certain observations about geometrical figures are indeed true. In the process we use special results called **theorems**.

GEOMETRY PACKAGE

Many amazing discoveries have been made by people who were simply drawing figures with rulers and compasses.

For example, **Pappus** drew two line segments. He placed three points A, B and C on one of them, and three points D, E and F on the other. He then joined A to E and F, B to D and F and C to D and E. He made an interesting observation about some of the points of intersection. Try this for yourself. Write down a **conjecture** which summarises your observation.
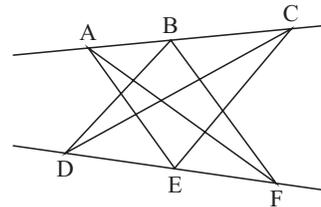
HISTORICAL NOTE

Around 300 BC, **Euclid of Alexandria** developed his theory of **Euclidean geometry**. This formed the basis of the **deductive geometry** we know today.

Euclid developed geometry through a number of **theorems** based on self evident truths called **axioms**.

Euclid's thirteen books, collectively called the *Elements*, formed the basis of all geometry until the 19th century. At this time, mathematicians such as **Gauss**, **Lobachevski** and **Bolyai** challenged one of the Euclidean axioms.
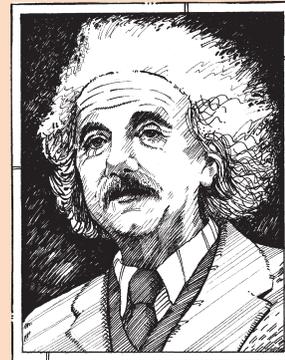
Albert Einstein

This challenge led to a major breakthrough into other branches of mathematics, and enabled **Albert Einstein** to produce his *Theory of Relativity* early last century.

OPENING PROBLEM

Kelly draws many different figures of the 'star' shape alongside. These figures vary in shape and size.

GEOMETRY PACKAGE

Each time, Kelly measures the five angles of the star with a protractor. She adds them together and always gets an answer of around 180°.

Kelly therefore *conjectures* that they actually do add to 180°.

Can you prove that Kelly's conjecture is true?
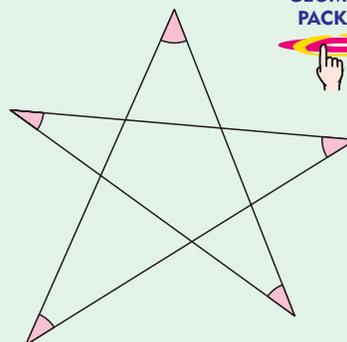
A REVIEW OF FACTS AND THEOREMS

In previous courses we established a number of **theorems** which we can use to solve geometrical problems.

The following table summarises these theorems:

Name	Statement	Figure
Angles on a line	The sum of the sizes of the angles on a line is 180°.	a° b° $a + b = 180$
Angles at a point	The sum of the sizes of the angles at a point is 360°.	b° a° c° $a + b + c = 360$
Vertically opposite angles	Vertically opposite angles are equal.	a° b° $a = b$
Corresponding angles	When two **parallel** lines are cut by a third line, then angles in corresponding positions are equal.	a° b° $a = b$
Alternate angles	When two **parallel** lines are cut by a third line, then angles in alternate positions are equal.	a° b° $a = b$
Co-interior or Allied angles	When two **parallel** lines are cut by a third line, then angles in co-interior positions are supplementary, or add to 180°.	a° b° $a + b = 180$
Angles of a triangle	The sum of the sizes of the interior angles of a triangle is 180°.	b° a° c° $a + b + c = 180$
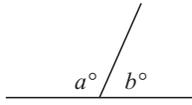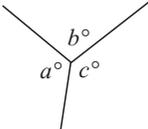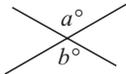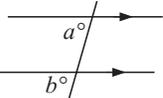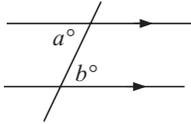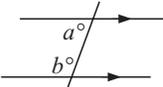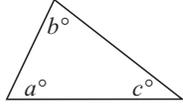
Name	Statement	Figure
Exterior angle of a triangle	The size of the exterior angle of a triangle is equal to the sum of the interior opposite angles.	$c = a + b$
Isosceles triangle	In an isosceles triangle: - base angles are equal - the line joining the apex to the midpoint of the base is perpendicular to the base and bisects the angle at the apex.	
Equal angles of a triangle	If a triangle has two equal angles then the triangle is isosceles.	
Parallelogram	In a parallelogram: - opposite sides are equal - opposite angles are equal.	
Diagonals of a parallelogram	The diagonals of a parallelogram bisect each other.	
Diagonals of a rhombus	The diagonals of a rhombus: - bisect each other at right angles - bisect the angles of the rhombus.	
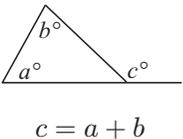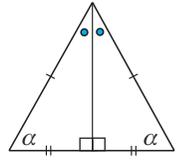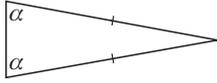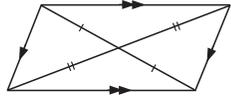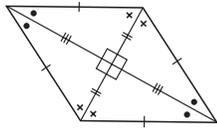
In geometrical figures where one or more angles or lengths are unknown, we can use the **theorems** to find the value of the unknowns. We can also use the theorems to solve other geometrical problems.

Theorems are special results which are true for a general case.

When using a theorem we should state the theorem's name as a reason to justify a particular step in the argument.

The figures we draw when solving problems by deductive geometry do not have to be to scale. However, they should be *reasonably* accurate sketches and should have all known information marked clearly on them.

Example 1   ◀》 **Self Tutor**

Find x:
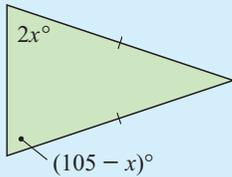
We have an isosceles triangle.

$\therefore$ $2x = 105 - x$ {isosceles triangle theorem}

$\therefore$ $3x = 105$

$\therefore$ $x = 35$

Sometimes we need to draw an additional line or lines so that we can solve a problem. These lines are often called **constructions**.

Example 2   ◀》 **Self Tutor**

Find a, stating appropriate reasons:
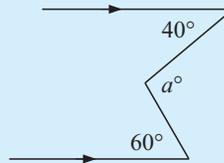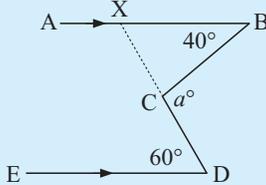
We label the figure and draw a construction line [CX].

$\quad$ $\widehat{BXD} = \widehat{XDE}$ {alternate angles}

$\therefore$ $\widehat{BXD} = 60^o$

$\therefore$ $a = 60 + 40$ {exterior angle of $\triangle$}

$\therefore$ $a = 100$

Example 3   ◀》 **Self Tutor**

Prove that the angle bisectors of angles ABD and CBD form a right angle.
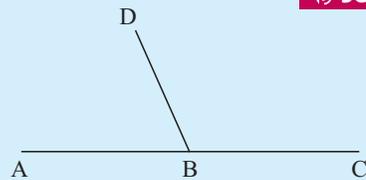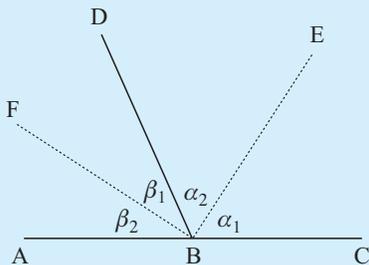
We draw angle bisectors [BE] and [BF].

$\therefore$ $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$

But $2\alpha + 2\beta = 180^o$ {sum of angles on a line}

$\therefore$ $\alpha + \beta = 90^o$

$\therefore$ $\widehat{EBF} = 90^o$

EXERCISE 24A

1 Find, giving reasons, the values of the unknowns in the given figures:

a

$40°$ $x°$

b

$110°$ $x°$

c

$x°$ $(x-20)°$ $(x-10)°$

d

$2x°$ $(x+35)°$

e

$x°$ $3x°$ $x°$

f

$(x+5)°$ $y°$ $(x-5)°$ $(x-20)°$

g

$a°$ $c°$ $b°$ $35°$

h

$(2x+30)°$ $(x+50)°$

i

$x°$ $(x+20)°$ $x°$ $x°$
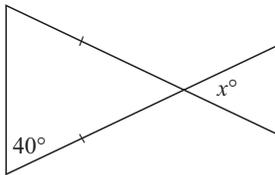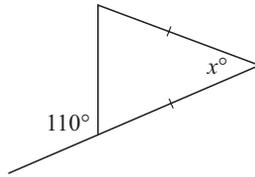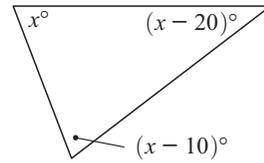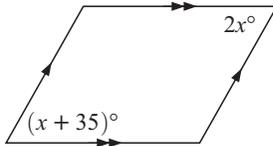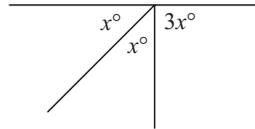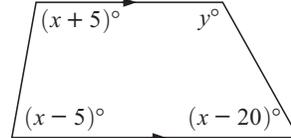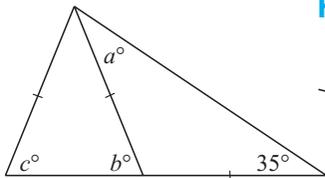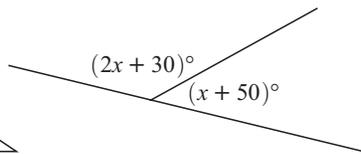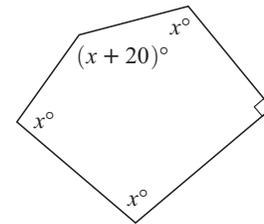
2 Find a in each of the following, stating reasons. You will need to construct an additional line in each figure.

a

$35°$ $a°$ $50°$

b

$114°$ $a°$ $127°$
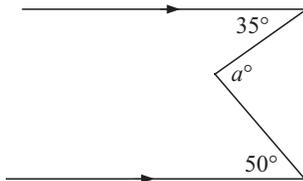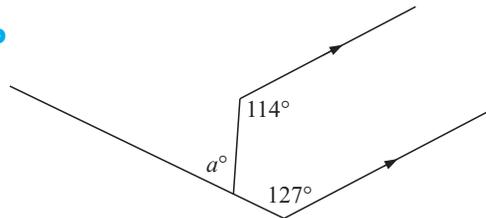
3 In the given figure we have two isosceles triangles PQS and SQR. $\widehat{QPS} = a°$ and $\widehat{QRS} = b°$.

P $a°$ Q $b°$ R S
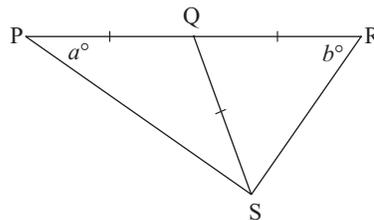
a State, with reasons, the sizes of angles PSQ and QSR.

b What is the value of $2a + 2b$?

c Deduce that $\widehat{PSR}$ is a right angle.

4 Alongside is a quadrilateral which has not been drawn accurately. However, its opposite angles are equal in size as shown.

A $β°$ B $α°$ D $α°$ $β°$ C
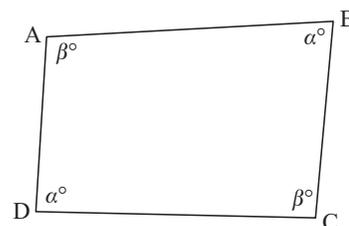
a Find the value of $2α + 2β$, giving reasons for your answer.

b Find the value of $α + β$.

c Give reasons why [AB] is parallel to [DC] and [AD] is parallel to [BC].

d Copy and complete: "If the opposite angles of a quadrilateral are equal, then ......"
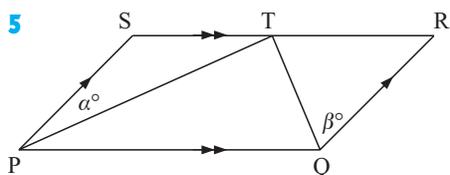
5   PQRS is a parallelogram. The angle bisectors of $\widehat{SPQ}$ and $\widehat{PQR}$ meet at T on the side [SR].
$\widehat{SPT} = \alpha^o$ and $\widehat{RQT} = \beta^o$.

a Find the sizes of angles TPQ and TQP.

b Find the value of $2\alpha + 2\beta$,   giving reasons for your answer.

c Explain why angle PTQ must be a right angle.

6   ABCD is a rhombus. M is the point on [AD] such that [CM] bisects angle ACD.
Show that angle DMC is 3 times larger than angle ACM.
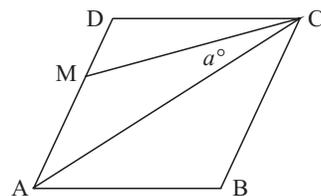
7   [AB] is the diameter of a semi-circle with centre O. P is any point on the semi-circle. Prove that angle POB is twice as large as angle PAB.

8   Triangle ABC is isosceles with   AB = AC.   [CB] is extended to D, and [AB] is extended to E so that   BE = DE.   Show that [DE] and [AC] are parallel.
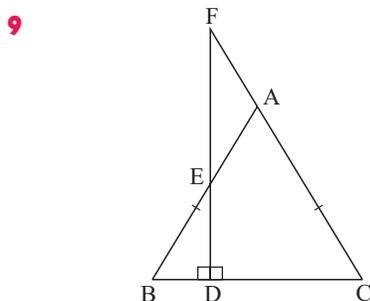
9   ABC is an isosceles triangle in which [AB] and [AC] are equal in length. [DE] is perpendicular to [BC] and is produced (extended) to meet [CA] produced at F.
Show that triangle AEF is also isosceles.

B   CIRCLE THEOREMS

Before we can talk about the properties and theorems of circles, we need to learn the appropriate language for describing them.

- A **circle** is the set of all points which are equidistant from a fixed point called the **centre**.

- The **circumference** is the distance around the entire circle boundary.

- An **arc** of a circle is any continuous part of the circle.

- A **chord** of a circle is a line segment joining any two points on the circle.

- A **semi-circle** is a half of a circle.
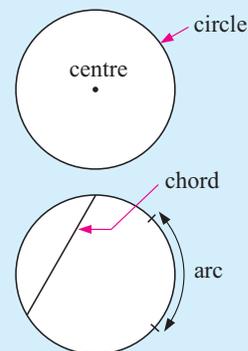
- A **diameter** of a circle is any chord passing through its centre.

- A **radius** of a circle is any line segment joining its centre to any point on the circle.

- A **tangent** to a circle is any line which touches the circle in exactly one point.

diameter

radius

tangent

point of contact
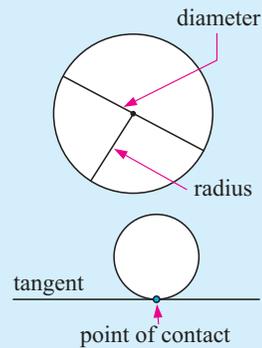
INVESTIGATION 1   PROPERTIES OF CIRCLES

This investigation is best attempted using the **computer package** on the CD. You can click on an icon to find each activity.

However, the investigation can also be done using a compass, ruler and protractor.

Part 1:   The angle in a semi-circle

What to do:

1 Draw a circle and construct a diameter. Label it as shown.

2 Mark any point P not at A or B on the circle. Draw [AP] and [PB].

3 Measure angle APB.

4 Repeat for different positions of P and for different circles. What do you notice?

5 Copy and complete: *The angle in a semi-circle is ......*
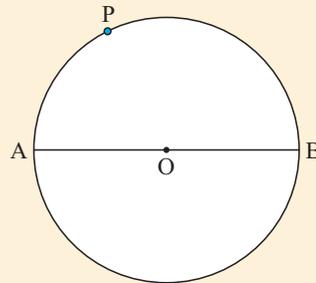
GEOMETRY
PACKAGE

Part 2:   Chords of a circle theorem

What to do:

1 Draw a circle with centre C. Construct any chord [AB].

2 Construct the perpendicular from C to [AB] which cuts the chord at M.

3 Measure the lengths of [AM] and [BM]. What do you notice?

4 Repeat the procedure above with another circle and chord.

5 Copy and complete:
The perpendicular from the centre of a circle to a chord ......
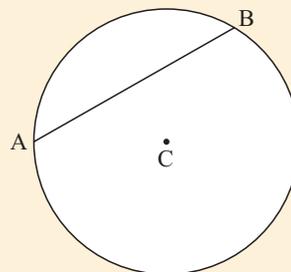
GEOMETRY
PACKAGE

Part 3:   Radius-tangent theorem

What to do:

1 Use a compass to draw a circle with centre O, and mark on it a point A.

2 At A, draw as accurately as possible a tangent [TA].

3 Draw the radius [OA].

4 Measure the angle OAT with a protractor.

5 Repeat the procedure above with another circle and tangent.

GEOMETRY PACKAGE

6 Copy and complete:
The tangent to a circle is ...... to the radius at the point ......
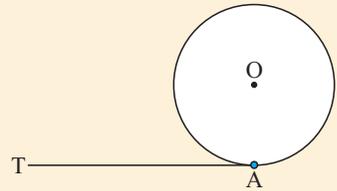
Part 4:   Tangents from an external point

What to do:

1 Use your compass to draw a circle, centre O.

2 From an external point P draw as accurately as possible the two tangents to the circle to meet it at A and B.

3 Measure [AP] and [BP].

GEOMETRY PACKAGE

4 Repeat with another circle of different size.

5 Copy and complete:
Tangents from an external point to a circle are ......
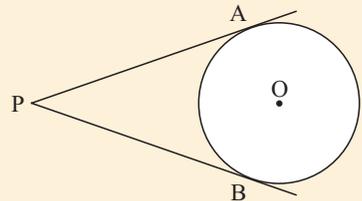
From the **Investigation** you should have discovered the following **circle theorems**.

Name of theorem	Statement	Diagram
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$\widehat{ABC} = 90^o$
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	$AM = BM$
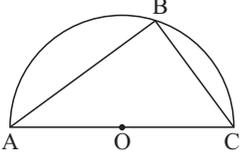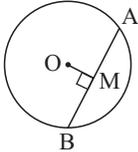
Name of theorem	Statement	Diagram
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	$\widehat{OAT} = 90^o$
Tangents from an external point	Tangents from an external point are equal in length.	$AP = BP$
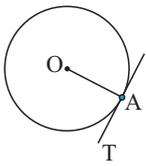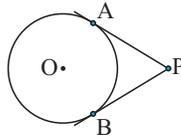
Two useful **converses** are:

- If line segment [AB] subtends a right angle at C then the circle through A, B and C has diameter [AB].

- The perpendicular bisector of a chord of a circle passes through its centre.
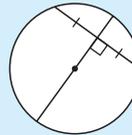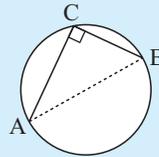
Example 4   🔊 **Self Tutor**

Find x, giving brief reasons for your answer.
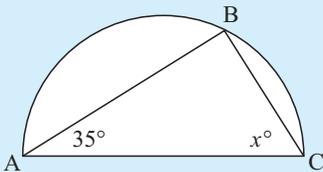
$\widehat{ABC}$ measures 90° {angle in semi-circle}

$\therefore$ $35 + 90 + x = 180$ {angles in triangle}

$\therefore$ $125 + x = 180$

$\therefore$ $x = 55$

Example 5   🔊 **Self Tutor**

A circle has a chord of length 8 cm. The shortest distance from the circle's centre to the chord is 2 cm. Find the length of the circle's radius to the nearest mm.

Suppose the radius has length r cm.

The shortest distance from the centre to the chord is the perpendicular distance.
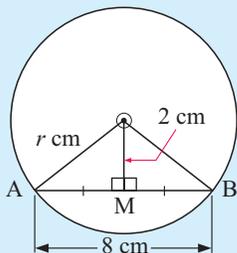
∴   M is the midpoint of [AB]   {chord of a circle}

∴   AM = 4 cm

In $\triangle$OAM, $r^2 = 2^2 + 4^2$ {Pythagoras}

$\quad\quad\quad$ ∴ $r^2 = 4 + 16$

$\quad\quad\quad$ ∴ $r^2 = 20$

$\quad\quad\quad$ ∴ $r = \pm\sqrt{20}$

$\quad\quad\quad$ ∴ $r \approx 4.5$ {as $r > 0$}

So, the radius is about 4.5 cm long.

EXERCISE 24B

1 Find x, giving brief reasons for your answers:

a
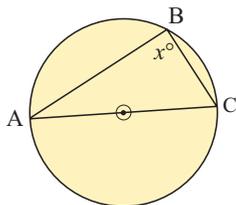
b
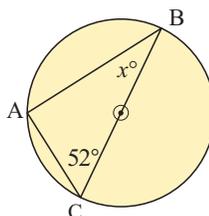
c
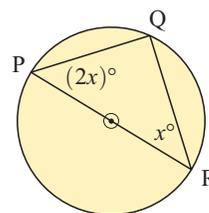
d
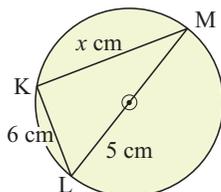
e
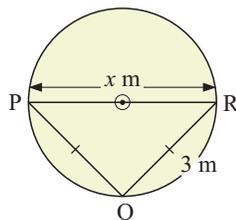
f
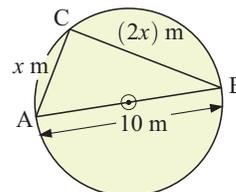
2 In each of the following, find x to 1 decimal place. Give brief reasons for your answers:

a
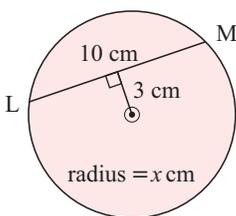
b
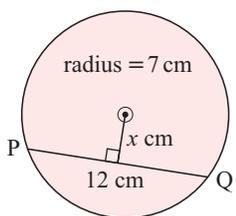
c
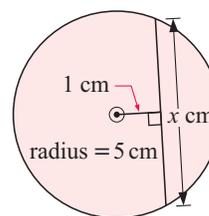
3 A circle has a chord of length 12 cm. The shortest distance from the chord to the circle's centre is 2 cm. How long, to the nearest mm, is the circle's radius?

4 A circle has a chord of length 15 cm. The radius of the circle is 9 cm. Find, to the nearest mm, the shortest distance from the chord to the circle's centre.

5 A circle of radius 5 cm has a chord such that the shortest distance from the circle's centre to the chord is 2 cm. How long is the chord, to the nearest mm?

Example 6   ◀) **Self Tutor**

A circle has a radius of 6 cm. At point T on the circle a tangent is drawn. The tangent is 10 cm long and ends at point P. Find the distance from P to the centre of the circle.

Let the distance from P to the centre of the circle be x cm.
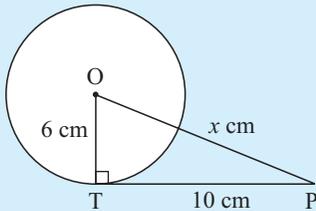
$$\widehat{OTP} = 90^o \qquad \text{\{radius-tangent theorem\}}$$
$$\therefore \quad OP^2 = OT^2 + TP^2 \qquad \text{\{Pythagoras\}}$$
$$\therefore \quad x^2 = 6^2 + 10^2$$
$$\therefore \quad x^2 = 136$$
$$\therefore \quad x = \pm\sqrt{136}$$
$$\therefore \quad x \approx 11.7 \qquad \text{\{OP} > 0\text{\}}$$

So, P is about 11.7 cm from the centre of the circle.

6 In the following diagrams, O is the centre of the circle, and [TP] and [SP] are tangents. Find the values of the unknowns, to 1 decimal place.

a

14 cm

x cm

P   11 cm   T

b

4 cm

x cm

T   5 cm   P

c

8.4 cm

x cm

P   T

d

T

8 cm

10 cm

P

y cm x cm

S

e

T

x^o

80° P

y^o

S

f

x cm

2 cm

P

4 cm

T
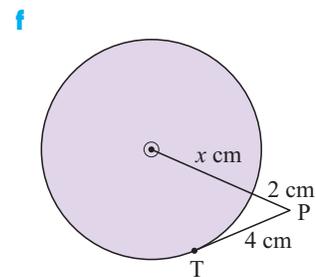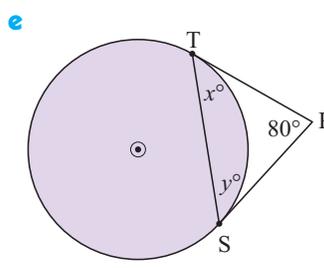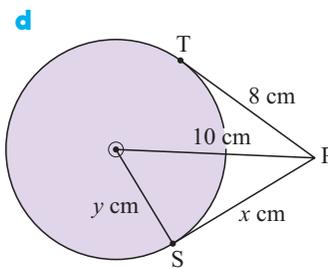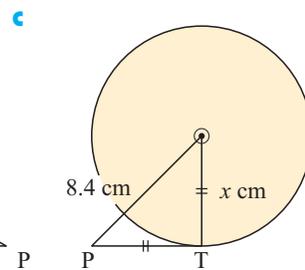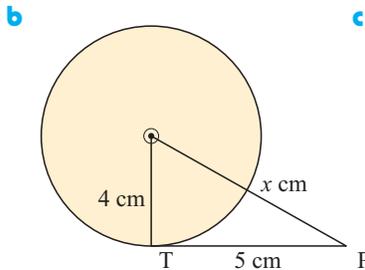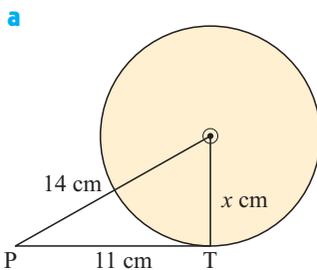
7 Point Q is 12 cm from the centre of a circle of radius 7 cm. A tangent is drawn from Q to touch the circle at P. How long is [PQ]?

8

B

Y

X

O

A   Z   C
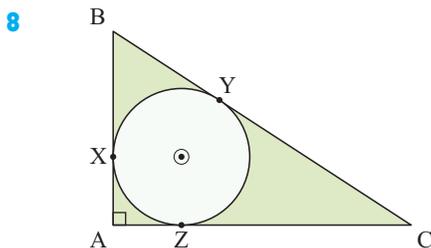
A circle is drawn within the right angled triangle ABC. The circle has radius 3 cm, and $BX = 4$ cm.

a What can be deduced about quadrilateral AXOZ?

b How far is C from Z? Show all working.

C   CONGRUENT TRIANGLES

Two triangles are **congruent** if they are identical in every respect except for position. The triangles have the same shape and size.

If one triangle was cut out with scissors and placed on the top of the other, they would match each other perfectly.
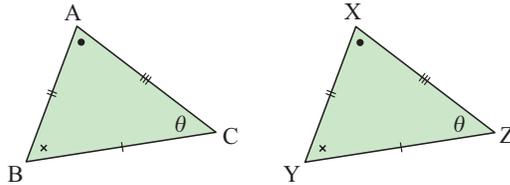
Triangles ABC and XYZ are congruent.

Notice that we label the vertices that are in corresponding positions in the same order.

We say triangle ABC is congruent to triangle XYZ rather than triangle YXZ or triangle ZYX.

There are four acceptable tests for the **congruence of two triangles**:

TESTS FOR TRIANGLE CONGRUENCE

Two triangles are congruent if one of the following is true:

- All corresponding sides are equal in length. (**SSS**)

- Two sides and the **included** angle are equal. (**SAS**)

- Two angles and a pair of **corresponding sides** are equal. (**AAcorS**)

- For right angled triangles, the hypotenuses and one pair of sides are equal. (**RHS**)

The information we are given will help us decide which test to use to prove two triangles are congruent.

Congruence is one of the main tools used to establish key results and theorems in deductive geometry.

For example, we can use congruence to prove the **isosceles triangle theorem**, which is:

In an isosceles triangle:
- the base angles are equal
- the line joining the apex to the midpoint of the base is perpendicular to the base and bisects the angle at its apex.

Proof:

Consider $\triangle ABC$ where $AB = AC$ and M is the midpoint of [BC].
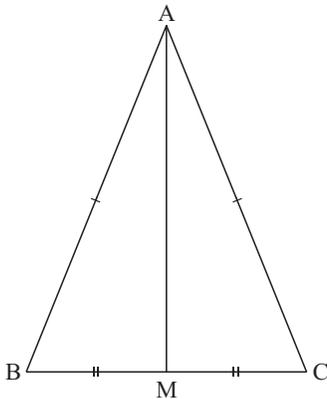
Triangles ABM and ACM are congruent (SSS)
since $AB = AC$ {given}
         $BM = MC$ {given}
and AM is common to both triangles.

Consequently, all corresponding angles are equal.

- $A\widehat{B}M = A\widehat{C}M$,   so the base angles are equal
- $A\widehat{M}B = A\widehat{M}C$,   and since these add to 180°, each must be a right angle
- $B\widehat{A}M = C\widehat{A}M$,   so the vertical angle is bisected.

Example 7   🔊 **Self Tutor**

Explain why $\triangle ABC$ and $\triangle DBC$ are congruent:
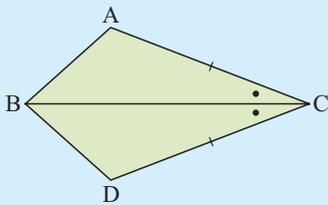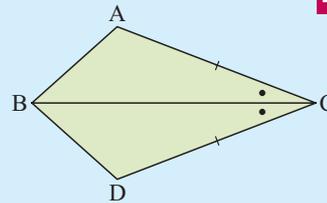
$\triangle$s ABC and DBC are congruent (SAS) as:

- $AC = DC$
- $A\widehat{C}B = D\widehat{C}B$, and
- [BC] is common to both.

Example 8   🔊 **Self Tutor**

Triangle ABC is isosceles with $AC = BC$. [BC] and [AC] are produced to E and D respectively so that $CE = CD$.

Prove that $AE = BD$.
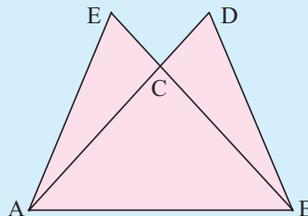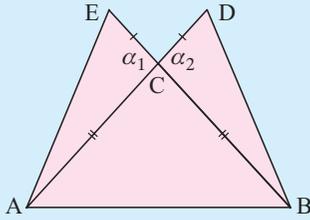
In triangles ACE and BCD:

- $AC = BC$ {given}
- $\alpha_1 = \alpha_2$ {vertically opposite}
- $CE = CD$ {given}

$\therefore$ the triangles are congruent (SAS) and in particular $AE = BD$.

EXERCISE 24C

1
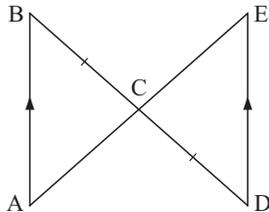
a Explain why triangles ABC and EDC are congruent.

b If $AC = 5$ cm and $B\widehat{A}C = 37^o$, find:

  i the length of [CE]

  ii the size of $D\widehat{E}C$.

2 Point P is equidistant from both [AB] and [AC]. Use congruence to show that P lies on the bisector of $B\widehat{A}C$.

Hint: To find the distance from P to either line we draw a perpendicular from the point to the line. So, the figure becomes:
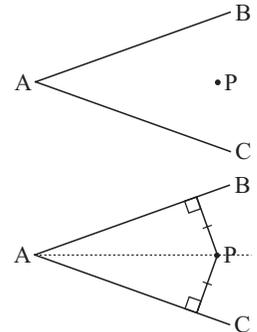
3 Two concentric circles are drawn. At P on the inner circle, a tangent is drawn which meets the other circle at A and B.

Use triangle congruence to prove that P is the midpoint of [AB].
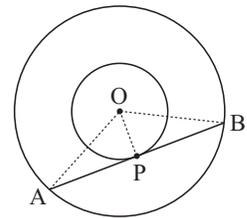
4   **a** Prove that triangles AMC and BMD are congruent.

  b Deduce that [AC] and [DB] are parallel and equal in length.

  c What can be deduced about the quadrilateral ACBD?
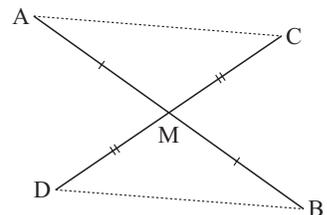
5 [AB] and [DC] are parallel and equal in length.

- **a** Join [BC], [AC] and [AD], and show that Δs ABC and CDA are congruent.
- **b** Now show that ABCD is a parallelogram.
- **c** Copy and complete:
  "*If a pair of opposite sides of a quadrilateral are parallel and equal in length, then the quadrilateral is ......*"
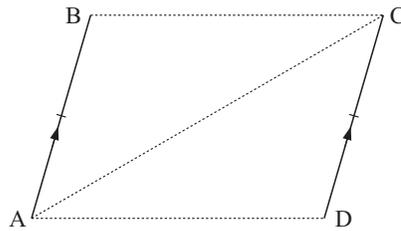
6 In ΔABC, [BM] is drawn perpendicular to [AC] and [CN] is drawn perpendicular to [AB].
Now if these perpendiculars are equal in length:

- **a** prove that Δs BCM and CBN are congruent
- **b** prove that ΔABC is isosceles.
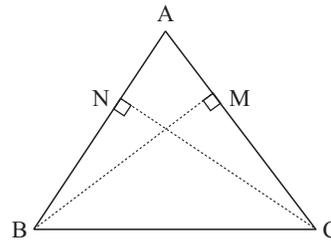
7 In ΔPQR, M is the midpoint of [QR]. [MX] is drawn perpendicular to [PQ], and [MY] is drawn perpendicular to [PR]. Now if the perpendiculars are equal in length:

- **a** prove that ΔMQX is congruent to ΔMRY
- **b** prove that ΔPQR is isosceles.
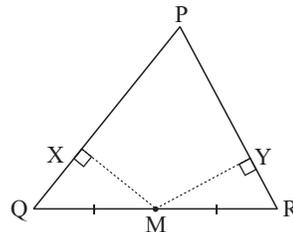
D   SIMILAR TRIANGLES

The word **similar** suggests a comparison between objects which have some, but not all, properties in common. Similar figures have the same **shape** but not necessarily the same size. A similar figure results when a figure undergoes an enlargement or reduction.

Common examples of similar figures include television images, photo enlargements, house plans, maps, and model cars.

> Two triangles are **similar** if one is an enlargement of the other.
>
> Consequently, similar triangles are **equiangular**, and have corresponding sides in the same **ratio**.

For example, ΔABC is *similar* to ΔPQR.

Since the sides are in the same ratio,

$$\frac{QR}{BC} = \frac{RP}{CA} = \frac{PQ}{AB} = \text{the enlargement factor.}$$
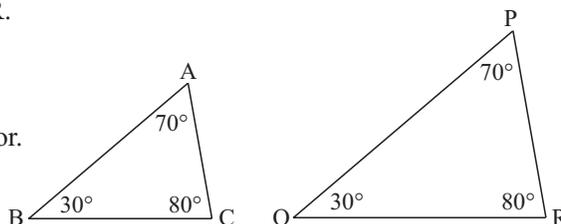
TESTS FOR TRIANGLE SIMILARITY

Two triangles are similar if either:

- they are equiangular,   or
- their side lengths are in the same ratio.

Note: Since the angle sum of any triangle is $180°$, if two angles of one triangle are equal in size to two angles of the other triangle then the remaining angles must also be equal.

Example 9   ◄⑴ **Self Tutor**

Show that the following figures possess similar triangles:

a

b

a
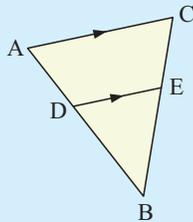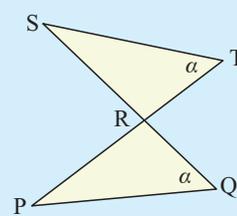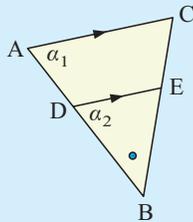
Δs ABC and DBE are equiangular as:

- $\alpha_1 = \alpha_2$ {equal corresponding angles}
- angle B is common to both triangles
  $\therefore$ the triangles are similar.

b
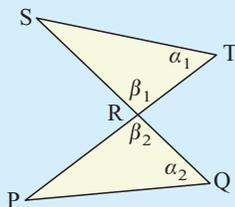
Δs PQR and STR are equiangular as:

- $\alpha_1 = \alpha_2$ {given}
- $\beta_1 = \beta_2$ {vertically opposite angles}
  $\therefore$ the triangles are similar.

EXERCISE 24D.1

1 Show that the following figures possess similar triangles:

a

b

c

d   **e**   **f**
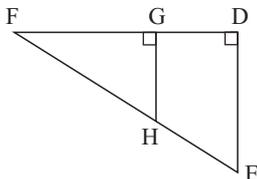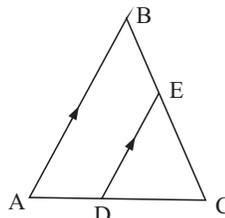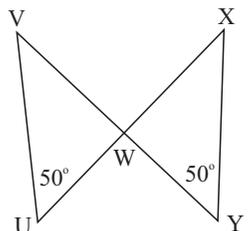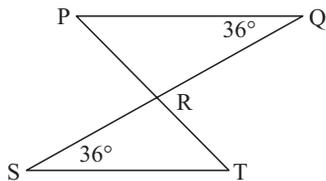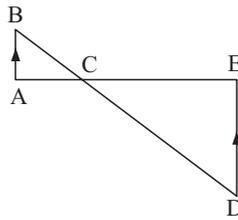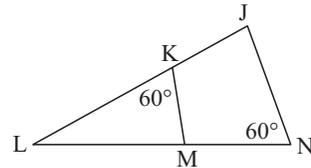
FINDING SIDE LENGTHS

In the following questions we will find the lengths of unknown sides of triangles. To do this, we first prove that two triangles are equiangular and therefore similar. We then use the property that the sides of similar triangles are in the same ratio.

Example 10 🔊 **Self Tutor**

Establish that a pair of triangles is similar, and hence find x:
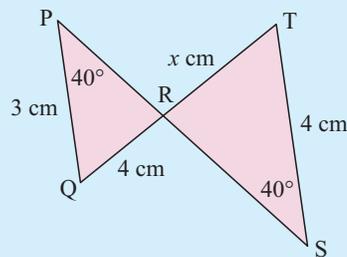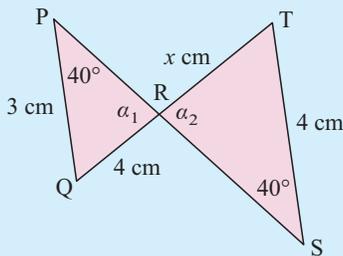
$Q\widehat{P}R = T\widehat{S}R$ {given}

$\alpha_1 = \alpha_2$ {vertically opposite angles}

So, Δs PQR and STR are similar and

$$\frac{PQ}{ST} = \frac{QR}{TR} = \frac{PR}{SR} \quad \text{\{same ratio\}}$$

$$\therefore \quad \frac{3}{4} = \frac{4}{x}$$

$$\therefore \quad 3x = 16$$

$$\therefore \quad x = \tfrac{16}{3}$$

When solving similar triangles it may be useful to construct a table. For example, for the previous example we could use the following steps:

Step 1: Label equal angles.

Step 2: Put the information in table form, showing the equal angles and the sides opposite these angles.

Step 3: Since the triangles are equiangular, they are similar.

Step 4: Use the columns to write down the equation for the ratio of the corresponding sides.

Step 5: Solve the equation.

	α	β	40°
ΔPQR	3	-	4
ΔSTR	4	-	x

from which $\dfrac{3}{4} = \dfrac{4}{x}$

$$\therefore \quad x = \tfrac{16}{3}$$

Example 11   ◀) **Self Tutor**

Establish that a pair of triangles is similar. Find x if $BE = 8$ cm:
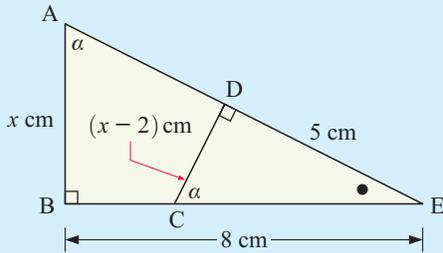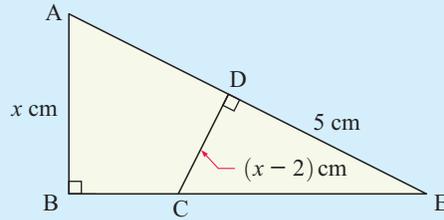
The triangles are equiangular and hence similar.

	α	90°	●
$\triangle ABE$	8	-	x
$\triangle CDE$	5	-	$x-2$

$$\therefore \quad \frac{8}{5} = \frac{x}{x-2} \quad \{\text{same ratio}\}$$

$$\therefore \quad 8(x-2) = 5x$$
$$\therefore \quad 8x - 16 = 5x$$
$$\therefore \quad 3x = 16$$
$$\therefore \quad x = \frac{16}{3}$$

EXERCISE 24D.2

1 In the following, establish that a pair of triangles is similar, and find x:

a

b

c

d

e

f

g

h

i
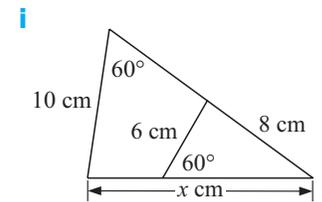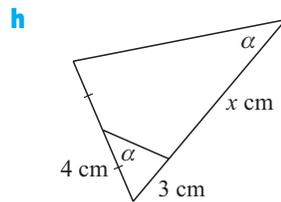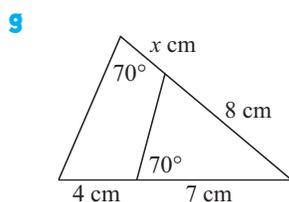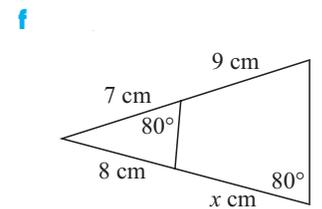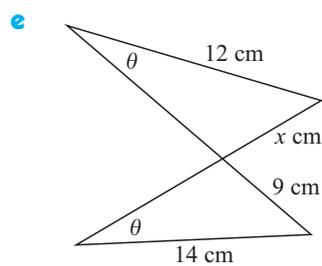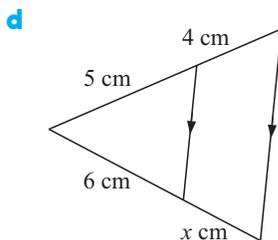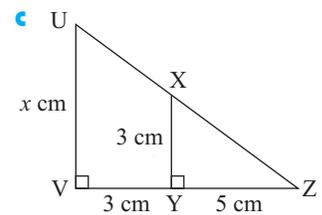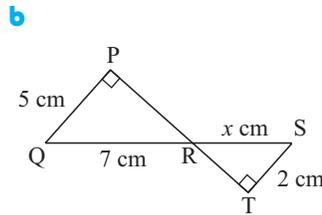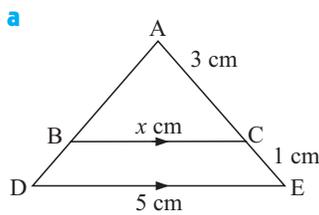
E   PROBLEM SOLVING WITH SIMILAR TRIANGLES

The properties of similar triangles have been known since ancient times. But even with the technologically advanced measuring instruments available today, similar triangles are important for finding heights and distances which would otherwise be difficult to measure.

Step 1: Read the question carefully and draw a sketch showing all the given information.

Step 2: Introduce a variable such as x to represent the quantity to be found.

Step 3: Set up an equation involving the variable and solve for the variable.

Step 4: Answer the question in a sentence.

Diagrams are very important. They often help you to solve the problem. Make sure your diagrams are large enough.

Example 12   ◀) Self Tutor

One sunny morning, Phan and Khuyen take a one metre long ruler and compare the length of its shadow with that of their school hall. Their measurements are shown in the diagram.

Use similar triangles to find the height of the hall.
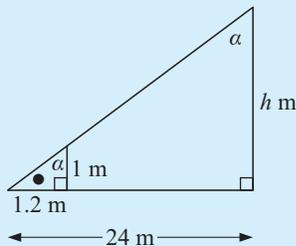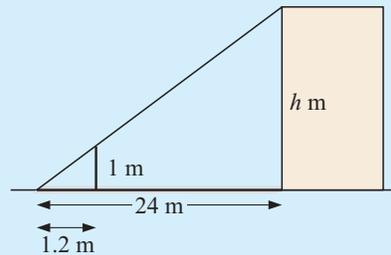
	●	90°	α
small △	1	-	1.2
large △	h	-	24

Since the triangles are similar,

$$\frac{h}{1} = \frac{24}{1.2}$$

$\therefore \quad h = 20$

So, the school hall is 20 m high.

EXERCISE 24E

1
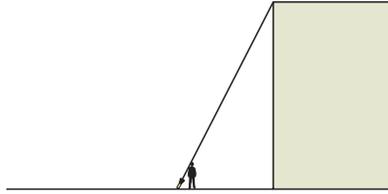
A spotlight is fixed on the ground 35 m from the town hall. It is switched on at night to illuminate the building. When Dirk stands 5 m in front of the light, his shadow is as tall as the building. If Dirk is 1.40 m tall, how tall is the town hall?

2 A sign for truck drivers states that the road ahead has a constant slope of 2 in 11. This means that for every 11 m horizontally it rises 2 m.

   a Over what horizontal distance will the road rise 300 m?

   b What is the length of the road [AB]?
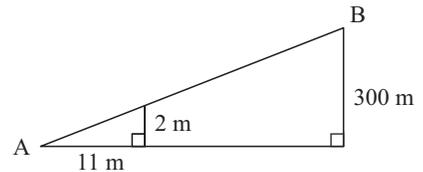
3
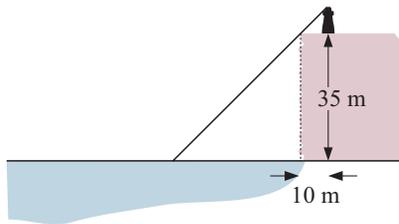
A lighthouse is situated 10 m back from a 35 m high cliff. If the beacon is 8 m above the base of the lighthouse, how far from the cliff does the shadow extend?

4 A slide projector is set up so that its lamp is 4 m from a screen.

The slide is placed 16 mm from the lamp so that [AD] is parallel to [BC].
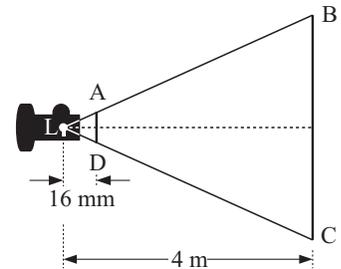
   a Explain why triangles LAD and LBC are similar.

   b If the picture on the slide is 22 mm wide and 16 mm high, what will the dimensions of the image on the screen be?

5
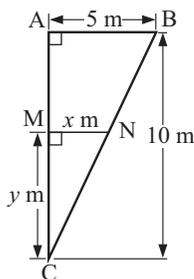
   a Show that Δs ABC and MNC are similar.

   b Hence, show that $y = 2x$.

   c The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.

   Find the volume of water in the cone in terms of x.
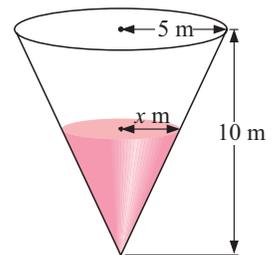
6 Enrico is standing at point A on one side of a river, directly opposite the tree T. The bank on his side is quite straight. 10 m along the bank is a post P. Enrico walks to P and then a further 6 m along the bank to C. He then walks directly away from the river until he reaches D which is in line with P and T. CD = 8 m.

   a Show that triangles CDP and ATP are similar.

   b How wide is the river at the point A?
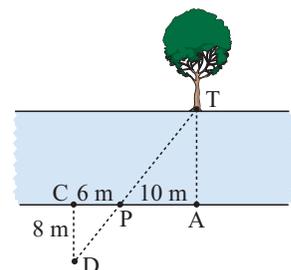
F   THE MIDPOINT THEOREM

INVESTIGATION 2   THE MIDPOINT THEOREM

Your task is to investigate the line joining the midpoints of two sides of a triangle.

Use of the **geometry package** on the CD is recommended, but you could also use a ruler and protractor.

GEOMETRY PACKAGE

What to do:

1   Construct a large triangle and label it like the triangle alongside.

2   As accurately as you can, mark the midpoints P and Q of [AB] and [AC] respectively.

3   Join [PQ] and [QB].

4   Measure the lengths of [PQ] and [BC]. What do you notice?

5   Measure $P\hat{Q}B$ and $Q\hat{B}C$. What conclusion can be drawn about [PQ] and [BC]?

6   Repeat the procedure with different sized triangles.

7   Copy and complete: "*The line joining the midpoints of two sides of a triangle is ...... to the third side and ...... its length.*"
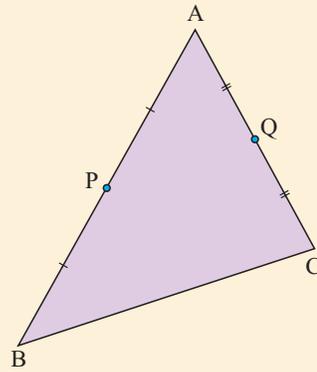
From the investigation you should have discovered:

THE MIDPOINT THEOREM

The line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
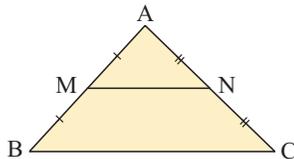
- [MN] is parallel to [BC].
- $MN = \frac{1}{2}(BC)$.

CONVERSE OF MIDPOINT THEOREM

The line drawn from the midpoint of one side of a triangle, parallel to a second side, bisects the third side.
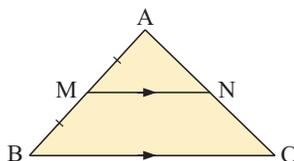
$AN = NC$.

EXERCISE 24F

1 Find the unknowns in the following, giving reasons.

a

b
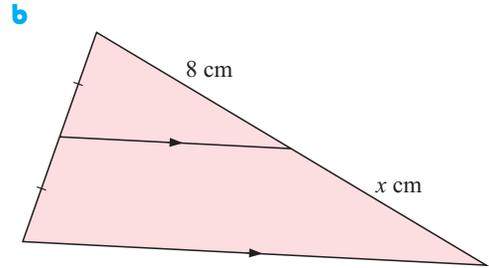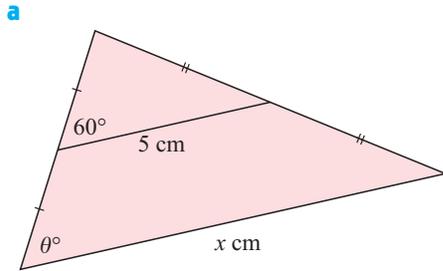
2 ABCD is a parallelogram whose diagonals meet at E. M is the midpoint of [AD]. Show that [ME] is parallel to [AB] and half its length.
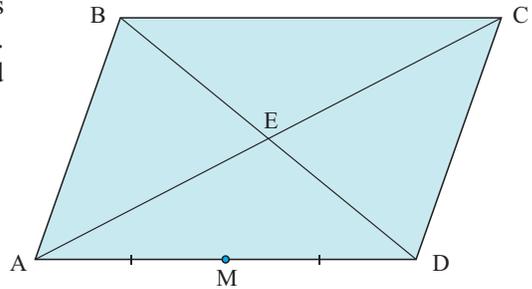
Example 13
🔊 **Self Tutor**

ABCD is a parallelogram. [AB] is produced to E such that AB = BE. [AD] is produced to meet [EC] produced at F. Prove that EC = CF.
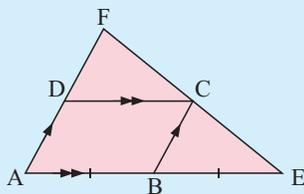
In △AEF, [BC] is parallel to [AF].
  {as ABCD is a parallelogram}

Since   B is the midpoint of [AE],
  C is the midpoint of [EF].
  {midpoint theorem converse}

Hence   EC = CF.

3
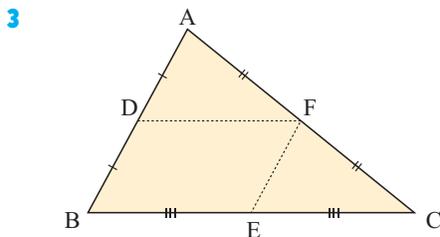
ABC is a triangle. D, E and F are the midpoints of its sides as shown. Show that DFEB is a parallelogram.

4 A and B are the midpoints of sides [PQ] and [PR] of △PQR. Y is any point on [QR]. Prove that X is the midpoint of [PY].
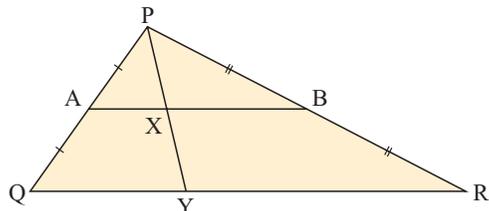
5 Stacey says "I have drawn dozens of different quadrilaterals. When I join the midpoints of adjacent sides of each one of them, the figure formed appears to be a parallelogram."

  a Draw two quadrilaterals of your own choosing to check Stacey's conjecture.

  b By drawing one diagonal of a labelled quadrilateral, show that Stacey's conjecture is correct.

6 To **prove** the midpoint theorem in $\triangle BAC$, we extend [PQ] to meet a line from C which is parallel to [BA] at R. Copy and complete the proof:
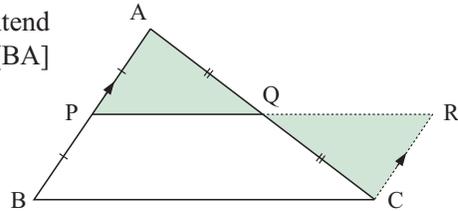
In $\triangle$s APQ and CRQ:

  - AQ = CQ   {..........} $^{(1)}$
  - $A\widehat{P}Q = C\widehat{R}Q$ {..........} $^{(2)}$
  - .... = ...   {vertically opposite} $^{(3)}$

∴   the triangles are congruent   {........} $^{(4)}$

Consequently, AP = CR, and as AP = BP, CR = ... $^{(5)}$

So, [BP] and [CR] are parallel and equal in length, and this is sufficient to deduce that BCRP is a .... $^{(6)}$

∴   [PQ] is parallel to ......$^{(7)}$ and PR = BC.

But, from the congruence,   PQ = QR,   and so   PQ = $\frac{1}{2}$ BC.

G   EULER'S RULE

Leonhard Euler, pronounced 'oiler', was one of the greatest mathematicians of all time. He made numerous interesting observations in geometry. One of them is known as **Euler's Rule**. It connects the number of vertices, edges and regions in a polygon.

INVESTIGATION 3   EULER'S RULE
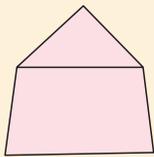
Consider the figure: It has 5 vertices,
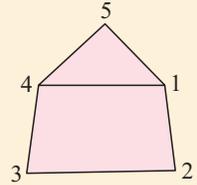
6 edges, and 3 regions.
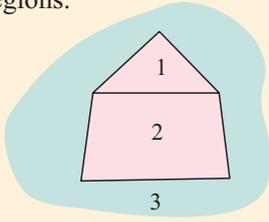
Outside the figure counts as a region.

What to do:

1 Consider the following figures:
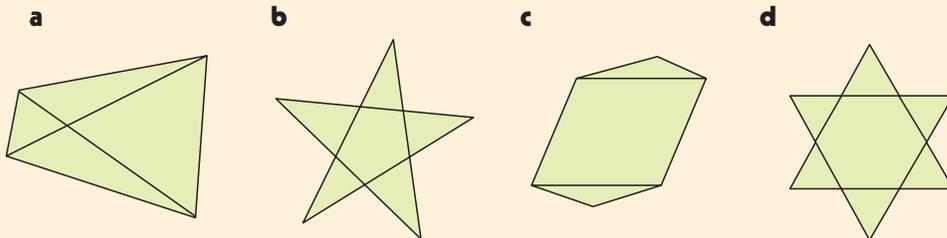
a　　　　　　　b　　　　　　　c　　　　　　　d

2 Copy and complete the table alongside. **e** to **h** are for four diagrams like those above, but of your choice.

Figure	Vertices (V)	Regions (R)	Edges (E)	V + R − 2
given example	5	3	6	6
a				
b				
c				
d				
e				
f				
g				
h				

3 Find the relationship between V, R and E.

From the previous investigation you should have discovered that:

> In any closed figure, the number of edges is always two less than the sum of the number of vertices and regions.
>
> $$E = V + R - 2$$

Note: Euler's rule applies even if the edges are *not straight lines*.

For example, has $V = 4$, $R = 5$ and $E = 7$
and $V + R - 2 = 9 - 2 = 7$ which checks.

EXERCISE 24G

1 Using Euler's rule, determine the number of:
- **a** edges for a figure with 5 vertices and 4 regions
- **b** vertices of a figure with 7 edges and 3 regions
- **c** regions for a figure with 10 edges and 8 vertices.

2 Draw a possible figure for each of the cases in **1**.

3 Draw two *different* figures which have 5 vertices and 7 edges.

ACTIVITY   NETS OF SOLIDS

Click on the icon for printable nets of the Platonic Solids. Make the solids from light card. Research the significance of these solids.

PRINTABLE WORKSHEET

FINDING THE CENTRE OF A CIRCLE

LINKS
click here

Areas of interaction:
Approaches to learning

REVIEW SET 24A

1   Find the value of x, giving reasons:

a
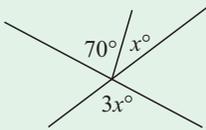
b
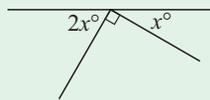
c
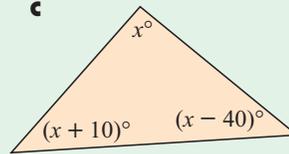
d
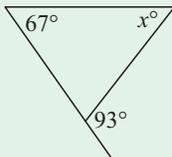
e
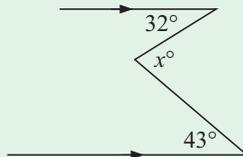
f
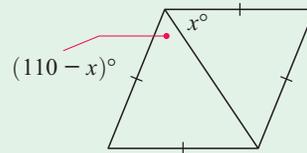
2   Find x:

a
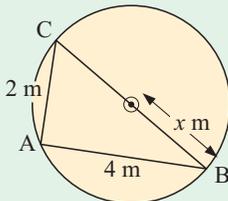
b
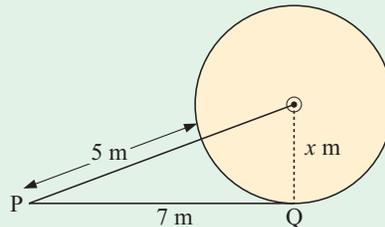
3   In the diagram alongside, establish that a pair of triangles is similar and find x:
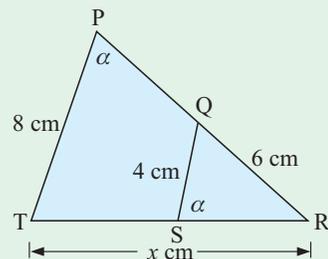
4
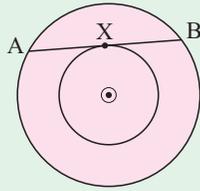
Two concentric circles have centre O. [AB] is a chord of the larger circle and is also a tangent to the smaller circle.

Prove that [AB] is bisected at X, i.e., that X is the midpoint of [AB].

5
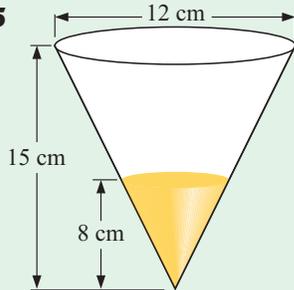
A conical flask has height 15 cm and base diameter 12 cm. Water is poured into the flask to a depth of 8 cm.

a Show that triangles ABC and MNC are similar.

b Hence show that $x = 3.2$.

c Find the diameter of the surface of the water.
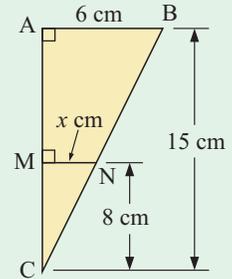
6
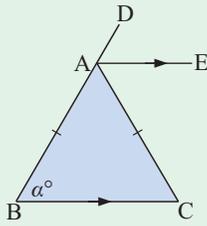
Triangle ABC is isosceles with AB = AC. [BA] is produced to D and [AE] is drawn parallel to [BC].

Given that $A\widehat{B}C = \alpha$:

a find $A\widehat{C}B$ in terms of α

b show that [AE] bisects $D\widehat{A}C$.

7 Find, giving full reasoning, the length of [AC]. Your answer must not involve x.
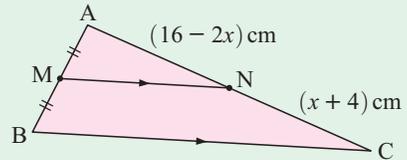
REVIEW SET 24B

1 Find the value of x, giving reasons:

a

b

c

d

e

f
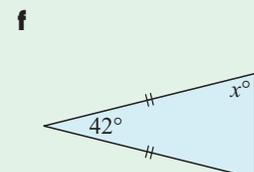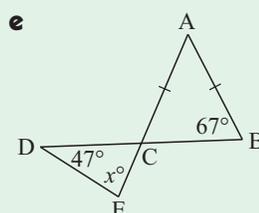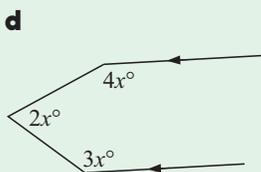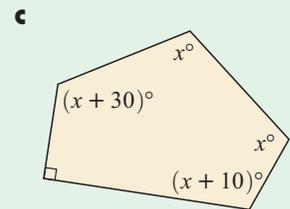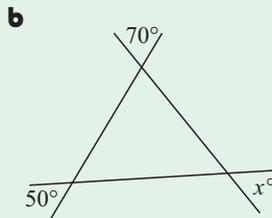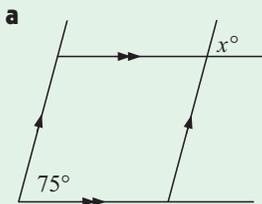
2 Find the values of the unknowns:

a

b

3
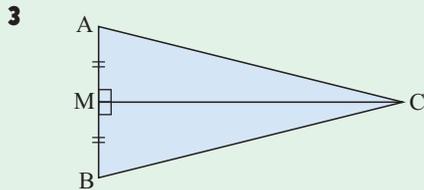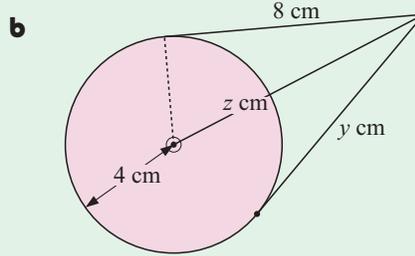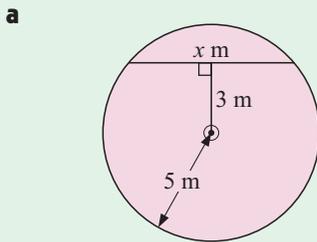
Using only congruence,
prove that triangle ABC is isosceles.

4 In each of the following, establish that a pair of triangles is similar, and find x:

a

b
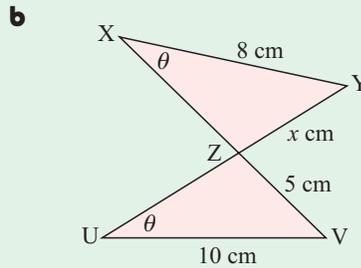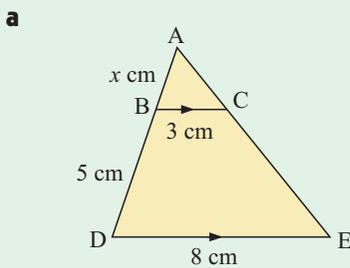
5 A cone of radius 10 cm and height 18 cm fits exactly over a cylinder so that the cylinder is just touching the inside surface of the cone. The radius of the cylinder is 4 cm.

   a Show that $\triangle$s ABC and AMN are similar.

   b Hence show that $x = 7.2$.

   c Find the height of the cylinder.

6
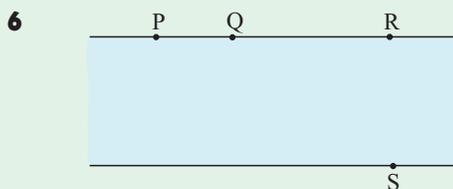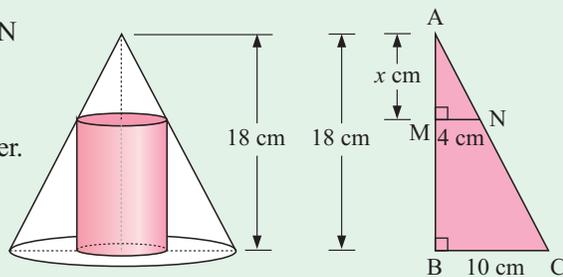
P, Q, R and S are markers on the banks of a canal which has parallel sides. R and S are directly opposite each other. PQ $= 30$ m and QR $= 100$ m.

When I walk 20 m from P directly away from the bank, I reach a point T where T, Q and S line up. How wide is the canal?

Chapter

25

Non-right angled triangle trigonometry

Contents:

OPENING PROBLEM

Andy is a surveyor who measures the exact dimensions of property boundaries. Andy normally uses a laser measuring device, but today he cannot because a large number of trees block his vision.

Instead, he measures the distance from point A to the corner of a shed C to be 124 m. He measures angle ACB to be $130°$, and the distance CB to be 105 m.

How can Andy use his measurements to find the length of the boundary [AB]?
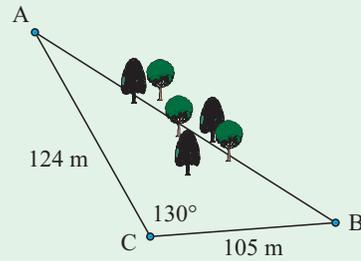
Andy's problem contains a non-right angled triangle. However, it can actually be solved using trigonometry.

In this chapter we will see how trigonometry is applied to non-right angled triangles.

A   THE UNIT QUARTER CIRCLE
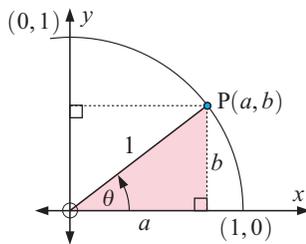
The circle of radius 1 unit with its centre at the origin O is called the **unit circle**. The part of it for which x and y are both positive is called the **unit quarter circle**.

Suppose [OP] can rotate about O and [OP] makes an angle θ with the x-axis as shown.

Let P have coordinates (a, b).

Notice that $\cos \theta = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{a}{1} = a$ and $\sin \theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{b}{1} = b$.

So,   the coordinates of P on the unit quarter circle are $(\cos \theta, \sin \theta)$ where θ is the angle measured from the positive x-axis to [OP].

For any sized angle θ we could use an accurate scale diagram to find the coordinates of P. You can see such a diagram on the page opposite.

Example 1   ◀) Self Tutor

Use the unit quarter circle to find:

a $\sin 40°$ b $\cos 30°$ c   the coordinates of P if $\theta = 50°$.

a   The y-coordinate at $40°$ is about 0.64
   $\therefore$ $\sin 40° \approx 0.64$
b   The x-coordinate at $30°$ is about 0.87
   $\therefore$ $\cos 30° \approx 0.87$
c   For $\theta = 50°$,   P is $(\cos 50°, \sin 50°) \approx (0.64, 0.77)$
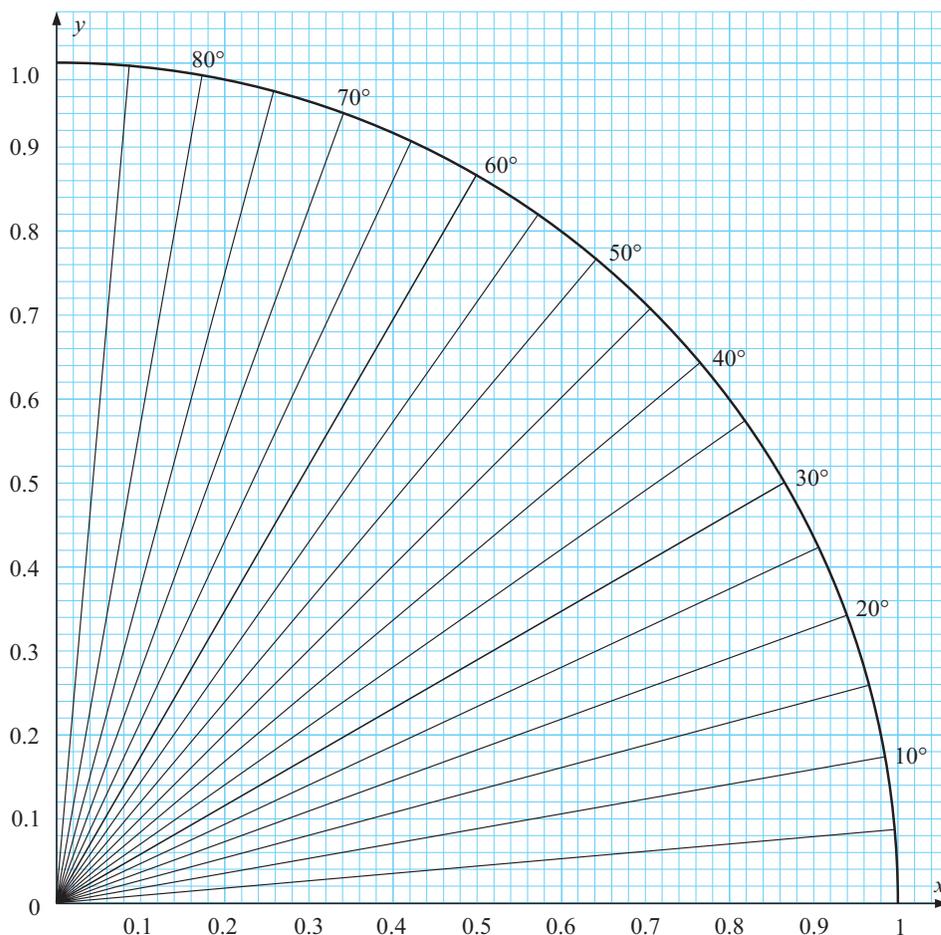
You have probably already noticed the difficulty of obtaining values from the unit quarter circle that are accurate to beyond 2 decimal places. Fortunately, we can obtain more accurate values using a calculator.

EXERCISE 25A.1

1 Use the unit quarter circle diagram to find the value of:

    a $\sin 0^o$ **b** $\sin 15^o$ **c** $\sin 25^o$ **d** $\sin 30^o$

    e $\sin 45^o$ **f** $\sin 60^o$ **g** $\sin 75^o$ **h** $\sin 90^o$

Make sure your calculator is in degrees mode.

Check your answers using your calculator.

2 Use the unit quarter circle diagram to find the value of:

    a $\cos 0^o$ **b** $\cos 15^o$ **c** $\cos 25^o$ **d** $\cos 30^o$

    e $\cos 45^o$ **f** $\cos 60^o$ **g** $\cos 75^o$ **h** $\cos 90^o$

Check your answers using your calculator.

3 Use the unit quarter circle diagram to find the coordinates of the point P on the unit circle where [OP] makes an angle of 55° with the x-axis. Use your calculator to check this answer.

4
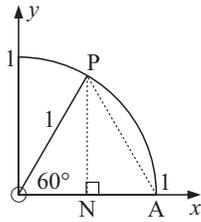
Use the diagram alongside to explain why $\cos 60^o = \frac{1}{2}$.

THE UNIT CIRCLE AND TANGENTS
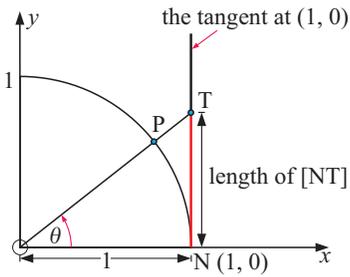
Now consider extending [OP] to meet the tangent at N(1, 0) at T.

The length of the part of the tangent [NT] is called the **tangent** of angle θ or $\tan \theta$.

We can use a scale diagram to approximate the tangent of an acute angle. For example, notice in the following diagram that $\tan 30^o \approx 0.58$ and $\tan 60^o \approx 1.73$.
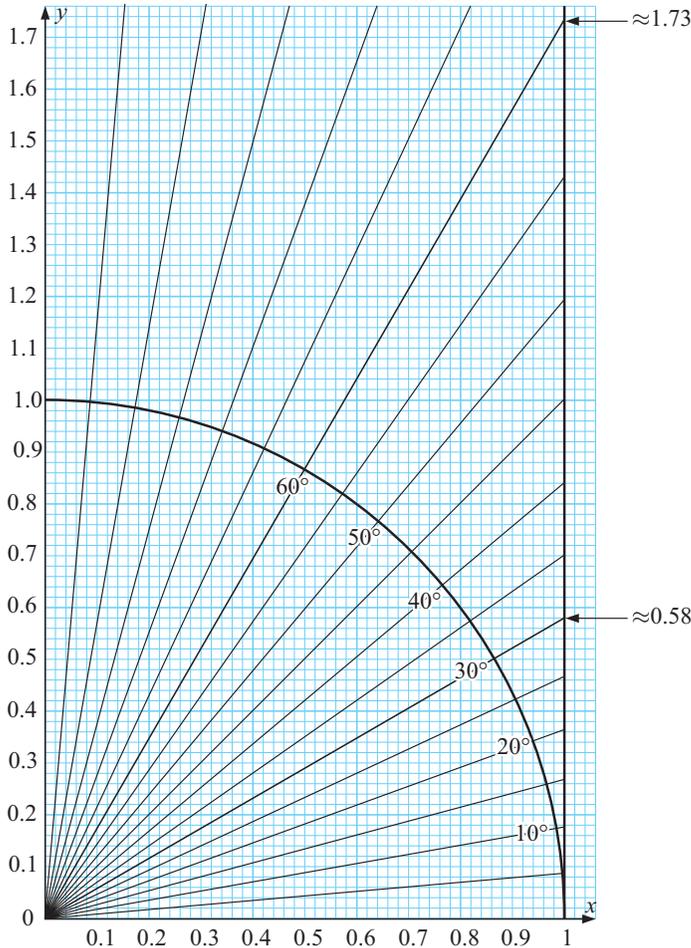
EXERCISE 25A.2

1 Use the unit quarter circle diagram to estimate the value of:

    a $\tan 0^o$ **b** $\tan 10^o$ **c** $\tan 20^o$ **d** $\tan 35^o$

    e $\tan 40^o$ **f** $\tan 45^o$ **g** $\tan 50^o$ **h** $\tan 55^o$

  Use your calculator to check your answers.

2 Explain why $\tan 45^o = 1$ exactly.

3 Why have you not been asked to find $\tan 80^o$ using the unit quarter circle diagram? Find $\tan 80^o$ using your calculator.

4
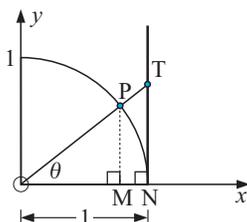
  a Find the coordinates of P in terms of θ.

  b Find the length of:

      i [OM]   **ii** [PM]   **iii** [TN]

  c Use similarity to show that $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$.

B   OBTUSE ANGLES

So far we have only considered angles between 0° and 90°, which are known as **acute** angles.

Obtuse angles have measurement between 90° and 180°. In order to display obtuse angles we can extend the unit quarter circle to the unit half circle shown in the diagram below.
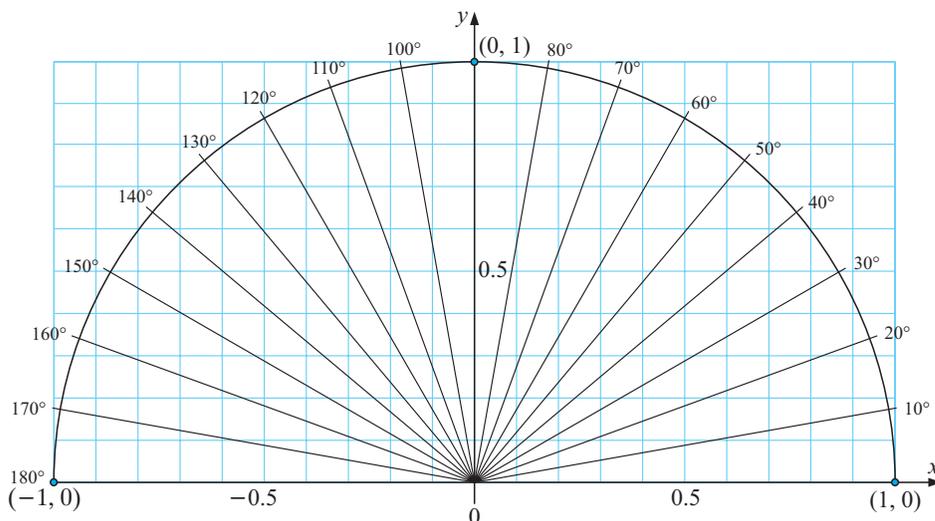
We can apply the same definitions for $\sin \theta$ and $\cos \theta$ as we did before.

> If P is any point on the unit circle and θ is the angle measured from the positive x-axis, then
>
> $\cos \theta$ is the x-coordinate of P and
> $\sin \theta$ is the y-coordinate of P.
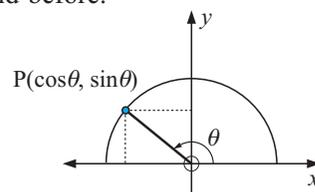
Example 2

◀) **Self Tutor**

Use the unit half circle to find:

a $\sin 140^o$ **b** $\cos 150^o$ **c** the coordinates of P if $\theta = 160^o$.

a The y-coordinate at 140° is about 0.64 $\therefore$ $\sin 140^o \approx 0.64$
b The x-coordinate at 150° is about -0.87 $\therefore$ $\cos 150^o \approx -0.87$
c For $\theta = 160^o$, P is $(\cos 160^o, \sin 160^o) \approx (-0.94, 0.34)$.

EXERCISE 25B

1 Use the unit half circle diagram to find the value of:

   a $\sin 110^o$ **b** $\sin 70^o$ **c** $\sin 120^o$ **d** $\sin 60^o$

   e $\sin 130^o$ **f** $\sin 50^o$ **g** $\sin 180^o$ **h** $\sin 0^o$

   Use your calculator to check your answers.

2 **a** Use your results from question **1** to copy
         and complete: $\sin(180° - \theta) = $ ....

   b Justify your answer using the diagram alongside.
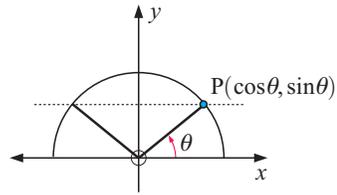
3 Use the unit half circle to find the value of:

   a $\cos 100^o$ **b** $\cos 80^o$ **c** $\cos 120^o$ **d** $\cos 60^o$

   e $\cos 130^o$ **f** $\cos 50^o$ **g** $\cos 180^o$ **h** $\cos 0^o$

   Use your calculator to check your answers.

4 **a** Use your results from question **3** to copy
         and complete: $\cos(180° - \theta) = $ ....

   b Justify your answer using the diagram alongside.
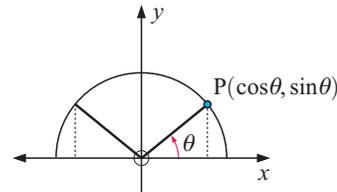
5 Find the obtuse angle which has the same sine as:

   a 26° **b** 45° **c** 69° **d** 86°

6 Find the acute angle which has the same sine as:

   a 98° **b** 127° **c** 156° **d** 168°

7 Without using your calculator, find:

   a $\sin 112^o$ if $\sin 68^o \approx 0.9272$ **b** $\sin 26^o$ if $\sin 154^o \approx 0.4384$

   c $\cos 168^o$ if $\cos 12^o \approx 0.9781$ **d** $\cos 49^o$ if $\cos 131^o \approx -0.6561$

   e $\sin 145^o$ if $\sin 35^o \approx 0.5736$ **f** $\cos 98^o$ if $\cos 82^o \approx 0.1392$.

C   AREA OF A TRIANGLE USING SINE

LABELLING CONVENTION

In the triangle ABC, we let the angle at A be A, the angle at B be B, and the angle at C be C.

The sides opposite A, B and C are labelled a, b and c respectively.
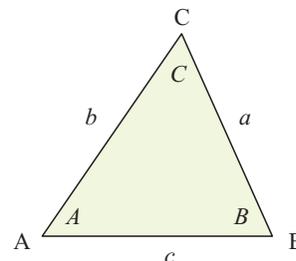
AREAS OF TRIANGLES

Until now we have only been able to calculate the area of a triangle given the lengths of one side and the altitude to that side.

We use   **area $= \frac{1}{2}bh$**.
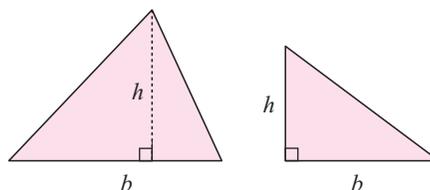
Now consider triangle ABC alongside.

Area of triangle ABC $= \frac{1}{2} \times$ AB $\times$ CN $= \frac{1}{2}ch$

However, $\sin A = \dfrac{h}{b}$

$\therefore \quad h = b \sin A$

$\therefore \quad$ area $= \frac{1}{2}c(b\sin A)$ So,   area $= \frac{1}{2}bc\sin A$
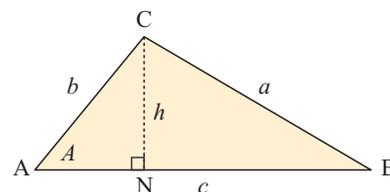
Similarly, if the altitudes from A and B were drawn, it could be shown that area $= \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

> The **area of a triangle** is a half of the product of two sides and the sine of the included angle.
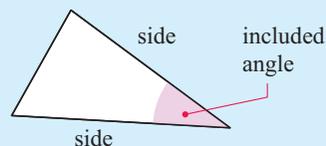
The formula   **area $= \frac{1}{2}ab\sin C$**   is worth remembering.

Example 3   ◀◎ **Self Tutor**

Find the area of $\triangle$ABC:

12 m

48°

C   15 m   A

B
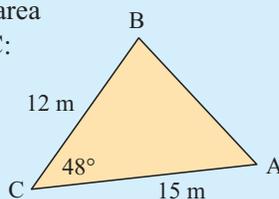
Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 12 \times 15 \times \sin 48^{o}$

≈ 66.9 m^2

EXERCISE 25C

1 Find the area of:

a

10 m

40°

8 m

b

87°

10.3 km   11.2 km

c

8 cm

121°

9 cm

d

12.8 m

30°

10.3 m

e

32°

3.8 m

f

1.7 km

132°

2.4 km

2

17 m

(1)

30°

20 m

17 m

150°   (2)

20 m

DEMO
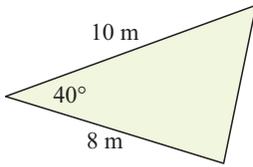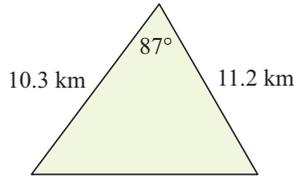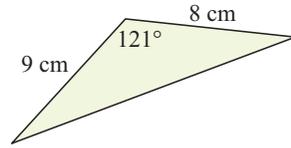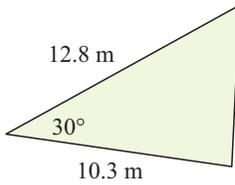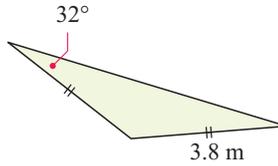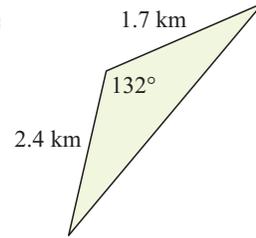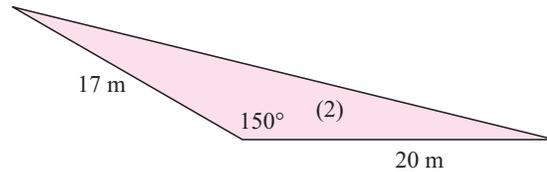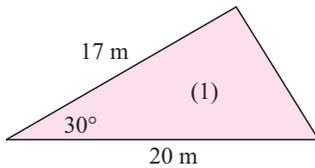
a Find the areas of triangles (1) and (2).

b Explain why the triangles have the same area.

3 The area of the illustrated rose garden is
127.3 m². Find x.

R

18 m

45°

P

Q x m
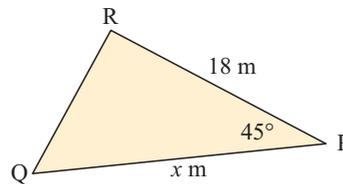
D   THE SINE RULE

Consider triangle ABC below:

A

c h b

B   C

B a C
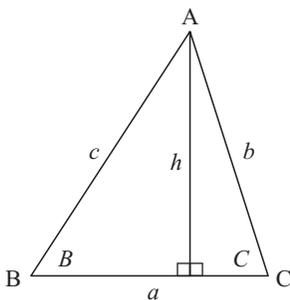
$$\sin B = \frac{h}{c} \qquad \text{and} \qquad \sin C = \frac{h}{b}$$

$$\therefore \quad h = c \sin B \qquad \text{and} \qquad h = b \sin C$$

$$\therefore \quad c \sin B = b \sin C$$

$$\therefore \quad \frac{c \sin B}{bc} = \frac{b \sin C}{bc} \qquad \{\text{dividing both sides by } bc\}$$

$$\therefore \quad \frac{\sin B}{b} = \frac{\sin C}{c}$$

Drawing the altitude from B to [AC] we could also show that:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Consequently, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

This result is known as the **sine rule**. It connects the lengths of the sides of any triangle with the sines of the opposite angles.

THE SINE RULE

In any triangle ABC with sides a, b and c units in length and opposite angles A, B and C respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
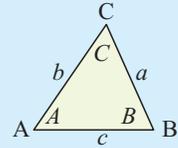
Proof: The area of any triangle ABC is given by $\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C.$

Dividing each expression by $\frac{1}{2}abc$ gives $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

If an **angle** needs to be found, we use $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

If a **side** needs to be found, we use $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$

FINDING SIDE LENGTHS

Example 4	◀) **Self Tutor**
Find the length of side [AB], correct to 1 decimal place.	Using the sine rule, $$\frac{AB}{\sin 68^o} = \frac{24.2}{\sin 52^o}$$ $\therefore \quad AB = \dfrac{24.2}{\sin 52^o} \times \sin 68^o$ $\therefore \quad AB \approx 28.5$ $\therefore \quad$ [AB] is approximately 28.5 m long.
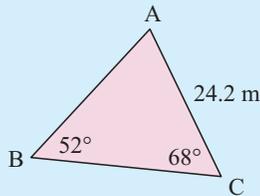
EXERCISE 25D.1

1 Find the value of x in:

a

b

c

d

51°

138 m

x m

44°

e

76°   4.83 m

x m

44°

f

31° x km

38°
213 km

2 Find *all* unknown sides and angles of:

a

θ

115 m x m

59°   71°

y m

b

83° x m

z m

ϕ

48°
215 m

c

ϕ 102°

x km

3.87 km

θ
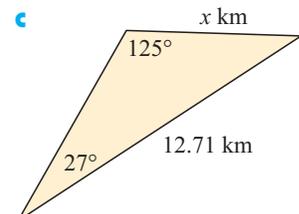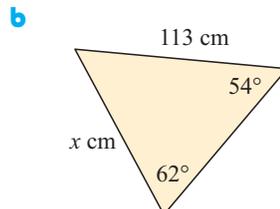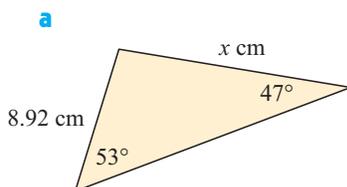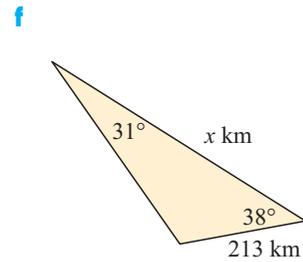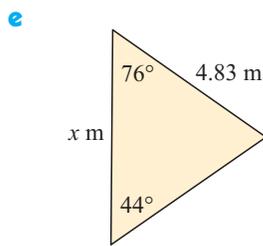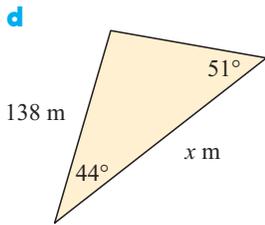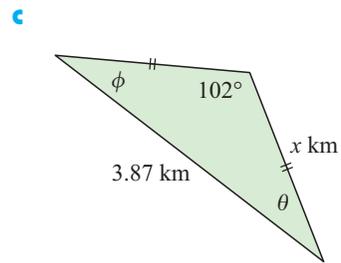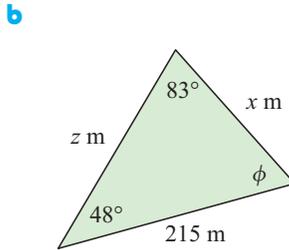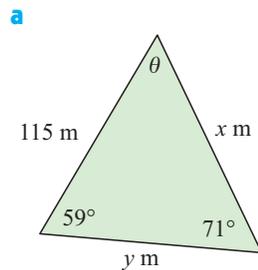
FINDING ANGLE SIZES

First of all we will consider finding the size of an angle in a diagram drawn approximately to scale. In such a case there is no confusion as to whether the unknown angle is acute or obtuse.

Example 5   ◀) **Self Tutor**

Find, correct to 1 decimal place, the value of θ in the following diagrams drawn to scale:

a

8 m   6 m

63°

θ

b

θ 23°

8 m   17 m
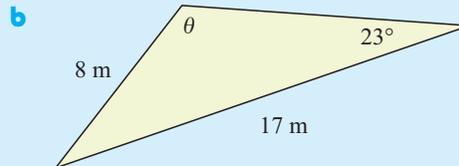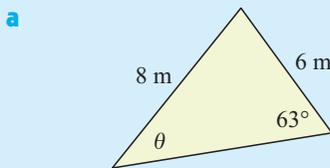
a Using the sine rule,

$$\frac{\sin \theta}{6} = \frac{\sin 63^o}{8}$$

$$\therefore \quad \sin \theta = \frac{6 \times \sin 63^o}{8}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{6 \times \sin 63^o}{8}\right)$$

{as θ is clearly acute}

$$\therefore \quad \theta \approx 41.9^o$$

b Using the sine rule,

$$\frac{\sin \theta}{17} = \frac{\sin 23^o}{8}$$

$$\therefore \quad \sin \theta = \frac{17 \times \sin 23^o}{8}$$

$$\therefore \quad \theta = 180^o - \sin^{-1}\left(\frac{17 \times \sin 23^o}{8}\right)$$

{as θ is clearly obtuse}

$$\therefore \quad \theta \approx 123.9^o$$

If a scale diagram is not provided, we must be careful to examine both acute and obtuse solutions. Sometimes other information given in the question may enable us to reject one of the solutions.

Example 6

◀) **Self Tutor**

Find, correct to 1 decimal place, the measure of angle B in triangle ABC given that $C\widehat{A}B = 33°$, $AC = 20$ cm   and $BC = 12.5$ cm.

Using the sine rule,

$$\frac{\sin B}{20} = \frac{\sin 33°}{12.5}$$

$$\therefore \quad \sin B = \frac{20 \times \sin 33°}{12.5}$$

Now $\sin^{-1}\left(\dfrac{20 \times \sin 33°}{12.5}\right) \approx 60.6°$

$$\therefore \quad B \approx 60.6° \quad \text{or} \quad 180° - 60.6°$$

$$\therefore \quad B \approx 60.6° \quad \text{or} \quad 119.4°$$

In this case both of these answers are feasible.

We can illustrate both answers
on an accurate scale diagram:
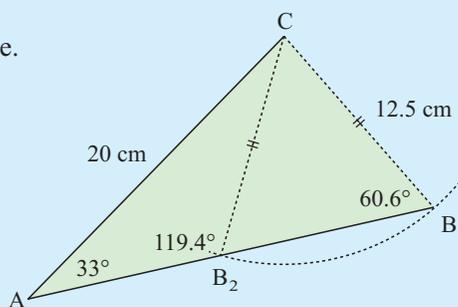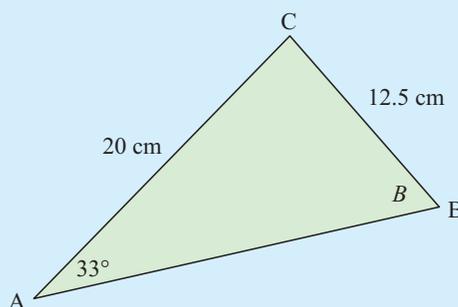
EXERCISE 25D.2

1 Find, correct to one decimal place, the value of θ in the following diagrams drawn roughly to scale.

a   113°, 8.1 m, θ, 14.9 m

b   138 m, 107 m, θ, 70°

c θ, 83 cm, 40°, 114 cm

d   71 m, θ, 31°, 128 m

e   35°, 3.87 km, θ, 2.62 km

f   1345 m, θ, 2170 m, 28°
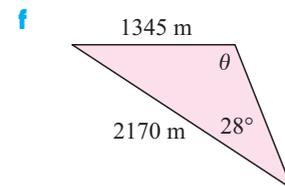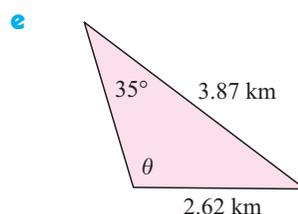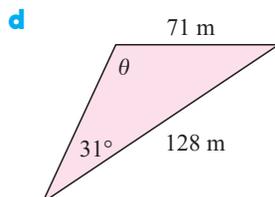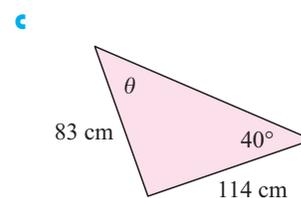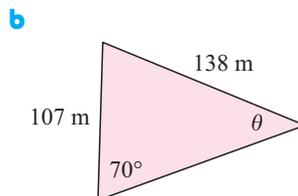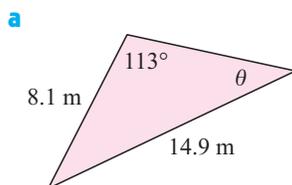
2 In triangle ABC, find correct to 1 decimal place the measure of:

   a angle A given   AC $= 4.9$ km,   BC $= 12.8$ km   and $A\widehat{B}C = 16^o$

   b angle B given   AB $= 28.6$ cm,   AC $= 34.9$ cm   and $B\widehat{C}A = 53^o$

   c angle C given   AB $= 147$ m,   AC $= 178$ m   and $A\widehat{B}C = 39^o$

   d angle B given   AB $= 8.3$ m,   AC $= 12.7$ m   and $A\widehat{C}B = 48^o$

   e angle A given   BC $= 125$ m,   AC $= 149$ m   and $A\widehat{B}C = 64.8^o$

   f angle B given   AC $= 38.1$ mm,   AB $= 28.1$ mm   and $A\widehat{C}B = 42.8^o$

   g angle C given   BC $= 5.62$ km,   AB $= 4.08$ km   and $C\widehat{A}B = 72.2^o$.

E   THE COSINE RULE

Consider triangle ABC shown.

Using Pythagoras' theorem, we find

$$b^2 = h^2 + x^2, \quad \text{so} \quad h^2 = b^2 - x^2$$

and $a^2 = h^2 + (c - x)^2$

Thus, $a^2 = (b^2 - x^2) + (c - x)^2$

$\therefore \quad a^2 = b^2 - x^2 + c^2 - 2cx + x^2$

$\therefore \quad a^2 = b^2 + c^2 - 2cx$ ...... (1)
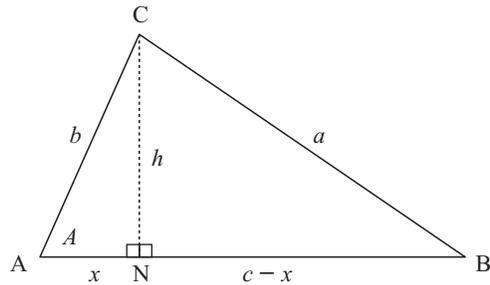
But in $\triangle$ACN, $\cos A = \dfrac{x}{b}$ and so $x = b \cos A$

So, in (1), $a^2 = b^2 + c^2 - 2bc \cos A$

THE COSINE RULE

In any triangle ABC with sides a, b and c units in length and opposite angles A, B and C respectively,

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$
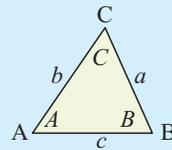
The **cosine rule** can be used to solve problems involving triangles given

- **two sides** and the **included angle**,   or   • **three sides**.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

are useful rearrangements of the cosine rule. They can be used if we are given all three side lengths of a triangle.

Example 7   ◄)) **Self Tutor**

Find, correct to 3 significant figures, the length of [AC].
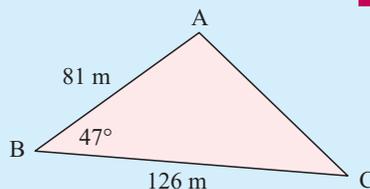
Using the cosine rule, $AC^2 = 126^2 + 81^2 - 2 \times 126 \times 81 \times \cos 47^o$

$\therefore \quad AC = \sqrt{126^2 + 81^2 - 2 \times 126 \times 81 \times \cos 47^o}$

$\therefore \quad AC \approx 92.3$

So, [AC] is approximately 92.3 m long.

Example 8   ◄)) **Self Tutor**

Find the size of $A\widehat{B}C$ in the given figure.
Give your answer correct to 1 decimal place.
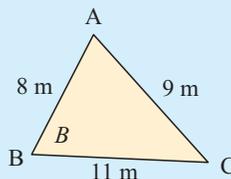
$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\therefore \quad \cos B = \dfrac{11^2 + 8^2 - 9^2}{2 \times 11 \times 8}$

$\therefore \quad B = \cos^{-1} \left(\dfrac{11^2 + 8^2 - 9^2}{2 \times 11 \times 8} \right)$

$\therefore \quad B \approx 53.8^o$

So, $A\widehat{B}C$ measures 53.8^o {to 1 decimal place}

EXERCISE 25E

1   Find the length of the remaining side of:

a

18 m
54°
20 m

b

9.3 m
32°
11.8 m

c

21 km
83°
17 km

d

114°   8 cm
7 cm

e

102 m
126°
117 m

f

7.6 km
9.2 km   141°
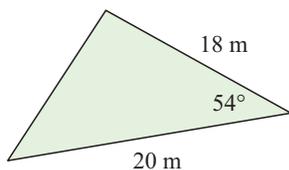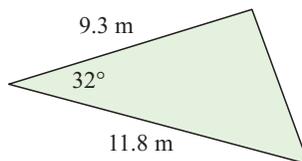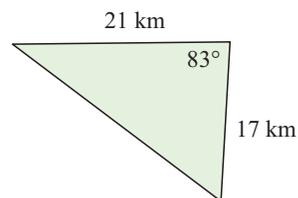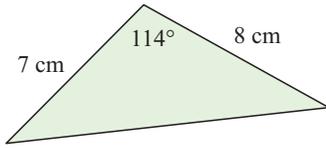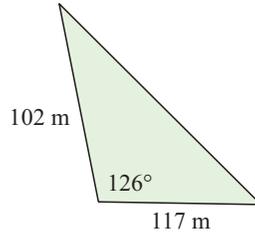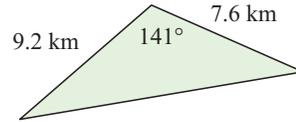
2 Find, correct to 1 decimal place, the unknown in:

a

7 m   8 m
θ
9 m

b

8 m
θ
6 m
10 m

c

8.1 km
ϕ
5.6 km
8.3 km

d

ϕ
5 m   4 m
7 m

e

14 m
α
32 m
47 m

f

127 m
β
83 m

3

A
θ
9.6 m   7.2 m
B   C
16.8 m
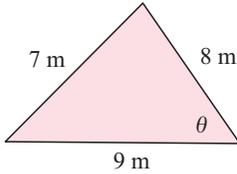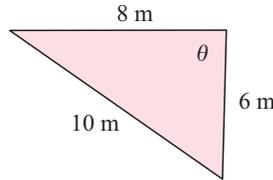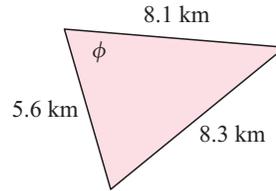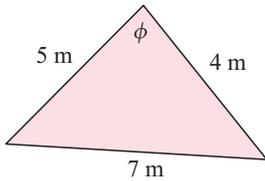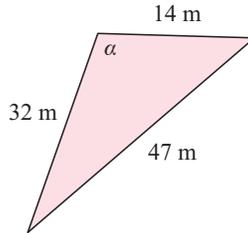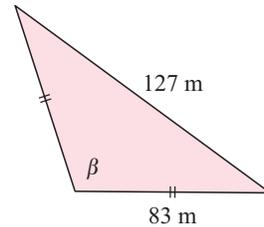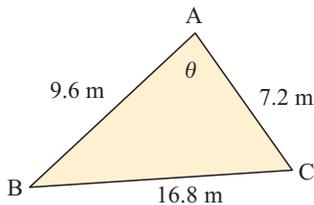
a Find θ to the nearest degree.
b Explain your answer to **a**.

4 Find the measure of all unknown sides and angles of:

a

β
6 m x m
α
44°   9 m

b

β 8 m
6 m θ
α 12 m
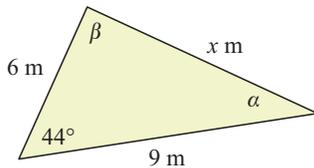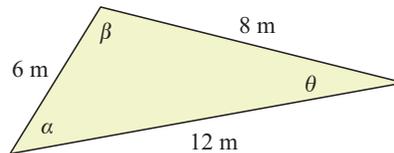
F   PROBLEM SOLVING WITH THE SINE AND COSINE RULES

In the following questions you should draw a diagram of the situation. The diagram should be reasonably accurate and all important information should be clearly marked on it.

Example 9 ◀⁣)) **Self Tutor**

A triangle has side lengths 11 cm, 13 cm and 14 cm. What is the size of the smallest angle of the triangle?

The smallest angle must be opposite the shortest side.
Let this angle be θ.

Now $\quad \cos\theta = \dfrac{13^2 + 14^2 - 11^2}{2 \times 13 \times 14} \qquad$ {cosine rule}

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{13^2 + 14^2 - 11^2}{2 \times 13 \times 14}\right)$

$\therefore \quad \theta \approx 47.9^o$

So, the smallest angle measures 47.9^o (to 3 s.f.)
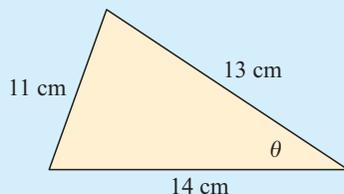
Example 10 ◀⁣)) **Self Tutor**

A ship sails for 58 km on the bearing 072°. Once it has passed a reef, it can then turn and sail for 41 km on the bearing 158°. How far is the ship from its starting point?
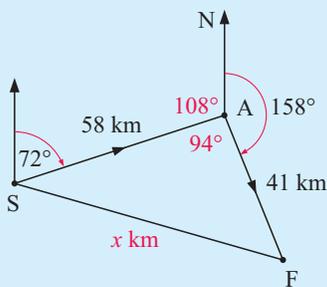
We suppose the ship starts at S, sails to A, then changes direction and sails to F.

$\widehat{SAN} = 180^o - 72^o = 108^o$
$\qquad\qquad$ {cointerior angles}

$\therefore \quad \widehat{SAF} = 360^o - 158^o - 108^o$
$\qquad\qquad = 94^o \quad$ {angles at a point}

Let $\quad SF = x$ km.

Using the cosine rule, $\qquad x^2 = 58^2 + 41^2 - 2 \times 58 \times 41 \times \cos 94^o$

$\therefore \quad x = \sqrt{58^2 + 41^2 - 2 \times 58 \times 41 \times \cos 94^o}$

$\therefore \quad x \approx 73.3$

$\therefore \quad$ the ship is about 73.3 km from the starting point.

EXERCISE 25F

1 a Find the smallest angle of a triangle with side lengths 5 cm, 6 cm and 8 cm.

$\quad$ **b** Find the largest angle of a triangle with side lengths 8 cm, 11 cm and 16 cm.

2 A and B are two radar stations 86 km apart along the coast. A ship located at point C is in distress. Radar station A says the angle CAB is 47°, while radar station B says that angle CBA is 68°.

$\quad$ **a** Draw a reasonably accurate diagram of the situation, showing all given information.

$\quad$ **b** Rescue squads are located at A and B. Which squad should be used if conditions for the rescue are ideal?

$\quad$ **c** What distance does the squad need to travel to get to the rescue site?

3 Fred walks for 83 m in the direction $111°$ and then for 78 m in the direction $214°$. How far is Fred from his starting point?

4
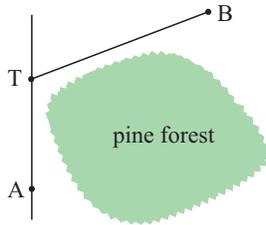
A and B are two towns on either side of a pine forest. To get from A to B by car you need to turn off the main road at T and then drive to B. The turn off point is 43 km from A and 61 km from B, and $\text{A}\widehat{\text{T}}\text{B} = 116°$.

Find the distance travelled by a helicopter flying directly from A to B.

5 The shortest distance from a tee T to the flag F on a golf hole is 318 m. Sharon drives a ball to her left to point X on the fairway. Given that X is 93 m from the flag and $\text{X}\widehat{\text{F}}\text{T}$ is $39°$, find the angle θ that Sharon's drive was off line.
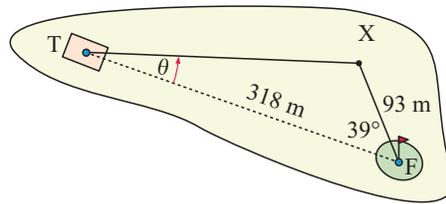
6 A field is triangular with sides of length 213 m, 178 m and 198 m. Find one corner angle, and hence find the area of the field in hectares.

7 A boat travels 13 km in the direction $138°$ and then a further 11 km in the direction $067°$. Find the distance and bearing of the boat from its starting point.

8 [XY] and [RS] are roads which intersect at C at an angle of $77°$. There is an explosion at a factory at point D on [CS].

Observers at A and B note that $\text{C}\widehat{\text{A}}\text{D}$ and $\text{C}\widehat{\text{B}}\text{D}$ are $28°$ and $38°$ respectively. A and B are 32.8 km apart.

  a Find θ.

  b Find the distance [BD].

  c How far is D from the intersection C?
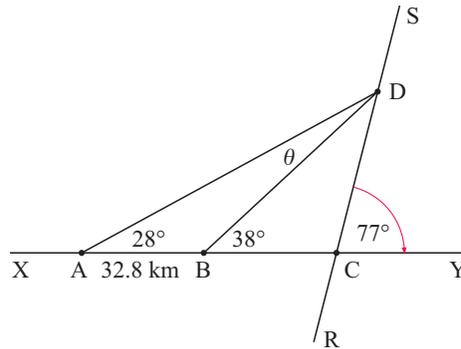
9 Answer the **Opening Problem** on page **502**.

REVIEW SET 25A

1
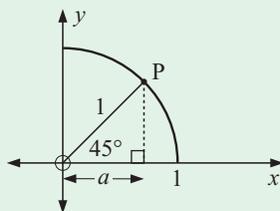
Use the given diagram and Pythagoras' theorem to show that $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$.

2 Copy and complete:

  a If $\sin \theta = a$ then $\sin(180° - \theta) = $ ......

  b If $\cos \theta = b$ then $\cos(180° - \theta) = $ ......

3
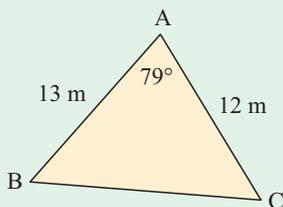
Find:
a the area of triangle ABC to the nearest m²
b the length of side [BC] correct to 1 decimal place.

4 Triangle PQR has area 82.2 m².
Find the length of [RQ].
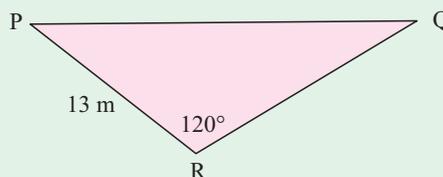
5 Find the value of x in:

a
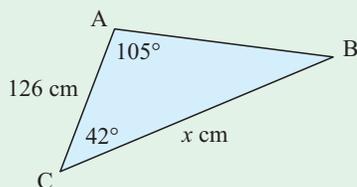
b
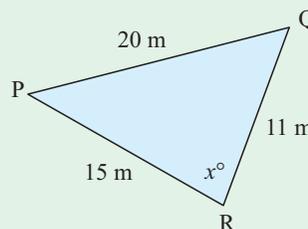
6 Triangle ABC has   AC = 20 m,   BC = 15.5 m,   and $\widehat{\text{CAB}} = 50°$.
Find the size of $\widehat{\text{ABC}}$.

7 A surveyor at point A measures off 200 m in the direction $138°$ to point B. The surveyor then measures off 150 m in the direction $256°$ to point C.
a How far is C from A?
b What is the bearing of C from A?

REVIEW SET 25B

1
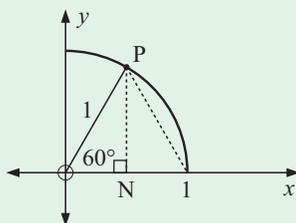
Use the given diagram and Pythagoras' theorem to show that $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$.

2 θ is an angle of a triangle. What can be deduced if:
a $\cos\theta = -\frac{1}{3}$ **b** $\sin\theta = \frac{1}{2}$?

3 Triangle ABC has   AB = 11 m,   BC = 13 m,   and   AC = 19 m.
a Find, correct to 2 decimal places, the size of the angle at B.
b Hence, find the area of $\triangle$ABC correct to 1 decimal place.
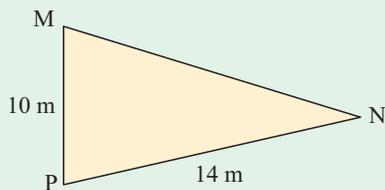
4   Triangle MNP has an area of 60 m^2.
Find the size of the acute angle at P.

M, 10 m, P, 14 m, N

5 Find all unknown sides and angles in:

θ, x m, y m, 38°, 43°, 23.8 m

6

C, x cm, 11 cm, E, α, 14 cm, D
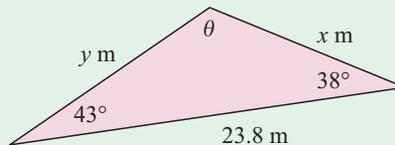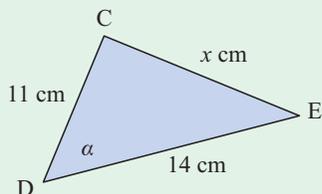
It is known that $\cos \alpha = \frac{1}{4}$.

a Show that $x = 4\sqrt{15}$ using the cosine rule.

b If $C\widehat{E}D = \beta$, show that $\cos \beta = k\sqrt{15}$ for some rational number k.

7 Triangle PQR has $Q\widehat{P}R = 60°$, PQ = 27 cm, and QR = 28 cm.

a If $Q\widehat{R}P = \theta$, find the two possible values that θ may take.

b Show that only one of the values in **a** is possible.

Chapter 26

Variation

PRINTABLE CHAPTER

Click on the icon to access this printable chapter

Contents: **A**   Direct variation
              B   Inverse variation

Chapter **27**

PRINTABLE CHAPTER

Click on the icon to access this printable chapter

Two variable analysis

Contents:

Chapter **28**

PRINTABLE CHAPTER

Click on the icon to access this printable chapter

Logic

Contents:

EXERCISE 1A

1   **a** double a **b** the product of p and q
   c the square root of m **d** the square of a **e** 3 less than a
   f the sum of b and c **g** the sum of double x and c
   h the square of twice a **i** twice the square of a
   j the difference between a and the square of c, where $a > c^2$
   k the sum of a and the square of b
   l the square of the sum of a and b

2   **a** $a + c$ **b** $p + q + r$ **c** ab **d** $r + s^2$
   e $(r + s)^2$ **f** $r^2 + s^2$ **g** $2a + b$ **h** $p - q$
   i $b^2 - a$ **j** $\dfrac{a+b}{2}$ **k** $a + \dfrac{b}{4}$ **l** $\sqrt{m+n}$
   m $x + \dfrac{1}{x}$ **n** $\frac{1}{4}(a+b)$ or $\dfrac{a+b}{4}$ **o** $\sqrt{x^2 + y^2}$

3   **a** L is equal to the sum of a and b.
   b K is equal to the average of a and b.
   c M is equal to 3 times d.
   d N is equal to the product of b and c.
   e T is equal to the product of b and the square of c.
   f F is equal to the product of m and a.
   g K is equal to the square root of the quotient of n and t.
   h c is equal to the square root of the sum of the squares of a and b.
   i A is equal to the average of a, b and c.

4   **a** $S = p + r$ **b** $D = b - a$ **c** $A = \dfrac{k+l}{2}$
   d $M = a + \dfrac{1}{a}$ **e** $K = t + s^2$ **f** $N = gh$
   g $y = x + x^2$ **h** $P = \sqrt{d + e}$

EXERCISE 1B

1   **a** 25   **b** 12   **c** 45   **d** -60 **e** 9   **f** -32
   g 20   **h** -25

2   **a** $-\frac{2}{3}$ **b** 1   **c** $\frac{5}{3}$ **d** $-\frac{13}{3}$ **e** 0   **f** $-\frac{1}{3}$
   g -7 **h** -5

3   **a** 1   **b** -64 **c** 25   **d** 49   **e** -65 **f** -125
   g 36   **h** 18

4   **a** 1   **b** $\sqrt{3} \approx 1.73$ **c** $\sqrt{3} \approx 1.73$ **d** $\sqrt{8} \approx 2.83$
   e 3   **f** $\sqrt{17} \approx 4.12$ **g** 2   **h** undefined

EXERCISE 1C.1

1   **a** $x = -5$ **b** $x = 9$ **c** $x = 4$ **d** $x = -9$
   e $x = 6$ **f** $x = -4$ **g** $x = 5$ **h** $x = \frac{1}{2}$

2   **a** $x = 48$ **b** $x = 12$ **c** $x = -10$ **d** $x = -18$
   e $x = -13$ **f** $x = 7$ **g** $x = 11$ **h** $x = 9$

EXERCISE 1C.2

1   **a** $x = 7$ **b** $x = -1$ **c** $x = -7$ **d** $x = 4$
   e $x = -3$ **f** $x = -2$

2   **a** $x = -9$ **b** $x = \frac{9}{4}$ **c** $x = 1\frac{1}{2}$ **d** $x = -\frac{4}{3}$
   e $x = \frac{1}{2}$ **f** $x = \frac{5}{4}$ **g** no soln   **h** $x = 0$
   i $x = 2\frac{1}{2}$ **j** $x = -\frac{13}{10}$

3   **a** True for all $x \in \mathbb{R}$ as both sides are $6x + 1$.
   b There are no values of x that make this equation true as $3 \neq 6$.
   c There are an infinite number of solutions in **a**.
     There are no solutions for part **b**.

EXERCISE 1D

1   **a** $x = \frac{8}{7}$ **b** $x = \frac{15}{4}$ **c** $x = -4$ **d** $x = 3\frac{1}{2}$
   e $x = \frac{15}{7}$ **f** $x = \frac{9}{4}$ **g** $x = \frac{6}{5}$ **h** $x = \frac{7}{5}$ **i** $x = \frac{2}{15}$

2   **a** $x = \frac{15}{2}$ **b** $x = 10$ **c** $x = \frac{15}{4}$ **d** $x = \frac{9}{7}$
   e $x = \frac{9}{14}$ **f** $x = -14$ **g** $x = -15$ **h** no soln

3   **a** $x = -4$ **b** $x = -\frac{1}{3}$ **c** $x = 8$ **d** $x = -10$
   e $x = 3$ **f** $x = -\frac{5}{3}$ **g** $x = 1$ **h** $x = 15$ **i** $x = 0$

4   **a** $x = 12$ **b** $x = -\frac{36}{5}$ **c** $x = -\frac{16}{5}$ **d** $x = \frac{13}{7}$
   e $x = 16$ **f** $x = \frac{40}{7}$ **g** $x = \frac{16}{3}$ **h** $x = \frac{11}{5}$
   i $x = \frac{55}{28}$ **j** $x = \frac{4}{5}$ **k** $x = -\frac{6}{5}$ **l** $x = \frac{17}{24}$

EXERCISE 1E

1   **a** $x < -\frac{2}{3}$ **b** $x > \frac{9}{5}$ **c** $x \leqslant \frac{1}{3}$
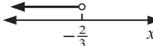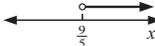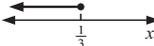
   d $x \geqslant -3$ **e** $x < 1$ **f** $x \leqslant -\frac{1}{5}$
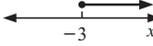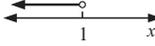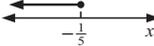
2   **a** $x \leqslant 4$ **b** $x > -5$ **c** $x > -4$
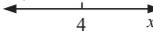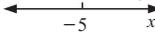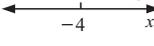
   d $x \geqslant \frac{7}{3}$ **e** $x < 1$ **f** $x \leqslant \frac{3}{5}$
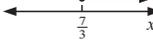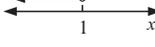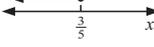
3   **a** $x > -\frac{7}{2}$ **b** $x > \frac{4}{3}$ **c** $x \leqslant \frac{1}{3}$
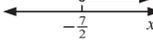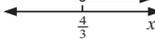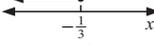
   d $x \geqslant 1$ **e** $x < 0$ **f** $x \leqslant -\frac{5}{3}$
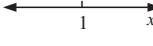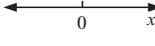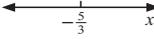
4   **a** no solutions as $1 \ngtr 6$ **b** true for all real x, i.e., $x \in \mathbb{R}$
   c true for all real x, i.e., $x \in \mathbb{R}$
   d No solutions in **a**, an infinite number of solutions for **b**, and an infinite number of solutions in **c** when $2x - 4 = 2(x - 2)$.

EXERCISE 1F

1 7   **2** 6   **3** $2\frac{1}{2}$ **4** 24   **5** 6 and 9   **6** 19   **7** $2\frac{1}{2}$
8 10 years old   **9** 6 years old   **10** 7 years old

EXERCISE 1G

1 35×5 cent, 40×10 cent   **2** 35×600 mL cartons
3 8×5 cent, 60×10 cent, 20×25 cent
4 $8500 \times$ €8, $17\,000 \times$ €15, $23\,000 \times$ €20
5 20 kg of brand A and 30 kg of brand B   **6** 7 kg
7 **1** \$3650   **2** $2.5x + 4(x + 100)$ dollars
    3 850 XBC, 950 NLG shares
8 \$6550
9 \$5000 in mining shares and \$10\,000 in technology shares
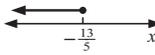**10** \$5000 in A, \$10\,000 in B, \$35\,000 in C   **11** £9000

EXERCISE 1H

1 16 km h^{-1} **2** 84 km h^{-1} **3** 10 km   **4** 50 km h^{-1}
5 48 km   **6** $3\frac{3}{7}$ km h^{-1}

EXERCISE 1I

1 250 mL   **2** 3 litres   **3** 48 litres   **4** $3\frac{1}{3}$ litres

REVIEW SET 1A

1 **a** $x^2 + 3$ **b** $(3 + x)^2$

2 **a** the sum of the square root of a and 3
b the square root of the sum of a and 3

3 3 **4** **a** $x = \frac{1}{5}$ **b** $x = \frac{47}{39}$

5 $x \leqslant -\frac{13}{5}$

6 $-\frac{27}{7}$

7 114 small cups, 342 medium cups, 202 large cups **8** 200 km

REVIEW SET 1B

1 **a** the average of a and b **b** the sum of a and a half of b

2 **a** $a + \sqrt{b}$ **b** $\sqrt{a + b}$ **3** 5

4 $x < \frac{1}{2}$
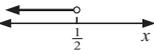
5 **a** $x = 3\frac{1}{2}$ **b** $x = -15$

6 $-2\frac{1}{2}$ **7** ¥1 700 000 **8** 24 km h^{-1}

EXERCISE 2A.1

1 **a** 16 **b** 125 **c** 64 **d** 343
e 1350 **f** 1008 **g** 37 125 **h** 5200

2 **a** 2×3^3 **b** $2^3 \times 3^2$ **c** $2^2 \times 5^2$ **d** $2^4 \times 3 \times 5$
e $2 \times 3^3 \times 5 \times 7$ **f** $2 \times 3^2 \times 7^2$ **g** $3^2 \times 11^2$ **h** $3^3 \times 5^3$

3 **a** $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64$
b $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$
c $5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625$
d $7^1 = 7, 7^2 = 49, 7^3 = 343$

4 **a** $n = 4$ **b** $n = 7$ **c** $n = 9$

5 **a** $n = 3$ **b** $n = 4$ **c** $n = 6$ **6** 4 **7** 1, 3

EXERCISE 2A.2

1 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1 **f** -1
g -1 **h** -8 **i** -8 **j** 8 **k** -25 **l** 125

EXERCISE 2A.3

1 **a** 512 **b** -3125 **c** -243 **d** 16 807 **e** 512
f 6561 **g** -6561 **h** 5.117 264 691
i $-0.764 479 956$ **j** $-20.361 584 96$

2 **a** $0.\overline{142857}$ **b** $0.\overline{142857}$ **c** $0.\overline{1}$ **d** $0.\overline{1}$
e 0.015 625 **f** 0.015 625 **g** 1 **h** 1

3 **a** It is the reciprocal of the number raised to the positive power.
b Any non-zero number raised to the power zero is 1.

EXERCISE 2B

1 **a** 7^5 **b** 5^7 **c** a^9 **d** a^5 **e** b^{13}
f a^{3+n} **g** b^{7+m} **h** m^9

2 **a** 5^7 **b** 11^4 **c** 7^3 **d** a^4 **e** b^3
f p^{5-m} **g** y^{a-5} **h** b^{2x-1}

3 **a** 3^8 **b** 5^{15} **c** 2^{28} **d** a^{10} **e** p^{20}
f b^{5n} **g** x^{3y} **h** a^{10x}

4 **a** 2^3 **b** 5^2 **c** 3^4 **d** 2^6 **e** 3^4 **f** 3^{a+2}
g 5^{t-1} **h** 3^{3k} **i** 2^{4-x} **j** 3^2 **k** 5^{4x-4}
l 2^2 **m** 2^{y-2x} **n** 2^{2y-3x} **o** 3^{2x} **p** 2^3

5 **a** a^3b^3 **b** a^4c^4 **c** b^5c^5 **d** $a^3b^3c^3$
e $16a^4$ **f** $25b^2$ **g** $81n^4$ **h** $8b^3c^3$
i $64a^3b^3$ **j** $\dfrac{a^3}{b^3}$ **k** $\dfrac{m^4}{n^4}$ **l** $\dfrac{32c^5}{d^5}$

6 **a** $8b^{12}$ **b** $\dfrac{9}{x^4y^2}$ **c** $25a^8b^2$ **d** $\dfrac{m^{12}}{16n^8}$
e $\dfrac{27a^9}{b^{15}}$ **f** $32m^{15}n^{10}$ **g** $\dfrac{16a^8}{b^4}$ **h** $125x^6y^9$

7 **a** a^2 **b** $8b^5$ **c** m^3n **d** $7a^5$ **e** $4ab^2$
f $\dfrac{9m^3}{2}$ **g** $40h^5k^3$ **h** $\dfrac{1}{m^5}$ **i** p^3

8 **a** 1 **b** $\frac{1}{3}$ **c** $\frac{1}{6}$ **d** 1 **e** 9 **f** $\frac{1}{9}$
g 8 **h** $\frac{1}{8}$ **i** 25 **j** $\frac{1}{25}$ **k** 100 **l** $\frac{1}{100}$

9 **a** 1 **b** 1 **c** 3 **d** 1 **e** 2 **f** 1 **g** $\frac{1}{25}$
h $\frac{1}{32}$ **i** 3 **j** $\frac{5}{2}$ **k** $\frac{3}{4}$ **l** 12 **m** $\frac{3}{2}$ **n** $\frac{4}{5}$
o $\frac{8}{7}$ **p** $\frac{7}{2}$ **q** $\frac{16}{9}$ **r** $\frac{8}{27}$ **s** $\frac{4}{25}$ **t** $\frac{9}{49}$

10 **a** $\dfrac{1}{2a}$ **b** $\dfrac{2}{a}$ **c** $\dfrac{3}{b}$ **d** $\dfrac{1}{3b}$ **e** $\dfrac{b^2}{4}$ **f** $\dfrac{1}{4b^2}$
g $\dfrac{1}{9n^2}$ **h** $\dfrac{n^2}{3}$ **i** $\dfrac{a}{b}$ **j** $\dfrac{1}{ab}$ **k** $\dfrac{a}{b^2}$ **l** $\dfrac{1}{a^2b^2}$
m $\dfrac{1}{2ab}$ **n** $\dfrac{2}{ab}$ **o** $\dfrac{2a}{b}$ **p** a^2b^3

11 **a** m s^{-1} **b** m^3 h^{-1} **c** cm^2 s^{-1} **d** cm^3 min^{-1}
e g s^{-1} **f** m s^{-1} s^{-1} or m s^{-2}

12 **a** 3^{-1} **b** 2^{-1} **c** 5^{-1} **d** 2^{-2} **e** 3^{-3} **f** 5^{-2}
g 2^{-3x} **h** 2^{-4y} **i** 3^{-4a} **j** 3^{-2} **k** 5^{-2} **l** 5^{-3}
m 2^4 **n** $2^0 = 3^0 = 5^0$ **o** $2^{-3} \times 3^{-3}$ **p** $2^4 \times 5^2$

13 25 days **14** 63 sums

EXERCISE 2C

1 **a** $x = 1$ **b** $x = 2$ **c** $x = 3$ **d** $x = 0$
e $x = -1$ **f** $x = -1$ **g** $x = -3$ **h** $x = 2$
i $x = 0$ **j** $x = -4$ **k** $x = 5$ **l** $x = 1$

2 **a** $x = 2\frac{1}{2}$ **b** $x = -\frac{2}{3}$ **c** $x = -\frac{2}{3}$ **d** $x = -\frac{1}{2}$
e $x = -1\frac{1}{2}$ **f** $x = -\frac{1}{2}$ **g** $x = -\frac{1}{3}$ **h** $x = \frac{5}{3}$
i $x = -1$ **j** $x = \frac{5}{2}$ **k** $x = -2$ **l** $x = -4$
m $x = 0$ **n** $x = -\frac{5}{2}$ **o** $x = -\frac{1}{2}$ **p** $x = -\frac{8}{3}$

EXERCISE 2D

1 **a** 2.3×10^2 **b** 5.39×10^4 **c** 3.61×10^{-2}
d 6.8×10^{-3} **e** 3.26×10^0 **f** 5.821×10^{-1}
g 3.61×10^8 **h** 1.674×10^{-6}

2 **a** 2300 **b** 0.023 **c** 564 000 **d** 0.000 793 1
e 9.97 **f** 60 400 000 **g** 0.4215 **h** 0.000 000 036 21

3 **a** 4×10^6 cells **b** 8×10^{-4} m **c** 6.38×10^6 m
d 2×10^{-5} cm

4 **a** 1 392 000 km **b** 0.018 cm **c** 32 000 000 bacteria
d 0.000 008 2 tonnes **e** 37 000 000 min **f** 0.000 021 m

5 **a** 1.2×10^{11} **b** 2.8×10^9 **c** 5.6×10^{-8}
d 5.4×10^{-6} **e** 9×10^{10} **f** 1.6×10^{15}
g 1.6×10^{-11} **h** 5.12×10^{-7} **i** 1.2×10^0 **j** 2×10^7

6 **a** 2×10^3 **b** 3×10^{-2} **c** 2×10^8
d 1×10^{-19} **e** 3.2×10^3 **f** 8×10^8

7 **a** 4.00×10^{13} **b** 2.59×10^7 **c** 7.08×10^{-9}
d 4.87×10^{-11} **e** 8.01×10^6 **f** 3.55×10^{-9}

8 $\approx 2.14 \times 10^{22}$ **9** $\approx 3.24 \times 10^8$

10 **a** 1000 times **b** 5×10^{-21}, 100 000 times

11 **a** **i** 2.88×10^8 km **ii** 7.01×10^{10} km **b** 0.9 s
c ≈ 26.6 times bigger **d** Microbe C, 32.9 times heavier

EXERCISE 2E

1 **a** 2 **b** $\frac{1}{2}$ **c** 3 **d** $\frac{1}{3}$ **e** 6 **f** $\frac{1}{6}$
g 2 **h** $\frac{1}{2}$ **i** 10 **j** $\frac{1}{10}$ **k** 3 **l** $\frac{1}{3}$

2 **a** $11^{\frac{1}{2}}$ **b** $11^{-\frac{1}{2}}$ **c** $12^{\frac{1}{2}}$ **d** $12^{-\frac{1}{2}}$ **e** $26^{\frac{1}{3}}$
f $26^{-\frac{1}{3}}$ **g** $7^{\frac{1}{4}}$ **h** $7^{-\frac{1}{5}}$

3 **a** $2^{\frac{1}{4}}$ **b** $2^{\frac{1}{5}}$ **c** $2^{\frac{1}{2}}$ **d** $2^{\frac{4}{3}}$ **e** $2^{-\frac{3}{4}}$
f $2^{-\frac{4}{3}}$ **g** $2^{-\frac{3}{7}}$ **h** $2^{-\frac{6}{5}}$

4 **a** $3^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $3^{-\frac{1}{4}}$ **d** $3^{-\frac{2}{3}}$

5 **a** 8 **b** 32 **c** 4 **d** 32 **e** 8 **f** 27
g $\frac{1}{27}$ **h** $\frac{1}{2}$ **i** $\frac{1}{8}$ **j** $\frac{1}{4}$ **k** $\frac{1}{81}$ **l** $\frac{1}{8}$

6 **a** 8 **b** 9 **c** 16 **d** 2.41 **e** 2.68 **f** 91.2
g 1.93 **h** 849 **i** 2.13 **j** 9.16 **k** 0.0313
l 0.004 12 **m** 0.339 **n** 0.182 **o** 0.215

REVIEW SET 2A

1 **a** -1 **b** 27 **c** $\frac{2}{3}$ **2** **a** $a^6 b^7$ **b** $\frac{2}{3x}$ **c** $\frac{y^2}{5}$

3 **a** 2^{-3} **b** 2^7 **c** 2^{12} **4** **a** $\frac{1}{b^3}$ **b** $\frac{1}{ab}$ **c** $\frac{a}{b}$

5 **a** 8×10^{15} **b** 7.5×10^{12} **6** **a** $x = -2$ **b** $x = \frac{3}{4}$

7 **a** 4 **b** $\frac{1}{9}$ **8** **a** 2.28 **b** 0.517 **c** 3.16
9 100 times **10** 1.45×10^{13}

REVIEW SET 2B

1 **a** 8 **b** $-\frac{4}{5}$ **2** **a** a^{21} **b** $p^4 q^6$ **c** $\frac{4b}{a^3}$

3 **a** 2^{-4} **b** 2^{x+2} **c** 2^{2x-3}

4 **a** $\frac{1}{x^5}$ **b** $\frac{2}{a^2 b^2}$ **c** $\frac{2a}{b^2}$ **5** **a** $x = 4$ **b** $x = -\frac{2}{5}$

6 **a** 6×10^{10} **b** 8.1×10^{-5} **7** **a** 8 **b** $\frac{1}{125}$
8 **a** 2.52 **b** 0.224 **c** 3.11 **9** 1 million times
10 4.19×10^{24}

EXERCISE 3A

1 **a** $a+9$ **b** $9+a$ **c** $m+3$ **d** $2x+1$ **e** $2f-3$
f $6a$ **g** $4a$ **h** $-4a$ **i** cannot be simplified
j $2d^2 + d$ **k** cannot be simplified **l** $-4x^2 + 5$
m $5a - 5$ **n** $4a$ **o** $5xy$ **p** $2x^2 z$

2 **a** 0 **b** $6a$ **c** cannot be simplified **d** $3xy$ **e** $-cd$
f $3p^2$ **g** $3x + 7$ **h** $4a - 2$ **i** $2x + 5y$ **j** $2m^2 + m$
k $3ab + 1$ **l** $2x - 5$ **m** $3x^2 + 2x$
n cannot be simplified **o** $x^2 - 10x$

3 **a** $3x + 4$ **b** cannot be simplified **c** $2ab$ **d** $5x^2 - 5$
e $3p^2 - 7$ **f** $a - 3$ **g** $-4a + b$ **h** cannot be simplified
i $a^2 + a^3$ **j** cannot be simplified **k** $xy^2 + 2x^2 y$
l $3x^3 - 3x^2$

EXERCISE 3B

1 **a** bc **b** $2ab$ **c** xy^2 **d** $2pq^2$

2 **a** $6x$ **b** $20x$ **c** $-14x$ **d** $-6x$ **e** $2x^2$ **f** $6x^2$
g $-2x^2$ **h** $-12x$ **i** $2x^2$ **j** $-3x^3$ **k** $2x^3$
l $-6d^2$ **m** a^2 **n** $4a^2$ **o** $2a^4$ **p** $-3a^3$

3 **a** $22x$ **b** $15x - 2y^2$ **c** $11x^2$ **d** $5ab$ **e** x^2
f $-xy$ **g** $7ab$ **h** $-2cd$ **i** $3ab - 2ac$

EXERCISE 3C

1 **a** $3x + 3$ **b** $10 - 2x$ **c** $-x - 2$ **d** $x - 3$
e $4a + 8b$ **f** $6x + 3y$ **g** $5x - 5y$ **h** $-6x^2 + 6y^2$
i $-2x - 8$ **j** $-6x + 3$ **k** $x^2 + 3x$ **l** $2x^2 - 10x$
m $-3x - 6$ **n** $-4x + 12$ **o** $x - 3$ **p** $-2x + 2y$
q $a^2 + ab$ **r** $ab - a^2$ **s** $2x^2 - x$ **t** $2x^3 - 2x^2 - 4x$

2 **a** $2x + 5$ **b** $1 - 4x$ **c** $3x - 1$ **d** $2 - 4x$ **e** x^2
f $6x - x^2$ **g** $2ab + a^2$ **h** $7x - 3x^2$ **i** $2x^2 - 10x$

3 **a** $5x - 2$ **b** $3a - 2b$ **c** $a + 2b$ **d** $15 - 3y$ **e** $-6y - 10$
f $15x - 8$ **g** $a + 5b$ **h** $x^2 + 6x - 6$ **i** $x^2 + 2x + 6$
j $2x^2 - x$ **k** $-2x^2 + 2x$ **l** $x^2 - y^2$ **m** $5 - 3x$
n $8x - 8$ **o** $6x^2 - 22x$

EXERCISE 3D

1 **a** $A_2 = ac$ **b** $A_2 = ad$ **c** $A_3 = bc$ **d** $A_4 = bd$
e $A = (a+b)(c+d)$
$(a+b)(c+d) = ac + ad + bc + bd$

2 **a** $x^2 + 10x + 21$ **b** $x^2 + x - 20$ **c** $x^2 + 3x - 18$
d $x^2 - 4$ **e** $x^2 - 5x - 24$ **f** $6x^2 + 11x + 4$
g $1 + 2x - 8x^2$ **h** $12 + 5x - 2x^2$ **i** $6x^2 - x - 2$
j $25 - 10x - 3x^2$ **k** $7 + 27x - 4x^2$ **l** $25x^2 + 20x + 4$

3 **a** $x^2 - 4$ **b** $a^2 - 25$ **c** $16 - x^2$ **d** $4x^2 - 1$
e $25a^2 - 9$ **f** $16 - 9a^2$

4 **a** $x^2 + 6x + 9$ **b** $x^2 - 4x + 4$ **c** $9x^2 - 12x + 4$
d $1 - 6x + 9x^2$ **e** $9 - 24x + 16x^2$ **f** $25x^2 - 10xy + y^2$

EXERCISE 3E

1 **a** $x^2 - 4$ **b** $x^2 - 4$ **c** $4 - x^2$ **d** $4 - x^2$
e $x^2 - 1$ **f** $1 - x^2$ **g** $x^2 - 49$ **h** $c^2 - 64$
i $d^2 - 25$ **j** $x^2 - y^2$ **k** $16 - d^2$ **l** $25 - e^2$

2 **a** $4x^2 - 1$ **b** $9x^2 - 4$ **c** $16y^2 - 25$
d $4y^2 - 25$ **e** $9x^2 - 1$ **f** $1 - 9x^2$
g $4 - 25y^2$ **h** $9 - 16a^2$ **i** $16 - 9a^2$

3 **a** $4a^2 - b^2$ **b** $a^2 - 4b^2$ **c** $16x^2 - y^2$
d $16x^2 - 25y^2$ **e** $4x^2 - 9y^2$ **f** $49x^2 - 4y^2$

EXERCISE 3F

1 **a** $A_1 = a^2$ **b** $A_2 = ab$ **c** $A_3 = ab$ **d** $A_4 = b^2$
e $A = (a+b)^2$, $(a+b)^2 = a^2 + 2ab + b^2$

2 **a** $x^2 + 10x + 25$ **b** $x^2 + 8x + 16$ **c** $x^2 + 14x + 49$
d $a^2 + 4a + 4$ **e** $9 + 6c + c^2$ **f** $25 + 10x + x^2$

3 **a** $x^2 - 6x + 9$ **b** $x^2 - 4x + 4$ **c** $y^2 - 16y + 64$
d $a^2 - 14a + 49$ **e** $25 - 10x + x^2$ **f** $16 - 8y + y^2$

4 **a** $9x^2 + 24x + 16$ **b** $4a^2 - 12a + 9$ **c** $9y^2 + 6y + 1$
d $4x^2 - 20x + 25$ **e** $9y^2 - 30y + 25$ **f** $49 + 28a + 4a^2$
g $1 + 10x + 25x^2$ **h** $49 - 42y + 9y^2$ **i** $9 + 24a + 16a^2$

5 **a** $x^4 + 4x^2 + 4$ **b** $y^4 - 6y^2 + 9$ **c** $9a^4 + 24a^2 + 16$
d $1 - 4x^2 + 4x^4$ **e** $x^4 + 2x^2 y^2 + y^4$ **f** $x^4 - 2a^2 x^2 + a^4$

6 **a** $-x^2 - 3x - 8$ **b** $x^2 + x + 2$ **c** $2x^2 + 6x + 5$
d $-6x - 13$ **e** $11 - 13x + 3x^2$
f $10x^2 - 7x - 5$ **g** $3x^2 - 2x - 10$ **h** $3x^2 - x - 10$
i $2x^2 + 2x + 5$ **j** $-6x - 3$

EXERCISE 3G

1 **a** $x^3 + 4x^2 + 5x + 6$ **b** $x^3 + 5x^2 + 2x - 8$
c $x^3 + 3x^2 + 3x + 2$ **d** $x^3 + 4x^2 - 6x - 5$
e $2x^3 + 3x^2 + 9x + 4$ **f** $3x^3 - 5x^2 - 7x + 6$
g $2x^3 + 3x^2 + 4$ **h** $6x^3 - 5x^2 + 5x - 2$

2 **a** $x^3 + 3x^2 + 3x + 1$ **b** $x^3 + 9x^2 + 27x + 27$
c $x^3 - 3x^2 + 3x - 1$ **d** $x^3 - 9x^2 + 27x - 27$
e $8x^3 + 12x^2 + 6x + 1$ **f** $27x^3 - 54x^2 + 36x - 8$

3 **a** $x^3 + 5x^2 + 6x$ **b** $x^3 - 3x^2 - 4x$ **c** $x^3 - 5x^2 + 6x$
d $2x^3 + 8x^2 + 6x$ **e** $-2x^3 + 10x^2 - 8x$ **f** $x^3 + x^2 - 6x$
g $6x - 9x^2 - 6x^3$ **h** $x - x^2 - 6x^3$ **i** $2x^4 - 4x^3 + 2x^2$

4 **a** $x^3 + 6x^2 + 11x + 6$ **b** $x^3 + x^2 - 10x + 8$
c $x^3 - 8x^2 + 19x - 12$ **d** $2x^3 + x^2 - 5x + 2$
e $3x^3 + 14x^2 + 17x + 6$ **f** $4x^3 + 16x^2 - x - 4$
g $-3x^3 + 7x^2 - 4$ **h** $-3x^3 + 10x^2 - x - 6$

EXERCISE 3H

1 **a** $x^3 + 3x^2 + 3x + 1$ **b** $a^3 + 9a^2 + 27a + 27$
c $x^3 + 15x^2 + 75x + 125$ **d** $x^3 - 3x^2 + 3x - 1$
e $x^3 - 6x^2 + 12x - 8$ **f** $x^3 - 9x^2 + 27x - 27$
g $27 + 27a + 9a^2 + a^3$ **h** $27x^3 + 54x^2 + 36x + 8$
i $8x^3 + 36x^2 y + 54xy^2 + 27y^3$

2 $(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$

3 **a** $x^4 + 4x^3 + 6x^2 + 4x + 1$
b $y^4 + 8y^3 + 24y^2 + 32y + 16$
c $81 + 108a + 54a^2 + 12a^3 + a^4$
d $b^4 + 16b^3 + 96b^2 + 256b + 256$
e $x^4 - 4x^3 + 6x^2 - 4x + 1$
f $y^4 - 8y^3 + 24y^2 - 32y + 16$
g $81 - 108a + 54a^2 - 12a^3 + a^4$
h $b^4 - 16b^3 + 96b^2 - 256b + 256$

4 $(a + b)^5 = a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$
$(a - b)^5 = a^5 - 5a^4 b + 10a^3 b^2 - 10a^2 b^3 + 5ab^4 - b^5$

REVIEW SET 3A

1 **a** $-32x$ **b** $10x^3$ **c** $24x^2$ **d** x^2 **e** $7ac$ **f** $-x^3$

2 **a** $-3x - 18$ **b** $2x^3 - 8x$ **c** $-x - 4$ **d** $-7x + 7$
e $8x - 3x^2$ **f** $4x^2 - x$ **g** $2x^3 + x^2 - x$ **h** $5a + 9b + ab$

3 **a** $3x^2 - 4x - 4$ **b** $4x^2 - 4x + 1$ **c** $16x^2 - 1$
d $x^2 - 10x + 25$ **e** $6x^2 - 29x + 35$ **f** $x^3 - 4x$
g $9x^2 + 30x + 25$ **h** $-x^2 + 4x - 4$ **i** $-2x^3 + 4x^2 - 2x$

4 **a** $-x^2 - 4x - 4$ **b** $x^3 + 6x^2 + 12x + 8$
c $3x^3 + 4x^2 + 17x - 14$ **d** $x^3 - 6x^2 + 11x - 6$
e $x^4 + 3x^3 + 3x^2 + x$ **f** $x^4 - 1$

5 The area of the figure is $(a + b)^2$ or the sum of the areas of its 4 parts, i.e., $a^2 + ab + ba + b^2$.

REVIEW SET 3B

1 **a** $-6x^3$ **b** $-6x^3$ **c** $40x^2$ **d** $4x^2$ **e** $9x^4$ **f** $-4x^3$

2 **a** $-14x + 35$ **b** $-x$ **c** $2x^2 - 5x$ **d** $16x - 3$
e $5x^3 - 6x^2 + 3x$ **f** $16a - 2b$

3 **a** $2x^2 - x - 15$ **b** $9x^2 - 12x + 4$ **c** $4x^2 - 9$
d $5x^2 - 11x + 2$ **e** $4x^2 - 12x + 9$ **f** $1 - 25x^2$
g $25 - 20x + 4x^2$ **h** $-x^2 - 4x - 4$ **i** $-3x + 6x^2 - 3x^3$

4 **a** $5x^2 - x + 7$ **b** $2x^3 - 9x^2 + 10x - 3$
c $x^3 + 9x^2 + 27x + 27$ **d** $x^3 + 4x^2 - 7x - 10$
e $2x^4 - 6x^3 + 6x^2 - 2x$ **f** $-x^4 + 8x^2 - 16$

5 **a** $16x^4 + 32x^3 + 24x^2 + 8x + 1$
b $x^4 - 12x^3 + 54x^2 - 108x + 81$

6 $(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$

EXERCISE 4A

1

$\sqrt{17}$

2 **a** 2 and 3,
i.e., $2^2 + 3^2 = 13$

     b
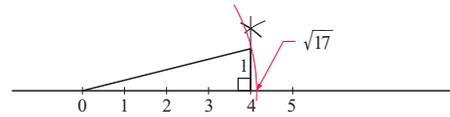
$\sqrt{13}$
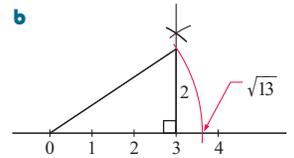
3 No, because we cannot write 6 as the sum of the squares of two positive integers.

4
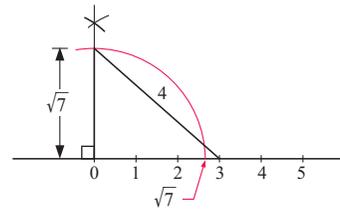
EXERCISE 4B.1

1 **a** $10\sqrt{2}$ **b** $3\sqrt{3}$ **c** $-\sqrt{5}$ **d** $\sqrt{2}$ **e** $2\sqrt{3} - 2$
f $-2\sqrt{2} - 3$ **g** $6\sqrt{2} - 2\sqrt{3}$ **h** $2\sqrt{7} - 3\sqrt{2}$ **i** $3\sqrt{3} - 1$
j $5 - \sqrt{3}$ **k** $4\sqrt{3} - 4\sqrt{2}$ **l** $5\sqrt{3} - 7$

2 **a** 3   **b** $3\sqrt{3}$ **c** $9\sqrt{3}$ **d** $\frac{1}{3}$ **e** 7   **f** $7\sqrt{7}$
g $\frac{1}{7}$ **h** $\frac{9}{7}$ **i** 5   **j** 25   **k** 5   **l** 20

3 **a** 8   **b** 32   **c** 12   **d** 27   **e** 20   **f** 45
g 28   **h** 40   **i** 490   **j** 24   **k** 30   **l** 70
m 32   **n** 147   **o** -6 **p** 30   **q** -42 **r** -22

4 **a** $2\frac{1}{2}$ **b** $1\frac{1}{3}$ **c** $1\frac{3}{5}$ **d** $2\frac{2}{3}$

5 **a** $\sqrt{6}$ **b** $\sqrt{14}$ **c** $\sqrt{34}$ **d** $\sqrt{21}$ **e** $10\sqrt{6}$
f 18   **g** $5\sqrt{14}$ **h** $6\sqrt{30}$ **i** $-10\sqrt{14}$
j $2\sqrt{21}$ **k** $24\sqrt{5}$ **l** $80\sqrt{6}$

6 **a** 2   **b** $\frac{1}{3}$ **c** $\sqrt{6}$ **d** $\frac{1}{5}$ **e** $\sqrt{15}$ **f** $\frac{1}{\sqrt{15}}$
g 3   **h** $\frac{1}{\sqrt{20}}$ **i** $3\sqrt{3}$ **j** 8   **k** 2   **l** $\frac{21}{2}$

7 **a** No   **b** No

EXERCISE 4B.2

1 **a** $3\sqrt{2}$ **b** $5\sqrt{2}$ **c** $6\sqrt{2}$ **d** $7\sqrt{2}$
e $9\sqrt{2}$ **f** $10\sqrt{2}$ **g** $100\sqrt{2}$ **h** $1000\sqrt{2}$

2 **a** $2\sqrt{3}$ **b** $3\sqrt{3}$ **c** $4\sqrt{3}$ **d** $10\sqrt{3}$

3 **a** $2\sqrt{5}$ **b** $4\sqrt{5}$ **c** $8\sqrt{5}$ **d** $10\sqrt{5}$

4 **a** $3\sqrt{11}$ **b** $2\sqrt{13}$ **c** $2\sqrt{10}$ **d** $3\sqrt{7}$
e $4\sqrt{3}$ **f** $5\sqrt{5}$ **g** $7\sqrt{3}$ **h** $5\sqrt{7}$
i $4\sqrt{11}$ **j** $5\sqrt{6}$ **k** $5\sqrt{11}$ **l** $20\sqrt{5}$

5 **a** $2 + \sqrt{2}$ **b** $3 - \sqrt{3}$ **c** $1 + \frac{3}{4}\sqrt{2}$ **d** $2 - \sqrt{2}$
e $2 + \sqrt{2}$ **f** $3 + \frac{1}{2}\sqrt{3}$ **g** $\frac{7}{4} - \frac{5}{8}\sqrt{2}$ **h** $\frac{1}{2} - \sqrt{2}$

EXERCISE 4C

1 **a** $12 + 4\sqrt{2}$ **b** $3\sqrt{2} + 3\sqrt{3}$ **c** $20 - 5\sqrt{7}$
   d $6\sqrt{11} - 24$ **e** $\sqrt{2} + 2$ **f** $2 - 5\sqrt{2}$
   g $2\sqrt{3} + 6$ **h** $3 - \sqrt{6}$ **i** $6\sqrt{5} - 5$
   j $10 - \sqrt{5}$ **k** $10 + \sqrt{15}$ **l** $2\sqrt{7} + 7 + \sqrt{14}$

2 **a** $-4\sqrt{2} - 2$ **b** $3\sqrt{2} - 2$ **c** $-2 + \sqrt{14}$
   d $-3\sqrt{3} - 3$ **e** $-5\sqrt{3} + 3$ **f** $-6 - \sqrt{15}$
   g $-2\sqrt{10} + \sqrt{15}$ **h** $-4 - 2\sqrt{6}$ **i** $-2\sqrt{3} + 4\sqrt{6}$
   j $-14 - 4\sqrt{7}$ **k** $11 - 2\sqrt{11}$ **l** $8 - 8\sqrt{2}$

3 **a** $8 + 5\sqrt{2}$ **b** $11 + 6\sqrt{2}$ **c** $\sqrt{2}$ **d** $9 + \sqrt{3}$ **e** 1
   f $4 - 3\sqrt{6}$ **g** $1 - \sqrt{7}$ **h** 9 **i** $21 + 12\sqrt{2}$ **j** $8 + 2\sqrt{2}$

4 **a** $4 + 2\sqrt{3}$ **b** $27 + 10\sqrt{2}$ **c** $11 - 6\sqrt{2}$ **d** $8 + 2\sqrt{7}$
   e $5 - 2\sqrt{6}$ **f** $21 - 8\sqrt{5}$ **g** $8 + 2\sqrt{15}$ **h** $15 - 6\sqrt{6}$
   i $9 - 6\sqrt{2}$ **j** $17 + 12\sqrt{2}$ **k** $17 - 12\sqrt{2}$ **l** $59 - 30\sqrt{2}$

5 **a** 7 **b** 2 **c** 22 **d** -13 **e** -2 **f** -46
   g -4 **h** -16 **i** 14 **j** 1 **k** -4 **l** 7

EXERCISE 4D.1

1 **a** $\frac{\sqrt{3}}{3}$ **b** $\sqrt{3}$ **c** $3\sqrt{3}$ **d** $\frac{11\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{9}$
   f $\sqrt{2}$ **g** $3\sqrt{2}$ **h** $6\sqrt{2}$ **i** $\frac{\sqrt{6}}{2}$ **j** $\frac{\sqrt{2}}{8}$
   k $\sqrt{5}$ **l** $3\sqrt{5}$ **m** $-\frac{3\sqrt{5}}{5}$ **n** $40\sqrt{5}$ **o** $\frac{\sqrt{5}}{15}$
   p $\sqrt{7}$ **q** $3\sqrt{7}$ **r** $\frac{2\sqrt{11}}{11}$ **s** $2\sqrt{13}$ **t** $\frac{\sqrt{3}}{9}$

EXERCISE 4D.2

1 **a** $\frac{3 - \sqrt{2}}{7}$ **b** $\frac{6 + 2\sqrt{2}}{7}$ **c** $\sqrt{5} - 2$ **d** $\sqrt{2} + 1$
   e $-3 - 2\sqrt{2}$ **f** $\frac{4\sqrt{3} + 3}{13}$ **g** $2\sqrt{2} + 4$ **h** $-7 - 3\sqrt{5}$

2 **a** $-\frac{9}{7} - \frac{3}{7}\sqrt{2}$ **b** $4 - 2\sqrt{2}$ **c** $-\frac{2}{23} - \frac{5}{23}\sqrt{2}$
   d $-4 + 2\sqrt{2}$

3 **a** $-2 - 2\sqrt{3}$ **b** $12 - 6\sqrt{3}$ **c** $3 + 2\sqrt{3}$ **d** $-\frac{1}{2} + \frac{5}{6}\sqrt{3}$

4 **b** **i** $\frac{2\sqrt{3} - 1}{11}$ **ii** $\frac{-6 - 5\sqrt{2}}{7}$

5 **b** **i** $\sqrt{3} - \sqrt{2}$ **ii** $\frac{-3 - \sqrt{15}}{2}$

REVIEW SET 4A

1 **a** 12 **b** 8 **c** $6\sqrt{10}$ **d** $\frac{7}{2}$

2 **a** $1^2 + 3^2 = (\sqrt{10})^2$ **b**
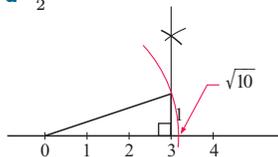
3 **a** $\sqrt{5}$ **b** $\sqrt{5}$ **c** $\sqrt{7}$ **d** $\frac{1}{\sqrt{10}}$

4 **a** $2\sqrt{2}$ **b** $3\sqrt{2}$ **5** $7\sqrt{2}$

6 **a** $2\sqrt{3} + 2$ **b** $3\sqrt{2} - 2$ **c** $8 + 2\sqrt{7}$ **d** $9 - 4\sqrt{5}$
   e 7 **f** $1 - 2\sqrt{2}$

7 **a** $2\sqrt{5}$ **b** $\frac{1 + \sqrt{3}}{2}$ **c** $\frac{-4 - \sqrt{7}}{3}$

8 **a** $-6 - 3\sqrt{5}$ **b** $\frac{5}{2} - \frac{1}{2}\sqrt{5}$

REVIEW SET 4B

1 **a** $\sqrt{6}$ **b** 2
   c 45 **d** $\frac{7}{3}$
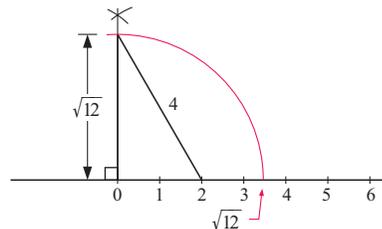
2 $4^2 - 2^2 = 12$

3 **a** $\sqrt{7}$ **b** $\frac{1}{2\sqrt{2}}$ **4** $-2\sqrt{3}$ **5** **a** $2\sqrt{3}$ **b** $3\sqrt{7}$

6 **a** $6 - 3\sqrt{3}$ **b** $\sqrt{14} - \sqrt{7}$ **c** $11 - 6\sqrt{2}$
   d $5 + 2\sqrt{6}$ **e** -1 **f** $3 + \sqrt{3}$

7 **a** $8\sqrt{3}$ **b** $\frac{4 + 3\sqrt{2}}{2}$ **c** $\frac{17 - 7\sqrt{5}}{4}$

8 **a** $\frac{45}{11} + \frac{9}{11}\sqrt{3}$ **b** $\frac{1}{2} - \frac{1}{2}\sqrt{3}$

EXERCISE 5A

1 **a** $7 \in K$ **b** $6 \notin M$ **c** $\{3, 4\} \subseteq \{2, 3, 4\}$
   d $\{3, 4\} \nsubseteq \{1, 2, 4\}$

2 **a** **i** $\{4, 5\}$ **ii** $\{2, 3, 4, 5, 6, 7, 8\}$
   b **i** $\varnothing$ **ii** $\{2, 3, 4, 5, 6, 7, 8\}$
   c **i** $\{2, 3, 4, 5\}$ **ii** $\{1, 2, 3, 4, 5, 6, 7\}$

3 **a** $\varnothing$ **b** yes **4** **a** true **b** true **5** **a** yes **b** no

6 **a** There are no elements in the empty set. So every element of the empty set must be an element of any other set.
   b $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ **c** 2^n

EXERCISE 5B

1 **a** $4 = \frac{4}{1}$ and $-7 = \frac{-7}{1}$
   Both are in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.
   b $\frac{3}{0}$ is not rational as 0 is the denominator, $(q \neq 0)$.

2 **a** $0.6 = \frac{3}{5}$ **b** $0.13 = \frac{13}{100}$ **c** $1\frac{1}{3} = \frac{4}{3}$
   d $-4\frac{1}{2} = \frac{-9}{2}$ **e** $0.\overline{3} = \frac{1}{3}$

3 $\mathbb{Z}^- \cup \mathbb{Z}^+$ is the set of all integers except zero, whereas $\mathbb{Z}$ is the set of all integers including zero.

4 **a** true **b** true **c** true **d** true **e** false **f** false

5 **a** $\mathbb{R}$ **b** $\varnothing$ **c** $\mathbb{Q}'$ **d** $\mathbb{R}$

EXERCISE 5C

1 **a** finite **b** infinite **c** infinite **d** infinite

2 **a** **i** B is the set of all integers x such that x lies between -3 and 4, both inclusive.
   ii $B = \{-3, -2, -1, 0, 1, 2, 3, 4\}$ **iii** $n(B) = 8$
   iv
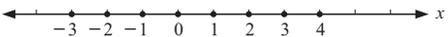
   b **i** B is the set of all natural numbers x such that x lies between -5 and -1, including -1.
   ii B is an empty set. **iii** $n(B) = 0$
   iv
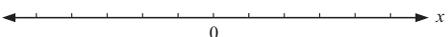
   c **i** B is the set of all real numbers x such that x lies between 2 and 3.
   ii/iii Cannot do this. There are an infinite number of elements.
   iv
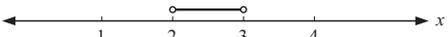
d i B is the set of all rational numbers between 1 and 2 inclusive.

    ii/iii Cannot do this. There are an infinite number of elements.

    iv
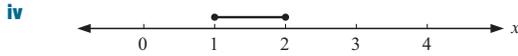
3 a $\{x \mid 100 < x < 300, \quad x \in \mathbb{Z}\}$
   b $\{x \mid x > 50, \quad x \in \mathbb{R}\}$ **c** $\{x \mid 7 \leqslant x \leqslant 8, \quad x \in \mathbb{Q}\}$

4 a yes   **b** yes   **c** no

EXERCISE 5D

1 a $\{3, 4, 6\}$

2 a $\mathbb{Z}^-$, (all the negative integers)   **b** {even positive integers}
   c $\{x \mid x > 10, \quad x \in \mathbb{Z}\}$ **d** $\{x \mid 3 < x < 5, \quad x \in \mathbb{Q}\}$

3 a i $\{1, 2, 3, 4, 6, 8, 12\}$ **ii** $\{4, 8, 12, 16, 20\}$
    iii $\{0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$
    iv $\{4, 8, 12\}$ **v** $\{1, 2, 3, 4, 6, 8, 12, 16, 20\}$
    vi $\{1, 2, 3, 6\}$
   b 4

4 a i $\{0, 1, 2, 3, 4, 5, 6, 7\}$ **ii** $\{1, 3, 6, 7\}$
    iii $\{0, 1, 4, 6\}$ **iv** $\{2, 5\}$ **v** $\{0, 2, 3, 4, 5, 7\}$
    vi $\{1, 6\}$

5 a i $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$
    ii $\{5, 6, 7, 8\}$ **iii** $\{3, 4, 9, 10, 11, 12, 13, 14\}$
    iv $\{3, 5, 6, 8, 9, 11, 12, 14\}$
    v $\{3, 4, 7, 9, 10, 11, 12, 13, 14\}$ **vi** $\{5, 6, 8\}$
   b i 9   **ii** 3

6 a false   **b** true

EXERCISE 5E.1

1 a

   b

   c

   d

2 a $A = \{1, 2, 4, 8\}, \quad B = \{2, 3, 5, 7, 11\}$
   b $A \cap B = \{2\}, \quad A \cup B = \{1, 2, 3, 4, 5, 7, 8, 11\}$
   c

3 a $R = \{2, 3, 5, 7, 11, 13, 17, 19\},$
    $S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$
   b $R \cap S = \varnothing$
    $R \cup S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,$
    $17, 18, 19\}$

c
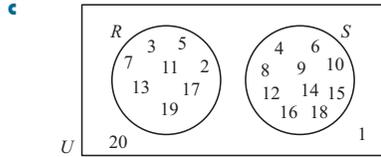
4 a $\{a, b, c, d, e, f, g, h, i, j, k\}$ **b** $\{a, c, f, i, j\}$
   c $\{c, d, g, h, j\}$ **d** $\{b, d, e, g, h, k\}$ **e** $\{a, b, e, f, i, k\}$
   f $\{c, j\}$ **g** $\{a, c, d, f, g, h, i, j\}$ **h** $\{b, e, k\}$

EXERCISE 5E.2

1 a

   b

   c

   d

   e

   f

2 a

   b

   c

   d

   e

   f

   g

   h

   i

3 **a**

b

c

d

e

f

g

h

i
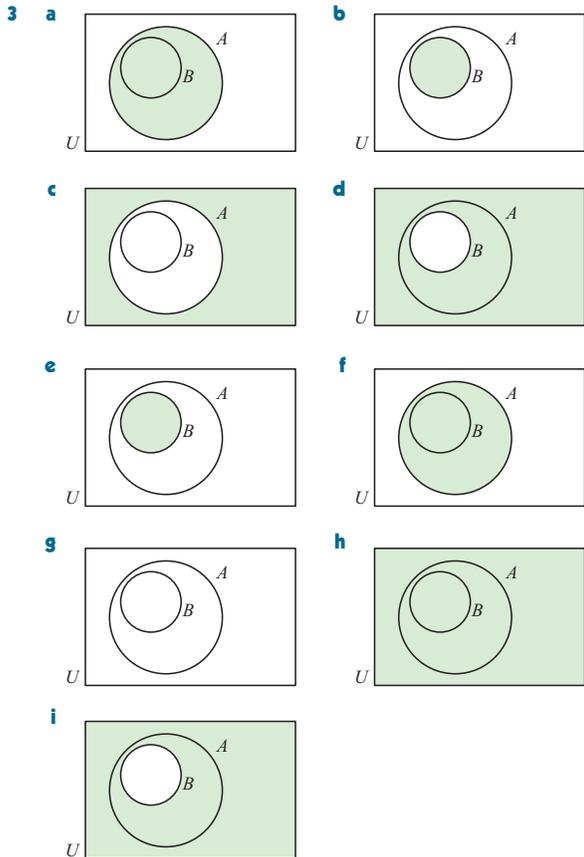
EXERCISE 5E.3

1 **a** 9 **b** 10 **c** 18 **d** 9 **e** 6 **f** 4

2 **a** 12 **b** 16 **c** 42 **d** 18 **e** 8 **f** 4

3 **a** $b+c$ **b** $c+d$ **c** b **d** $a+b+c$ **e** $a+c+d$ **f** d

4 **a** **i** $2a-4$ **ii** $5a-4$ **iii** $4a+5$ **iv** $6a+1$ **b** $a = 7$

5 **a** **i** $a+b+c$ **ii** $a+b+c$
   b $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

6 **a** 15 **b** 7 **7** **a** 19 **b** 4

8 **a** 15 **b** 23 **c** 21 **d** 3 **e** 33 **f** 2 **g** 50 **h** 6

9 **a**

b

c

d

e

f

g

h

10 **a**

b 78
c **i** 10
   ii 16
   iii 51
   iv 45

11 **a** 2 riders **b** 11 riders **c** 13 riders

12 **a**

b **i** 7%
   ii 84%
   iii 39%
   iv 77%

13 **a**

b **i** 26%
   ii 73%
   iii 29%
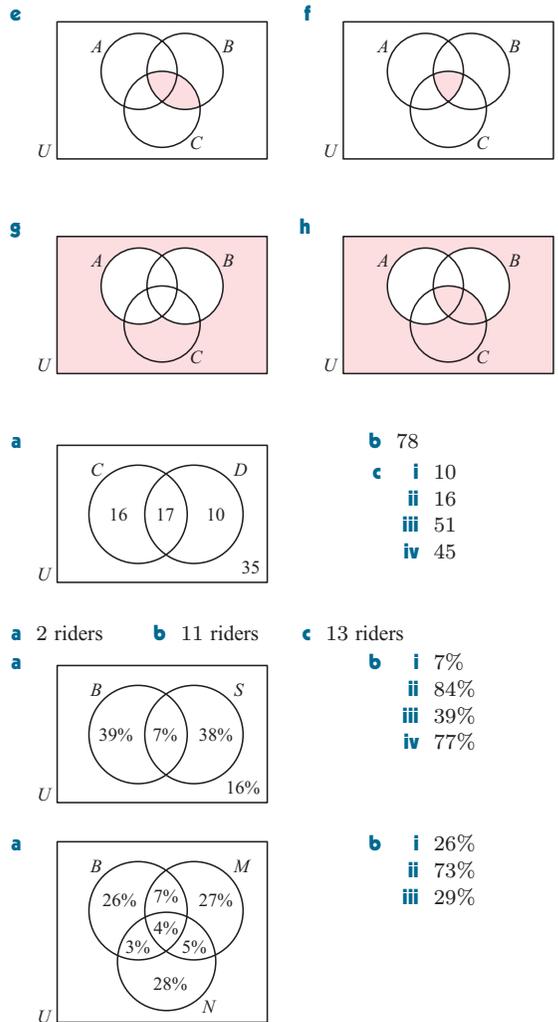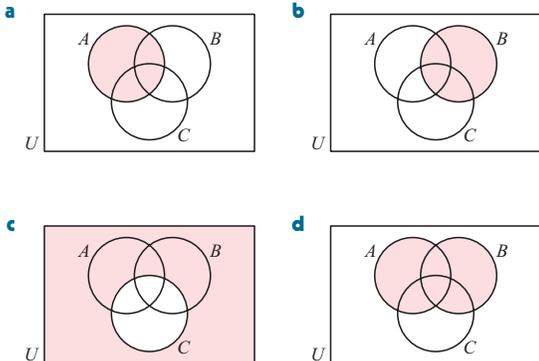
REVIEW SET 5A

1 **a** $\{4, 5, 6, 7, 8\}$ **b** no **c** 5

2 **a** $\{x \mid x > 10, \quad x \in \mathbb{Z}\}$ **b** $\{x \mid -2 < x < 3, \quad x \in \mathbb{Q}\}$

3 $\{5, 7\}$

4 **a** $\{-5, -4, -3, -1, 2, 4\}$ **b** $\{0, 3\}$
   c $\{-4, -2, -1, 0, 1, 2, 3, 4, 5\}$ **d** $\{-4, -1, 2, 4\}$
   e $\{-5, -3, -2, 0, 1, 3, 5\}$

5 **a** **i** $\{a, d, g, h, i\}$ **ii** $\{c, d, f, i\}$ **iii** $\{a, b, e, g, h, j\}$
   iv $\{d, i\}$ **v** $\{a, c, d, f, g, h, i\}$ **vi** $\{b, e, j\}$
   b 7

6 **a**
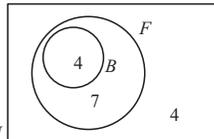
b **i** 7 people
   ii 4 people

7 **a** **i** $\{a, b, c, d, e, f, g\}$ **ii** $\{a, b, c, d, f\}$ **iii** $\{a, c\}$
   b **i** false **ii** true **iii** false **iv** false

8 8 (when $x = 2$)

REVIEW SET 5B

1 a finite   **b** 7

2 $\varnothing$, $\{p\}$, $\{q\}$, $\{r\}$, $\{p, q\}$, $\{p, r\}$, $\{q, r\}$, $\{p, q, r\}$

3 a $F = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
$M = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}$
  b no
  c i $\{3, 6, 9, 12, 18\}$
      ii $\{1, 2, 3, 4, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$

4 a $S = \{x \mid 0 < x < 3, \ x \in \mathbb{Q}\}$
  b i no   **ii** yes   **iii** no   **iv** yes   **v** no

5 a i $\varnothing$ **ii** U **iii** B
  b
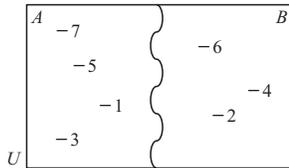
6 a i $A \subseteq B$ **ii** B and C are disjoint
      iii A and C are disjoint
  b i   **ii**
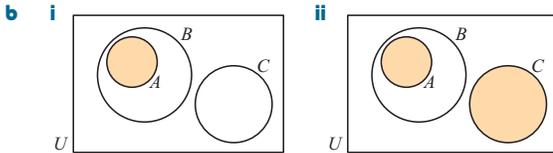
7 9   **8** 3

EXERCISE 6A.1

1 a 5 units   **b** 4 units   **c** 3 units   **d** $\sqrt{41}$ units
  e 5 units   **f** $\sqrt{13}$ units

2 a $\sqrt{34}$ units   **b** $\sqrt{29}$ units   **c** $\sqrt{34}$ units   **d** $\sqrt{26}$ units
  e $\sqrt{17}$ units   **f** $\sqrt{13}$ units

EXERCISE 6A.2

1 a $\sqrt{10}$ units   **b** $\sqrt{13}$ units   **c** $4\sqrt{2}$ units   **d** $\sqrt{41}$ units
  e $5\sqrt{2}$ units   **f** $\sqrt{13}$ units   **g** $\sqrt{10}$ units   **h** $2\sqrt{10}$ units
  i $2\sqrt{41}$ units   **j** $2\sqrt{6}$ units

2 a $AB = \sqrt{17}$ units, $BC = \sqrt{17}$ units, $AC = \sqrt{18}$ units
     isosceles
  b $XY = \sqrt{13}$ units, $YZ = \sqrt{32}$ units, $XZ = \sqrt{53}$ units
     scalene
  c $PQ = \sqrt{8}$ units, $QR = \sqrt{8}$ units, $PR = \sqrt{8}$ units
     equilateral
  d $EF = \sqrt{73}$ units, $FG = \sqrt{34}$ units, $EG = \sqrt{29}$ units
     scalene
  e $HI = \sqrt{40}$ units, $IJ = \sqrt{32}$ units, $HJ = \sqrt{40}$ units
     isosceles
  f $WX = \sqrt{48}$ units, $XY = \sqrt{48}$ units, $WY = \sqrt{48}$ units
     equilateral

3 a $PQ = \sqrt{10}$ units, $QR = \sqrt{20}$ units, $PR = \sqrt{10}$ units
  b $PQ^2 + PR^2 = QR^2$ $\therefore$ the triangle is right angled at P.

4 a $AB = 5$ units, $BC = 7$ units, $AC = \sqrt{74}$ units
     $AB^2 + BC^2 = AC^2$ $\therefore$ the triangle is right angled at B.
  b $AB = \sqrt{26}$ units, $BC = \sqrt{52}$ units, $AC = \sqrt{26}$ units
     $AB^2 + AC^2 = BC^2$ $\therefore$ the triangle is right angled at A.

5 $PQ = \sqrt{13}$ units, $QR = \sqrt{13}$ units, $PR = \sqrt{26}$ units
Triangle PQR is isosceles as $PQ = QR$, and is right angled at
Q as $PQ^2 + QR^2 = PR^2$.

6 a $a = 2$ **b** $b = 3$ or -5 **c** $c = 2$ or -4

EXERCISE 6B

1 a $(4, 2)$ **b** $(3\frac{1}{2}, 3)$ **c** $(3, -1)$ **d** $(1, -2)$
  e $(-3, -1)$ **f** $(-2, 1)$ **g** $(0, 1)$ **h** $(\frac{1}{2}, 1)$ **i** $(0, 0)$

2 a $(3, 5)$ **b** $(4, 0)$ **c** $(-1, 1)$ **d** $(3, 1\frac{1}{2})$
  e $(-1\frac{1}{2}, 2)$ **f** $(1, 1)$ **g** $(2\frac{1}{2}, 4)$ **h** $(-3\frac{1}{2}, 2)$
  i $(\frac{a+2}{2}, 2)$ **j** $(0, 2b)$

3 a $Q(-1, 11)$ **b** $Q(1, -10)$ **c** $Q(6, 0)$ **d** $Q(-1\frac{1}{2}, 6)$

4 a $A(10, -9)$ **b** $A(-6, -8)$

5 a $B(0, 5)$ **b** $D(-2, 9)$ **6 a** $A(-2, 7)$ **b** 10 units

7 a $X(3\frac{1}{2}, 4\frac{1}{2})$ **b** $C(6, 7)$ **8** $N(4, 3\frac{1}{4})$

EXERCISE 6C.1

1 a $\frac{1}{2}$ **b** $\frac{1}{5}$ **c** undefined   **d** 0   **e** $-\frac{3}{2}$ **f** $-\frac{1}{2}$
  g -3 **h** $-\frac{2}{5}$

2 a   **b**   **c**   **d**

  e   **f**   **g**

3 a   **b**
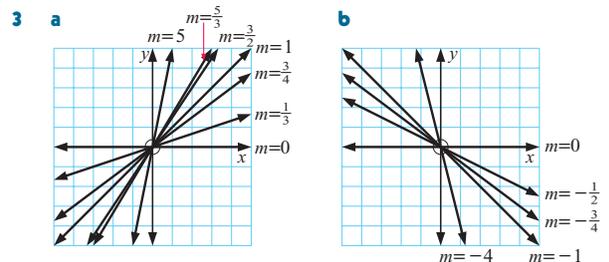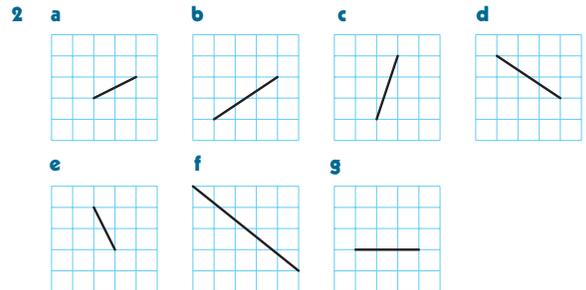
EXERCISE 6C.2

1 a 5   **b** 4   **c** $\frac{1}{4}$ **d** $-\frac{3}{4}$ **e** undefined   **f** 0   **g** $\frac{7}{2}$
  h $-\frac{7}{2}$ **i** 2

2 a -3 **b** $\frac{7}{3}$ **c** $-\frac{1}{2}$ **d** $\frac{1}{11}$ **e** undefined   **f** 0   **g** $\frac{2}{3}$

3 a $\frac{4}{5}$ **b** $-\frac{5}{4}$

4 a $a = \frac{5}{2}$ **b** $a = -\frac{2}{3}$ **c** $a = -\frac{1}{2}$ **d** $a = -\frac{7}{2}$

5 a $b = -\frac{1}{2}$ **b** $b = -3$

6 a i 5   **ii** $-\frac{1}{5}$ **b** perpendicular   **c** right angled at B

7 a $\dfrac{a-4}{3}$ **b** $a = 5\frac{1}{2}$ **c** $a = 5$

8 a $\dfrac{-2}{b+1}$ **b i** $b = 2$ **ii** $b = \frac{5}{3}$

9 a $a = -3$ **b** $a = \frac{6}{5}$

EXERCISE 6D

1 a 8 **b** The sprinter's speed is 8 m s^{-1}.
  c The speed is constant as the gradient is constant.

2 a 72 km h^{-1} **b** **i** 85 km h^{-1} **ii** 85 km h^{-1}
  c O to A (0 to 2 hours) and B to C (5 to 7 hours)

3 a If no hours are worked then no wages are received.
  b gradient is 15; the wage is £15 per hour
  c **i** £90   **ii** £270

4 a A has gradient $\frac{35}{3} = 11\frac{2}{3}$, B has gradient $\frac{75}{8} = 9\frac{3}{8}$
  b Gradient is number of km travelled per litre of petrol.
  c $25.98

5 a $3 initial charge
  b AB has gradient $\frac{3}{2}$, BC has gradient $\frac{3}{5}$, these values give the charge per km.
  c gradient is $\frac{6}{5}$, average charge is $1.20 per km.

EXERCISE 6E

1 a $AB = \sqrt{20}$ units, $BC = \sqrt{8}$ units, $AC = \sqrt{20}$ units
    $AB = AC$ so the triangle is isosceles.
  b M(2, 3)
  c gradient of [AM] = 1, gradient of [BC] = -1
    Gradients are $-$ve reciprocals
    so [AM] $\perp$ [BC]
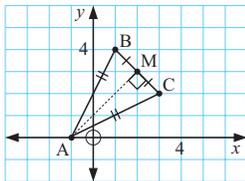
2 a gradient [AB] = 0, gradient [DC] = 0 $\therefore$ [AB] $\parallel$ [DC]
    gradient [AD] = $\frac{9}{15}$, gradient [BC] = $\frac{9}{15}$ $\therefore$ [AD] $\parallel$ [BC]
    $\therefore$ ABCD is a parallelogram.
  b $AB = DC = 9$ units, $BC = AD = \sqrt{306}$ units
  c **i** $(4, 2\frac{1}{2})$ **ii** $(4, 2\frac{1}{2})$
  d The diagonals of a parallelogram bisect each other.

3 a M(3, 2), N(4, -1), gradient MN = gradient AC = -3
  b $MN = \sqrt{10}$ units, $AC = \sqrt{40} = 2\sqrt{10}$ units

4 a $AB = \sqrt{20}$ units, $AC = \sqrt{20}$ units, $BC = \sqrt{40}$ units
    $AB = AC$ and $AB^2 + AC^2 = BC^2$
    $\therefore$ isosceles and right angled at A
  b M(2, 3)
  c gradient [AM] = $-\frac{1}{3}$, gradient [BC] = 3
    gradients are negative reciprocals so [AM] $\perp$ [BC].
  d
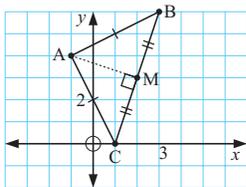
5 a $AB = BC = CD = DA = 5$ units $\therefore$ ABCD is a rhombus
  b midpoint [AC] = midpoint [BD] = (4, 2)
  c gradient [AC] = -2, gradient [BD] = $\frac{1}{2}$
    gradients are negative reciprocals so [AC] $\perp$ [BD]

d
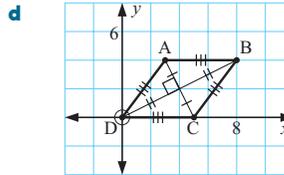
EXERCISE 6F

1 a gradient is undefined **b** gradient is 0 **c** gradient is undefined

d gradient is 0 **e** gradient is undefined **f** gradient is 0

2 a $y = 2$ **b** $x = 3$ **c** $y = 0$ **d** $y = -3$
  e $x = -7$ **f** $y = -4$

EXERCISE 6G.1

1 a $m = 3$, $c = 2$ **b** $m = 7$, $c = 5$ **c** $m = -2$, $c = 1$
  d $m = \frac{1}{3}$, $c = 6$ **e** $m = -1$, $c = 6$ **f** $m = -2$, $c = 3$
  g $m = -1$, $c = 10$ **h** $m = \frac{1}{2}$, $c = 1$ **i** $m = \frac{3}{2}$, $c = 2$
  j $m = 0$, $c = 0$ **k** $m = -\frac{1}{2}$, $c = \frac{3}{2}$ **l** $m = -\frac{1}{2}$, $c = \frac{7}{4}$

2 a $y = x - 1$ **b** $y = \frac{1}{2}x + 1$ **c** $y = -2x + 2$
  d $y = -x$ **e** $y = \frac{1}{3}x$ **f** $y = 3x - 1$

EXERCISE 6G.2

1 a

b

c

d

e

f

g

$y = -\frac{1}{2}x - 1$
$(-2, 0)$, -1

h

$y = -\frac{2}{3}x - 3$
-3, $(3, -5)$

2 a

$y = 2x + 1$
2, 1, 1

b

4, -2, 1
$y = 4x - 2$

c

$y = \frac{1}{2}x$
1, 2

d

$y = -x - 2$
-2, 1, -1

e

3, 1, -2
$y = -2x + 3$

f

2, -3
$y = -\frac{2}{3}x + 2$
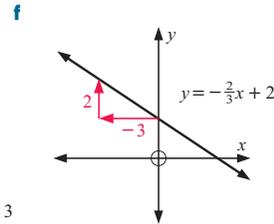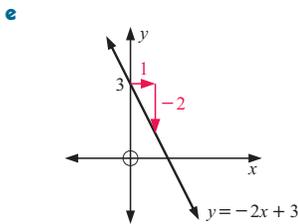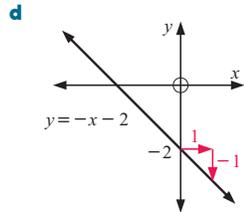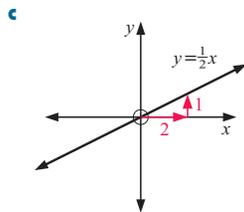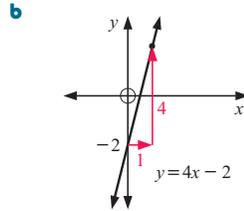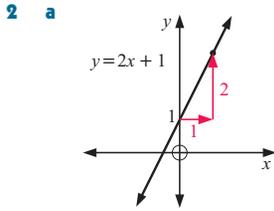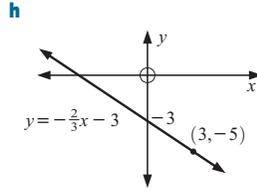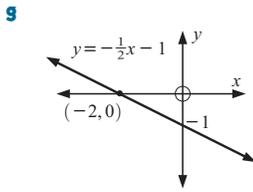
EXERCISE 6G.3

1 a $y = 2x + 1$ **b** $y = -x + 1$ **c** $y = -3x + 10$
  d $y = \frac{2}{3}x + \frac{7}{3}$ **e** $y = -\frac{1}{2}x - \frac{1}{2}$ **f** $y = -3$
  g $y = \frac{2}{3}x + \frac{17}{3}$ **h** $y = -\frac{4}{5}x + \frac{6}{5}$ **i** $y = -\frac{3}{4}x + \frac{3}{2}$
  j $y = \frac{4}{7}x - \frac{26}{7}$

2 a $y = x - 4$ **b** $y = -x + 4$ **c** $y = 6x + 16$
  d $y = -9x + 6$ **e** $y = -\frac{1}{3}x - \frac{7}{3}$ **f** $y = -3x - 7$

3 a $y = -\frac{1}{2}x + 4$ **b** $y = 3x + 1$ **c** $y = -\frac{3}{4}x + 3$
  d $y = -\frac{5}{4}x - \frac{5}{2}$ **e** $y = -2x - 2$ **f** $y = \frac{1}{3}x - \frac{7}{3}$

EXERCISE 6H.1

1 a $x - 3y = -11$ **b** $3x - 5y = -11$
  c $2x + 3y = 12$ **d** $4x - 5y = 21$
  e $3x - y = -10$ **f** $2x + y = 5$

2 a $x - y = 4$ **b** $x + y = 4$ **c** $x = -2$
  d $9x + y = 6$ **e** $3x - y = -5$ **f** $x + y = -5$

3 a 2 **b** 0 **c** -3 **d** undefined **e** -4
  f $\frac{2}{3}$ **g** $\frac{3}{4}$ **h** $-\frac{3}{5}$ **i** -3 **j** $-\frac{2}{3}$ **k** $-\frac{3}{5}$
  l $-\frac{4}{7}$ **m** $\frac{1}{2}$ **n** $\frac{3}{4}$ **o** $\frac{5}{6}$

EXERCISE 6H.2

1 a

2, $x + 3y = 6$, 6

b

4, -6, $3x - 2y = 12$

c

$2x + 5y = 10$, 2, 5

d

2, $4x + 3y = 6$, $1\frac{1}{2}$

e

5, $x + y = 5$, 5

f

$x - y = -3$, 3, -3

g

$3x - y = -6$, 6, -2

h

7, $7x + 2y = 14$, 2

i

3, -4, $3x - 4y = -12$
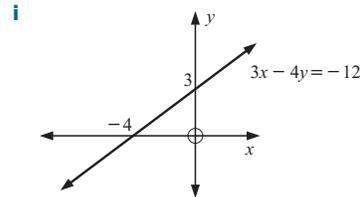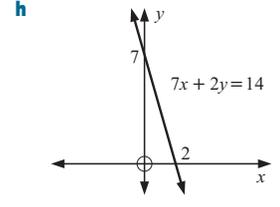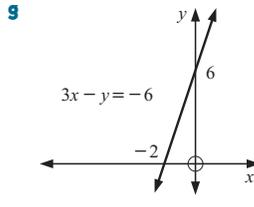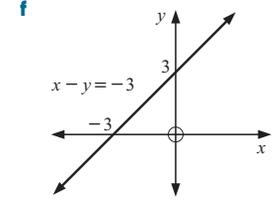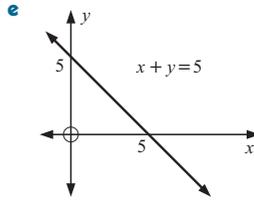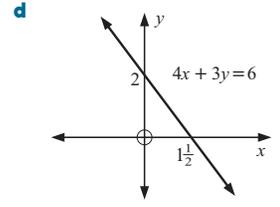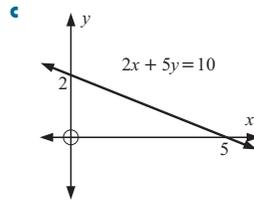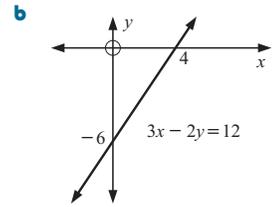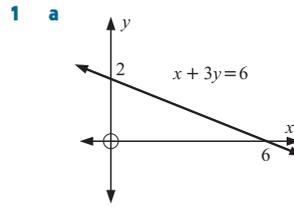
EXERCISE 6I

1 a yes **b** no **c** yes **2 a** $k = -4$ **b** $k = 1$
3 a $a = 7$ **b** $a = 9$ **c** $a = 5$ **d** $a = 7$
4 a $b = -3$ **b** $b = -\frac{9}{4}$ **c** $b = -\frac{7}{5}$ **d** $b = \frac{5}{4}$
  e $b = 3$ **f** $b = \frac{2}{3}$

EXERCISE 6J

1 a $(-1, 2)$ **b** $(2, 4)$ **c** $(3, 1)$ **d** $(-2, 3)$ **e** $(0, 6)$
  f $(-1, 2)$ **g** $(2, 1)$ **h** no point of intersection
  i infinitely many points of intersection
2 a None, as the lines are parallel.
  b Infinitely many, as the lines are coincident.
  c If $k = 5$, infinitely many, as the lines are coincident;
    if $k \neq 5$, none, as the lines are parallel.

REVIEW SET 6A

1 a $2\sqrt{17}$ units **b** $(2, 2)$ **c** -4 **d** C$(-1, 14)$
2 $a = -2$ or 4
3 PQ $=$ PR $= \sqrt{20}$ units, QR $= 4$ units, isosceles triangle

4 **a** $a = -\frac{18}{5}$ **b** $a = 10$ **5** **a** M$(-1, 2)$ **b** A$(-2, 3)$

6 $n = 21$ **7** **a** $m = \frac{2}{5}$, $c = 3$ **b** $m = \frac{2}{3}$, $c = -\frac{8}{3}$

8 **a** $y = 0$ **b** $y = -x$ **c** $y = \frac{1}{2}x + 3$

9 **a** $y = 5x - 2$ **b** $y = \frac{3}{4}x - \frac{13}{4}$ **c** $y = -\frac{4}{5}x + \frac{1}{5}$

10
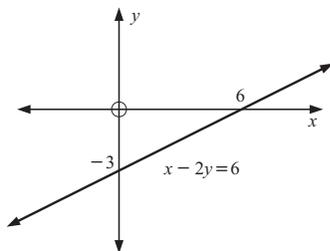
11 no **12** $(4, 2)$

13 **a** $\frac{1000}{3}$

  b The gradient represents the number of litres of water run from the tank per hour.

  c rate is constant as gradient is constant

REVIEW SET 6B

1 **a** $\sqrt{58}$ units **b** $(\frac{1}{2}, \frac{1}{2})$ **c** $-\frac{7}{3}$ **d** $7x + 3y = 5$

2 PQ $= \sqrt{10}$ units, QR $= \sqrt{17}$ units, PR $= 5$ units $\therefore$ scalene

3 $k = 0$ or 4

4 **a** Cyril's airstrip is closer by 17.9 km. **b** ≈ 569 km
  c $(54.5, -51.5)$

5 **a** A has gradient 14, B has gradient 10
  b Gradient is the number of km travelled per litre of petrol used.
  c $33.43

6 **a** OA $=$ AB $=$ BC $=$ CO $= 5$ units
  b $m_{[OB]} = \frac{1}{2}$, $m_{[AC]} = -2$ and these are $-$ve reciprocals
    $\therefore$ [OB] $\perp$ [AC].

7 **a** $d = \frac{8}{5}$ **b** $d = -10$ **8** $a = 1$

9 **a** $m = \frac{1}{3}$, $c = \frac{2}{3}$ **b** $m = -\frac{3}{2}$, $c = 4$

10 **a** $y = 5x - 1$ **b** $5x - 2y = 10$ **c** $2x + 3y = 7$

11 $k = \frac{37}{7}$

12

13 **a**

  b $(3, 2)$
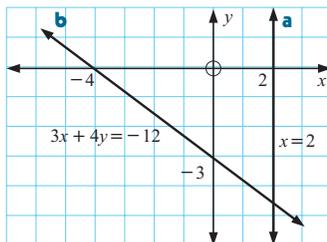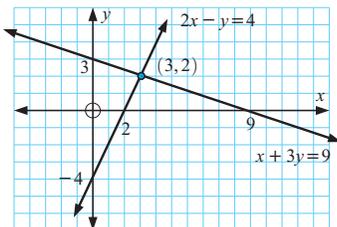
1 **a** AE $= 9$,   %E $\approx 3.09\%$ **b** AE $= 0.05$,   %E $\approx 1.69\%$
  c AE $= \$177$,   %E $\approx 0.556\%$
  d AE $\approx 0.001\,26$,   %E $\approx 0.0402\%$

2 **a** 0.5 mm **b** 0.5 kg **c** 0.5 cm **d** 50 mL

3 **a** between 153.5 cm and 154.5 cm
  b AE $= 0.5$ cm (or 5 mm) **c** %E $\approx 0.325\%$

4 **a** smallest possible length $= 13.75$ m
    smallest possible width $= 7.25$ m
  b perimeter $= 42.2 \pm 0.2$ m **c** %E $\approx 0.474\%$

5 **a** Yes **b** Boundaries are 99.69 m^2 and 101.80 m^2.

EXERCISE 7B.1

2 **a** 2610 m **b** 4.3 m **c** 8.65 m **d** 70 cm
  e 11.5 km **f** 367.5 cm **g** 0.381 m **h** 682 mm
  i 567 000 cm **j** 28.6 m **k** 24.3 m **l** 328 000 mm

3 **a** 13 700 m **b** 13.7 km **c** 1 370 000 cm

4 **a** **i** 12 000 000 mm **ii** 12 000 m **iii** 12 km
  b 20 000 nails

5 **a** 60 m **b** 1 800 000 cm

EXERCISE 7B.2

1 **a** 29.2 cm **b** 37.2 cm **c** 6.96 m

2 **a** 84.2 cm **b** 16.8 cm **c** 30.6 cm

3 **a** 37 cm **b** 40 km **c** 36 cm **d** 451 m **e** 61.7 cm
  f 34.6 cm **g** 94.2 cm **h** 13.7 cm **i** 37.7 cm

4 **a** $P = 4a + 2b + 2c$ **b** $P = 12a$
  c $P = 2r + \frac{3}{2}\pi r$ **d** $P = 2\pi x + 2y$

5 700 m **6** $4370 **7** **a** 965.6 m **b** 13 rolls
8 133 times **9** €889.88 **10** 407 m **11** ≈ 10.3 m
12 61 cm **13** 25.5 cm square **14** **a** 74.34 m **b** 6.28 m
15 ≈ 124 hours **16** $1531.53 **17** ≈ 1 h 35 min

EXERCISE 7C.1

1 **a** 3.84 ha **b** 257 000 m^2 **c** 35 000 cm^2 **d** 50 m^2
  e 1800 mm^2 **f** 4.6 cm^2 **g** 9 000 000 m^2 **h** 3500 ha
  i 5 km^2

3 **a** 387 cm^2 **b** 147 cm^2 **c** 30 cm^2 **d** 66.0 cm^2
  e 36 cm^2 **f** 6963 m^2

4 **a** 87.3 cm^2 **b** 5.53 m^2 **c** 73.3 cm^2 **5** 2.85 m^2
6 46.2% **7** £13 600 **8** 12 flights **9** 25.5%
10 0.636 : 1 **11** **a** 7500 washers **b** 3050 washers
12 23.9^o **13** $\sqrt{2}$

14 **a** $A = \frac{1}{2}\pi r^2$ **b** $A = \pi R^2 - \pi r^2$
  c $A = (x + 2a)^2 - \pi a^2$ **d** $A = \pi a^2 + 2ab$
  e $A = a\sqrt{x^2 - a^2}$ **f** $A = \sqrt{3}a^2$

EXERCISE 7C.2

1 6 units^2 **2** **a** 27.9 m^2 **b** 3.7 km^2 **3** 26.7 m^2

EXERCISE 7D.1

1 **a** 104 cm^2 **b** 4400 cm^2 **c** 41.5 m^2

2 **a** 360 cm^2 **b** 816 m^2 **c** 9835 cm^2

3 $A = 2ab + 2bc + 2ac$

4 **a** 36 m^2 **b** 392 m^2 **c** $2778.30

EXERCISE 7D.2

1 **a** 1640 cm^2 **b** 758 cm^2 **c** 452 cm^2 **d** 942 m^2
  e 603 cm^2

2 $10 698 **3** €2562 **4** 141 spheres **5** 3.69 m^2
6 $30.31 **7** **a** 5.64 m **b** 21.8 cm **c** 25.8 m

EXERCISE 7E.1

1 **a** $1\,900\,000$ cm^3 **b** 0.034 m^3 **c** 2 cm^3
d 840 L **e** 4.12 L **f** $0.006\,25$ kL
g $180\,000$ cm^3 **h** 0.0065 kL **i** 0.0048 m^3

2 56 bottles **3** 245 L

EXERCISE 7E.2

1 **a** 1800 cm^3 **b** 1567 cm^3 **c** 1.51 m^3 **d** 20.8 cm^3

2 **a** $1\,350\,000$ m^2 **b** 2.295×10^6 kL

3 **a** 45 m^2 **b** 360 kL **4** €595

5 **a** 18.1 m^3 **b** 0.08 m^3 min^{-1} **c** 3 h 46 min

6 **a** 100 cm^3 **b** 4.19 cm^3 **c** 23 spheres **d** 3.66 cm^3

7 617.9 km

8 **a**
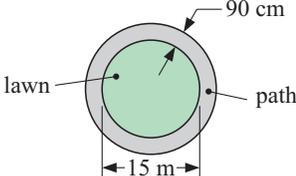
b 45.0 m^2
c 4.50 m^3

9 1.84 m

10 **a** 3.93 m^3 **b** 9.05 m^3 **c** 13.0 m^3 **d** $\$1842$

11 **a** $V = 6x^3$ **b** $V = \frac{1}{2}abc$ **c** $V = 4\pi r^3$ **d** $V = \pi x^3$

e $V = \dfrac{a^2 h}{3}$ **f** $V = \frac{1}{2}\pi x^3$ **g** $V = 4a^2 b + \pi a^3$

h $V = \dfrac{5\pi x^3}{3}$ **i** $V = 2\sqrt{3}x^3$

12 **a** **i** 2.82 cm **ii** 94.0 cm^3 **b** **i** 1.86 m **ii** 43.5 m^2

13 107 cm **14** $15\,915$ handles

REVIEW SET 7A

1 **a** 5300 m **b** 2 m^2 **c** $5\,000\,000$ cm^3 **d** 480 cm^3

2 **a** 47.1 cm **b** 92.8 m **c** 18 cm

3 **a** 105 cm^2 **b** 30 cm^2 **c** 18.3 cm^2 **4** 262°

5 %E $\approx 0.368\%$ **6** **a** 400 cm^3 **b** 340 cm^2 **7** 227 cm^2

8 5305 cans

9 **a** $A = \dfrac{(a+b)h}{2} + bc$ **b** $A = 4x^2 + \dfrac{\pi x^2}{2}$ **10** 10.8 cm

REVIEW SET 7B

1 **a** 3.2 m **b** $150\,000$ m^2 **c** $3\,600\,000$ mm^3 **d** 4.5 m^3

2 **a** 565 m **b** £6927 **3** 84 ± 1.5 cm

4 **a** $\frac{3}{8}$ **b** 52.3 cm **c** 170 cm^2 **5** $\$48\,768$

6 **a** 5196 cans **b** 154 m^2 **7** 41.8 m^3 **8** 21.7 m^2

9 8 times

EXERCISE 8A

1 **a** $x(3x+5)$ **b** $x(2x-7)$ **c** $3x(x+2)$
d $4x(x-2)$ **e** $-x(2x-9)$ **f** $-3x(x+5)$
g $4x(2x-1)$ **h** $-5x(2x+1)$ **i** $4x(3-x)$
j $x(x^2+x+1)$ **k** $x(2x^2+11x+4)$ **l** $a(b+c+d)$
m $ax(x+2)$ **n** $ab(b+a)$ **o** $ax^2(x+1)$

2 **a** $(x+2)(x-3)$ **b** $(x-1)(x-4)$ **c** $(x+1)(x+3)$
d $(x-2)(x+1)$ **e** $(x+3)(x+4)$ **f** $(x+4)(x+6)$
g $(x-3)(x-4)$ **h** $(x+4)(x+2)$ **i** $(x-4)(x-9)$
j $(x+2)(x+5)$ **k** $(x+1)^2(x+2)$
l $(a+b)(a^2+b^2+2ab+1)$ **m** $(x+1)(2x+3)$
n $(x-2)(3x-7)$ **o** $2(a+b)(2a+2b-1)$

EXERCISE 8B

1 **a** $(x+2)(x-2)$ **b** $(2+x)(2-x)$
c $(x+9)(x-9)$ **d** $(5+x)(5-x)$
e $(2x+1)(2x-1)$ **f** $(3x+4)(3x-4)$
g $(2x+3)(2x-3)$ **h** $(6+7x)(6-7x)$

2 **a** $3(x+3)(x-3)$ **b** $-2(x+2)(x-2)$
c $3(x+5)(x-5)$ **d** $-5(x+1)(x-1)$
e $2(2x+3)(2x-3)$ **f** $-3(3x+5)(3x-5)$

3 **a** $(x+\sqrt{3})(x-\sqrt{3})$ **b** no linear factors
c $(x+\sqrt{15})(x-\sqrt{15})$ **d** $3(x+\sqrt{5})(x-\sqrt{5})$
e $(x+1+\sqrt{6})(x+1-\sqrt{6})$ **f** no linear factors
g $(x-2+\sqrt{7})(x-2-\sqrt{7})$
h $(x+3+\sqrt{17})(x+3-\sqrt{17})$ **i** no linear factors

4 **a** $(x+3)(x-1)$ **b** $4(x+2)(x-1)$
c $(x-5)(x+3)$ **d** $3(x+1)(3-x)$
e $(3x+2)(x-2)$ **f** $(2x+3)(4x-3)$
g $(3x-1)(x+3)$ **h** $8x(x-1)$ **i** $-3(4x+3)$

EXERCISE 8C

1 **a** $(x+3)^2$ **b** $(x+4)^2$ **c** $(x-3)^2$
d $(x-4)^2$ **e** $(x+1)^2$ **f** $(x-5)^2$
g $(y+9)^2$ **h** $(m-10)^2$ **i** $(t+6)^2$

2 **a** $(3x+1)^2$ **b** $(2x-1)^2$ **c** $(3x+2)^2$
d $(5x-1)^2$ **e** $(4x+3)^2$ **f** $(5x-2)^2$
g $-(x-1)^2$ **h** $-2(x+2)^2$ **i** $-3(x+5)^2$

EXERCISE 8D

1 **a** $(b+c)(x+y)$ **b** $(p+q)(2x+3)$ **c** $(2a+b)(3x+2)$
d $(a+b)(m-n)$ **e** $(r-s)(3d+1)$ **f** $(a+b)(2c-5)$
g $(x+5)(x+7)$ **h** $(x-2)(x-6)$ **i** $(x+3)^2$
j $(x+1)(x+8)$ **k** $(x+1)(2x+3)$ **l** $(x-1)(3x+1)$
m $(x+5)(2x+3)$ **n** $(3x+2)(2x-1)$ **o** $(x+2)(4x+1)$
p $(2x+1)(3x+5)$ **q** $(2x+3)(3x-2)$ **r** $(x-2)(4x+1)$
s $(x+11)(3x+4)$ **t** $(3x-2)(6x+1)$ **u** $(2x-7)(5x+2)$

2 **a** $(x-1+a)(x-1-a)$ **b** $(x+a)(x-a+1)$
c $-(x+2+b)(x+2-b)$ **d** $-(x-3+c)(x-3-c)$
e $(x-y)(x+y-1)$ **f** $2(a-b)(a+2b)$
g $(x+2+m)(x+2-m)$ **h** $(x+a+b)(x+a-b)$
i $(x+y)(x-y-3)$

EXERCISE 8E

1 **a** $(x+2)(x+1)$ **b** $(x+3)(x+2)$ **c** $(x-3)(x+2)$
d $(x+5)(x-2)$ **e** $(x+7)(x-3)$ **f** $(x+4)^2$
g $(x-7)^2$ **h** $(x+7)(x-4)$ **i** $(x+5)(x+2)$
j $(x-8)(x-3)$ **k** $(x+11)(x+4)$ **l** $(x+7)(x-6)$
m $(x-8)(x+7)$ **n** $(x-9)^2$ **o** $(x-8)(x+4)$

2 **a** $2(x-4)(x+1)$ **b** $3(x+4)(x-1)$
c $5(x+3)(x-1)$ **d** $4(x+5)(x-4)$
e $2(x-5)(x+3)$ **f** $3(x+7)(x-3)$
g $-2(x-5)(x+4)$ **h** $-3(x-2)^2$
i $-7(x+4)(x-1)$ **j** $-(x+2)(x+1)$
k $-(x-3)(x-2)$ **l** $-(x-3)(x-6)$
m $5(x+5)(x-2)$ **n** $-2(x+7)(x-3)$
o $-(x-8)(x+4)$

EXERCISE 8F

1 **a** $3x(x+3)$ **b** $(2x+1)(2x-1)$ **c** $5(x+\sqrt{3})(x-\sqrt{3})$
d $x(3-5x)$ **e** $(x+8)(x-5)$ **f** $2(x+4)(x-4)$
g does not factorise **h** $(x+5)^2$ **i** $(x-3)(x+2)$
j $(x-13)(x-3)$ **k** $(x-12)(x+5)$ **l** $(x-4)(x+2)$
m $(x+5)(x+6)$ **n** $(x+8)(x-2)$ **o** $(x-8)(x+3)$
p $3(x+6)(x-4)$ **q** $4(x-5)(x+3)$ **r** $3(x-11)(x-3)$
s $-(x-7)(x-2)$ **t** $-(x+4)(x+9)$ **u** $-2(x+9)(x-2)$

EXERCISE 8G

1 **a** $(2x+3)(x+1)$ **b** $(2x+5)(x+1)$ **c** $(7x+2)(x+1)$
d $(3x+4)(x+1)$ **e** $(3x+1)(x+4)$ **f** $(3x+2)(x+2)$
g $(4x+1)(2x+3)$ **h** $(7x+1)(3x+2)$ **i** $(3x+1)(2x+1)$
j $(6x+1)(x+3)$ **k** $(5x+1)(2x+3)$ **l** $(7x+1)(2x+5)$

2 **a** $(2x+1)(x-5)$ **b** $(3x-1)(x+2)$ **c** $(3x+1)(x-2)$
d $(2x-1)(x+2)$ **e** $(2x+5)(x-1)$ **f** $(5x+1)(x-3)$
g $(5x-3)(x-1)$ **h** $(11x+2)(x-1)$ **i** $(3x+2)(x-3)$
j $(2x+3)(x-3)$ **k** $(3x-2)(x-5)$ **l** $(5x+2)(x-3)$
m $(3x-2)(x+4)$ **n** $(2x-1)(x+9)$ **o** $(2x-3)(x+6)$
p $(2x-3)(x+7)$ **q** $(5x+2)(3x-1)$ **r** $(21x+1)(x-3)$
s $(3x-2)^2$ **t** $(4x-5)(3x+8)$ **u** $(8x-3)(2x+5)$

3 **a** $-(3x+7)(x-2)$ **b** $-(5x-1)(x-2)$
c $-(4x-3)(x+3)$ **d** $-(3x-2)^2$
e $-(4x+1)(2x+3)$ **f** $-(6x+1)(2x-3)$

REVIEW SET 8A

1 **a** $3x(x+4)$ **b** $3x(5-x)$
c $(x+3)(x-1)$ **d** $(2x+3)(2x-3)$
e $6(x+2y)(x-2y)$ **f** $(x+\sqrt{13})(x-\sqrt{13})$
g $(x+1+\sqrt{6})(x+1-\sqrt{6})$ **h** $(x+4)^2$ **i** $(x-5)^2$

2 **a** $2(x+1)(x+3)$ **b** $5(x-1)^2$
c $(x+2)(a+b)$ **d** $(c-d)(2x-1)$
e $(x+4)(3x+2)$ **f** $(2x+3)(3x-1)$
g $(x-8)(x-3)$ **h** $(x+2+a)(x+2-a)$

3 **a** $(2x+1)(x+8)$ **b** $(2x-1)(x+8)$
c $(2x-1)(x-8)$ **d** $(2x-5)(3x+2)$
e $(3x+2)(4x-1)$ **f** $(2x-3)(6x+5)$

REVIEW SET 8B

1 **a** $4x(x-2)$ **b** $8x(2-x)$ **c** $2x(2x-1)$
d $(3+5x)(3-5x)$ **e** $2(3+a)(3-a)$
f $(x+\sqrt{23})(x-\sqrt{23})$
g $(x+2+\sqrt{3})(x+2-\sqrt{3})$ **h** $(x-6)^2$ **i** $2(x+2)^2$

2 **a** $3(x-3)(x+1)$ **b** $7(x+2)^2$ **c** $(m+n)(x-y)$
d $(3a+b)(a-2b)$ **e** $(2x+3)(x+4)$ **f** $(2x+3)(2x-1)$

3 **a** $(3x+1)(x-6)$ **b** $(3x-1)(x-6)$ **c** $(3x-1)(x+6)$
d $(4x+1)(3x+1)$ **e** $(x-2)(12x+1)$ **f** $(3x+2)^2$

EXERCISE 9A

1 **a** discrete numerical **b** continuous numerical
c categorical **d** categorical **e** categorical
f continuous numerical **g** continuous numerical
h discrete numerical **i** discrete numerical

2 **a** continuous numerical **b** categorical **c** categorical
d continuous numerical **e** discrete numerical
f discrete numerical **g** categorical

3 **a**
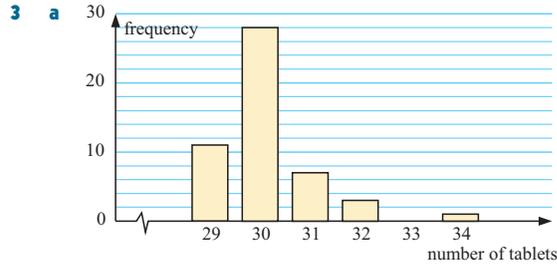
b discrete numerical **c** positively skewed

4 **a**
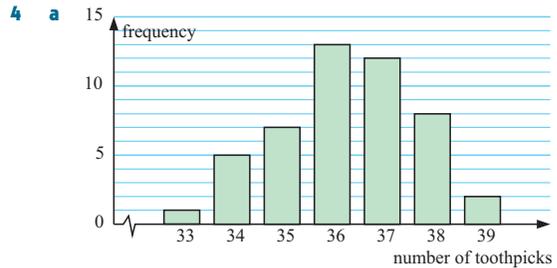
b discrete numerical **c** slightly negatively skewed

5 **a**
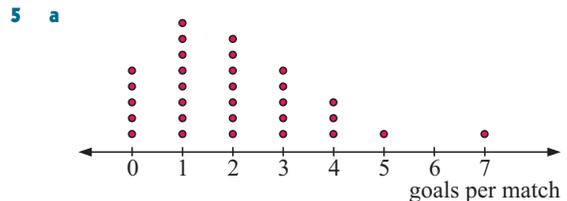
b positively skewed with 7 a possible outlier

6 **a**

Stem	Leaf
0	7
1	8
2	7 9
3	2 2 5 6 8 9 9
4	0 3 4 4 5 6 6 7 8

1 | 8 means 18 marks

b 45% **c** 20% **d** 90% **e** negatively skewed

7 **a**

Stem	Leaf
87	
87*	8
88	2
88*	5 5 6 8 9 9
89	1 2 3 3 4 4 4 4
89*	5 5 6 6 7 8 8 9
90	0 1 1 1 1 3 3 3 3 4 4 4 4
90*	5 6 7 7 7 8 8 9
91	0 0 3 3 3
91*	5 6 7
92	1 4 4
92*	7 8
93	1
93*	

87* | 8 means 878 peanuts

b 60% **c** $13\frac{1}{3}\%$ **d** slightly positively skewed
e No, only $88\frac{1}{3}\%$ of jars have between 880 and 920 peanuts.

EXERCISE 9B

1 **a** Heights can take any value from 170 cm to 205 cm.

b

Heights of basketball players
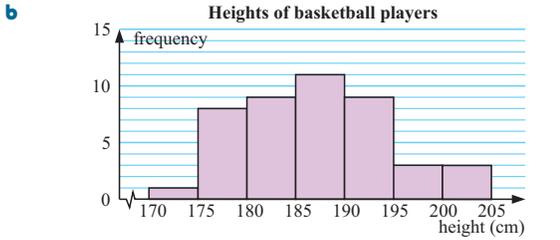
c The modal class is $(185 - < 190)$ cm as this occurred the most frequently.

d slightly positively skewed

2 a discrete (as is measured to nearest kg)

b

Stem	Leaf
0	3 6 8 8 8 8
1	0 0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6 1	2 means 12 kg

c positively skewed

d The modal weight loss was between 10 and 20 kg.

3 a 700 vehicles **b** $\approx 25.7\%$ **c** $\approx 38.6\%$ **d** $\approx 15.7\%$
e $14 850

EXERCISE 9C

1 Team A 91.25, Team B 91.75, $\therefore$ Team B

2 a 29 **b** 107 **c** 149.5

3 a 1.7 siblings **b** 1.5 siblings **4** ≈ 203 mm

5 a mean = £163 770, median = £147 200 (differ by £16 570)
The mean is affected by extreme values whereas the median is not. Thus the mean has been 'pushed up' higher than the median.
b i mean selling price **ii** median selling price

6 a mean = 59.45 nails, median = 59 nails
b Both the mean and median number of nails per pack are under the store's claim of 60 nails per pack.
c The mean, as it takes into account all of the data and there are no extreme values.

7 mean ≈ 34.6 chocolates, median ≈ 35 chocolates

8 a ≈ 70.9 g **b** ≈ 210 g **c** 139 g

9 17.3 goals per game **10** ≈ 17.7 **11** 10.1 **12** 6 and 12

EXERCISE 9D

1 a i 14 **ii** 4 **iii** 10 **iv** 13
b i 52 **ii** 45.5 **iii** 6.5 **iv** 21
c i 31.5 **ii** 19.5 **iii** 12 **iv** 38
d i 3 **ii** 1 **iii** 2 **iv** 5

2 a median = 2.05 minutes, $Q_3 = 3.15$ minutes,
$Q_1 = 1.1$ minutes
b range = 5.6 minutes, IQR = 2.05 minutes
c i "50% of the waiting times were greater than 2.05 minutes."
ii "75% of the waiting times were less than 3.15 minutes."
iii "The minimum waiting time was 0 minutes and the maximum waiting time was 5.6 minutes. The waiting times were spread over 5.6 minutes."

3 a 6.0 **b** 10.1 **c** 8.15 **d** 7.5 **e** 8.9 **f** 4.1
g 1.4

EXERCISE 9E

1 a i 98% **ii** 30% **b** 73% **c** 68% **d** 75%
e 22% **f** 82% and 98% **g** no
h The median divides the 'box' and the minimum and maximum values disproportionately, clearly showing a negatively skewed distribution.

2 a 12 **b** lower boundary = 13.5, upper boundary = 61.5
c 13.2 and 65 would be outliers

3 a median = 6, $Q_1 = 5$, $Q_3 = 8$ **b** 3
c lower boundary = 0.5, upper boundary = 12.5
d yes, 13 **e**
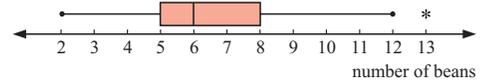
**4
a** $Min_x = 33$, $Q_1 = 35$, $Q_2 = 36$, $Q_3 = 37$, $Max_x = 40$
b i 7 **ii** 2 **c** no
d
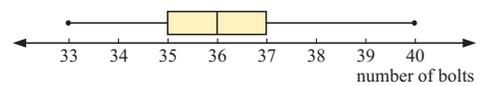
EXERCISE 9F

1 a a frequency histogram **b** continuous **c** $(40 - < 50)$ kg
d

Weight (kg)	Freq.	Midpoint of interval (kg)	Product
10 - < 20	5	15	75
20 - < 30	10	25	250
30 - < 40	25	35	875
40 - < 50	40	45	1800
50 - < 60	15	55	825
60 - < 70	5	65	325
Total	100		4150

e 41.5 kg
f i ≈ 42.6 kg **ii** ≈ 34.1 kg **iii** ≈ 48.9 kg
g ≈ 14.7 kg

2 a No, all we know is that the oldest person is between $(80 - < 90)$ years old.
b
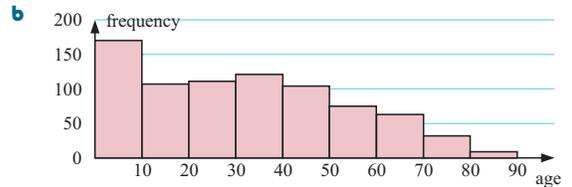
c 10 years **d** $(0 - < 10)$ years   (170 times)
e ≈ 32.2 years
f i ≈ 30.7 years **ii** ≈ 12.6 years **iii** ≈ 48.2 years
g ≈ 35.6 years
h The distribution is positively skewed with more younger people and less older people in the town.

EXERCISE 9G

1 a

Trout length (cm)	Freq.	Cum. Freq.
$21 \leqslant x < 24$	1	1
$24 \leqslant x < 27$	7	8
$27 \leqslant x < 30$	9	17
$30 \leqslant x < 33$	9	26
$33 \leqslant x < 36$	7	33
$36 \leqslant x < 39$	5	38
$39 \leqslant x < 42$	1	39
$42 \leqslant x < 45$	1	40

b

Cumulative frequency graph of lengths of trout
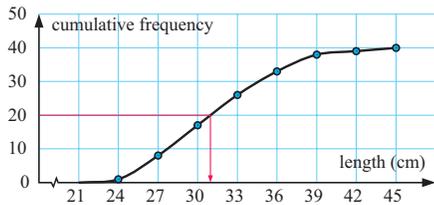
c median ≈ 31.0 cm   **d** median $= 30$ cm
e $Q_1 = 27.7$, $Q_3 = 34.7$, IQR $= 7.0$

2 a median ≈ 63.9 marks   **b** 53 students   **c** 60 students
d 25 students   **e** 76 out of 100   **f** IQR $= 24.4$

3 a median ≈ 44.8 min   **b** 14 of them   **c** 42.8 min
d 9.4 min, from 40 min to 49.4 min

4 a median ≈ 46.5 m   **b** 40 students
c 28 students   **d** 10 students

5 mean ≈ 170 cm,   median ≈ 171 cm, $Q_1 \approx 164$ cm,
$Q_3 \approx 177$ cm,   IQR ≈ 12.7 cm

REVIEW SET 9A

1 12.4 cm

2 a continuous   **b**

Stem	Leaf
0	4 8 9
1	3 5 7
2	1 5 6
3	1 4 5 6 7 9
4	1

c 62.5% $1 \mid 3$ means 1.3 cm

3 a 7
b $\text{Min}_x = 6$, $Q_1 = 7$, median $= 8$, $Q_3 = 9$, $\text{Max}_x = 13$
c

4 15
5 a 2 kg   **b** 2 - $<$ 4 kg   **c** $\overline{x} \approx 4.76$ kg
d

Frequency histogram of cat mass

e

Cumulative frequency graph of cat data
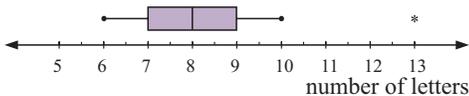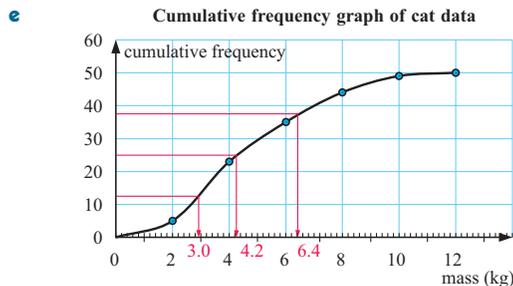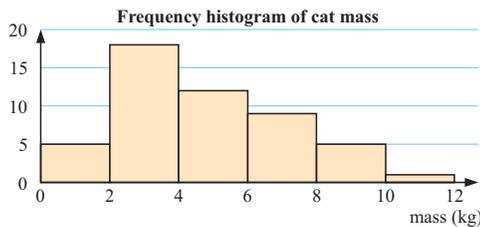
f i median ≈ 4.2 kg   **ii** $Q_1 \approx 3.0$ kg   **iii** $Q_3 \approx 6.4$ kg
g IQR ≈ 3.4 kg

REVIEW SET 9B

1 a

Stem	Leaf
17	2 5 6 7 9 9
18	0 0 1 1 2 3 3 4 5 6 6 7 8 8 9
19	0 1 2 2 3 4 6
20	0 5 20 $\mid$ 5 means 205 cm

b $\approx 26.7\%$
2 20.7 cm
3 a a frequency histogram   **b** continuous   **c** (55 - $<$ 60) kg
d

Weights (kg)	Freq.	Midpoints	Product
40 - $<$ 45	4	42.5	170
45 - $<$ 50	12	47.5	570
50 - $<$ 55	13	52.5	682.5
55 - $<$ 60	25	57.5	1437.5
60 - $<$ 65	17	62.5	1062.6
65 - $<$ 70	9	67.5	607.5
Total	80		4530

e ≈ 56.6 kg
4 a median $= 101.5$, $Q_1 = 98$, $Q_3 = 105.5$
b 7.5, the range of the middle 50% of scores
c lower boundary $= 86.75$,   upper boundary $= 116.75$
d yes, 81 and 84
e
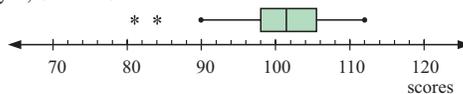
5 a 5 seconds   **b** 35 - $<$ 40 seconds
c

Frequency histogram of swim times
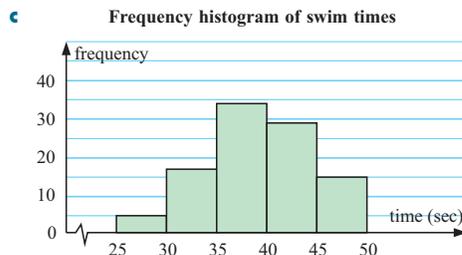
d **Cumulative frequency graph of swim times**
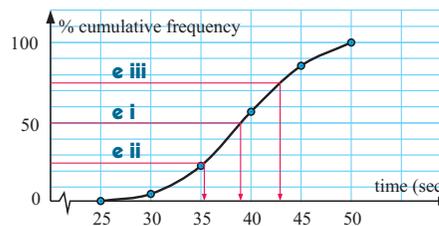
e   i ≈ 39.0 sec   **ii** ≈ 35.3 sec   **iii** ≈ 43.0 sec
f ≈ 7.83 seconds

EXERCISE 10A

1 a 0.47   **b** 0.53   **2 a** 0.17   **b** 0.157   **c** 0.327
3 a 0.78   **b** 0.22   **4 a** 0.242   **b** 0.502   **c** 0.256

EXERCISE 10B

1 a 0.487   **b** 0.051   **c** 0.731
2 a 43 days   **b** 127 calls
  c   i 0.047   **ii** 0.186   **iii** 0.465
3 a 0.089   **b** 0.126
4 a 0.957   **b** 0.0889   **c** 0.144   **d** 0.0935   **e** 0.0327

EXERCISE 10C

1 **a** 76.4 years   **b** 82 years   **c** 78.6 years   **d** 83.1 years

2 **a** 98.6%   **b** 87.1%   **c** 48.5%

3 **a** 99.2%   **b** 92.3%   **c** 66.3%

4 1260 males and 1635 females = 2895 people   **5** 0.004 82

EXERCISE 10D

1 **a** {H, T}   **b** {1, 2, 3, 4, 5, 6}
   c {BB, BG, GB, GG}   **d** {HH, HT, TH, TT}
   e {ABC, ACB, BAC, BCA, CAB, CBA}
   f {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}
   g {HHHT, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
   h {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

2 **a**
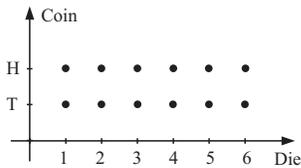
   b
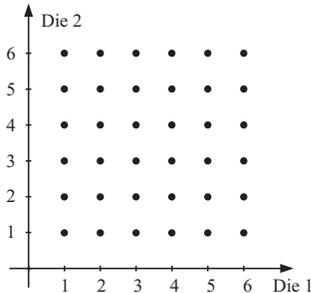
   c
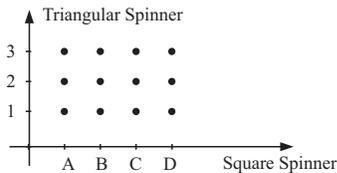
EXERCISE 10E

1 **a** $\frac{1}{3}$ **b** 0   **c** 0   **d** $\frac{1}{2}$ **e** $\frac{5}{6}$

2 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **3** **a** $\frac{1}{8}$ **b** $\frac{1}{2}$ **c** 0   **d** $\frac{5}{8}$

4 **a** $\frac{4}{7}$ **b** $\frac{3}{7}$ **c** 1   **d** 0

5 **a** $\frac{1}{13}$ **b** $\frac{12}{13}$ **c** $\frac{1}{2}$ **d** $\frac{1}{4}$ **e** $\frac{1}{52}$ **f** $\frac{4}{13}$

6 **a** $\frac{9}{80}$ **b** $\frac{1}{8}$ **c** $\frac{3}{10}$ **d** $\frac{9}{80}$

7 **a** $\frac{1}{2}$ or $\frac{31}{365}$ (more accurate) or $\frac{124}{1461}$ (more accurate again)
   b $\frac{1}{12}$ or $\frac{28}{365}$ (more accurate) or $\frac{113}{1461}$ (more accurate again)

EXERCISE 10F

1 **a** $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** $\frac{3}{4}$

2

   a $\frac{1}{12}$ **b** $\frac{7}{12}$ **c** $\frac{2}{3}$ **d** $\frac{5}{12}$ **e** $\frac{1}{4}$ **f** $\frac{3}{4}$
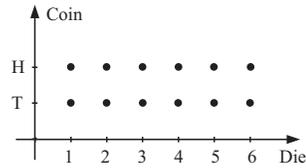
3
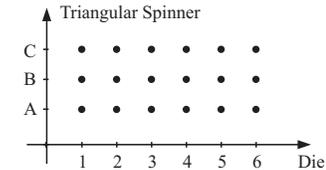
   a $\frac{1}{18}$ **b** $\frac{1}{6}$ **c** $\frac{2}{9}$

4 **a** $\frac{1}{36}$ **b** $\frac{1}{18}$ **c** $\frac{5}{9}$ **d** $\frac{11}{36}$ **e** $\frac{5}{18}$ **f** $\frac{25}{36}$ **g** $\frac{1}{6}$
   h $\frac{2}{9}$ **i** $\frac{5}{18}$ **j** $\frac{13}{18}$

EXERCISE 10G

1 **a** $\frac{6}{7}$ **b** $\frac{36}{49}$ **c** $\frac{216}{343}$ **2** **a** $\frac{1}{8}$ **b** $\frac{1}{8}$

3 **a** 0.0096   **b** 0.8096   **4** **a** $\frac{1}{16}$ **b** $\frac{15}{16}$

5 **a** 0.56   **b** 0.06   **c** 0.14   **d** 0.24

6 **a** $\frac{8}{125}$ **b** $\frac{12}{125}$ **c** $\frac{27}{125}$

EXERCISE 10H

1 **a**

   b

   c

   d

   e

   f
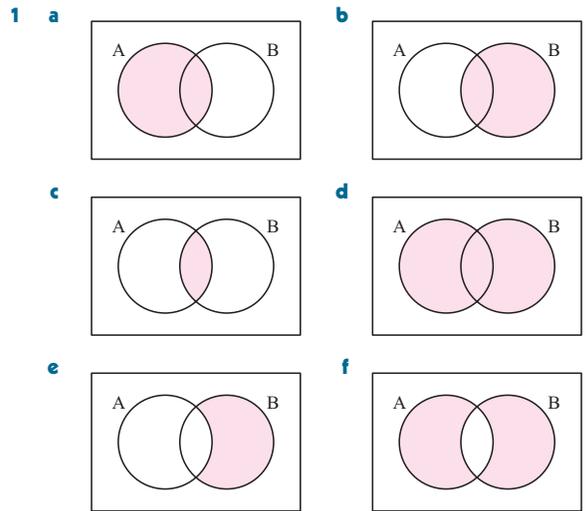
2 **a** 29   **b** 17   **c** 26   **d** 5

3 **a** 65   **b** 9   **c** 4   **d** 52

4 **a** $\frac{19}{40}$ **b** $\frac{1}{2}$ **c** $\frac{4}{5}$ **d** $\frac{5}{8}$ **e** $\frac{13}{40}$

5 **a** $\frac{19}{25}$ **b** $\frac{13}{25}$ **c** $\frac{6}{25}$ **6** **a** $\frac{7}{15}$ **b** $\frac{1}{15}$ **c** $\frac{2}{15}$

7 **a** **i** $\dfrac{b+c}{a+b+c+d}$ **ii** $\dfrac{b}{a+b+c+d}$

   iii $\dfrac{a+b+c}{a+b+c+d}$ **iv** $\dfrac{a+b+c}{a+b+c+d}$

b P(A or B) = P(A) + P(B) − P(A and B)

8 a i $\frac{1}{52}$ **ii** $\frac{4}{13}$

EXERCISE 10I

1 27 saves   **2** 15 days   **3** 30 doubles   **4** 200 times

5 a i 0.55   **ii** 0.29   **iii** 0.16

    b i 4125   **ii** 2175   **iii** 1200

6 a $3.50   **b** No   **7 a** $2.75   **b** $3.75   **8** $1.50

REVIEW SET 10A

1 a 0.269   **b** 0.548   **2** 39 goals

3 Two events are independent if the occurrence of one does not affect the occurrence of the other.

4 a
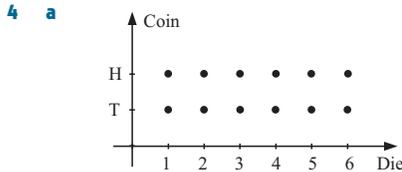
    b i $\frac{1}{4}$ **ii** $\frac{5}{12}$ **iii** $\frac{1}{3}$ **iv** $\frac{3}{4}$

5 a {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

    b i $\frac{3}{8}$ **ii** $\frac{7}{8}$

6 a 0.63   **b** 0.03   **c** 0.97   **d** 0.07

7 a 0.932   **b** 0.748   **c** 0.0697

REVIEW SET 10B

1 a i 0.0571   **ii** 0.571   **iii** 0.943

    b The answers in **a** should be fairly reliable as the data was collected over a reasonable length of time (35 days). However, the sample size is small.

2 62 waves   **3 a** $\frac{1}{3}$ **b** $\frac{1}{2}$ **4 a** $\frac{9}{25}$ **b** $\frac{4}{25}$ **c** $\frac{12}{25}$

5 a $7   **b** No, as Zelda should expect to lose $1 each game.

6 a 0.960   **b** 0.653   **c** 0.235

7
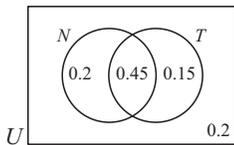
a 0.65
b 0.2
c 0.8
d 0.35

EXERCISE 11A.1

1 a 1.2   **b** 0.8   **c** 1.45   **d** 0.55   **e** 0.92   **f** 1.03
    g 2   **h** 7   **i** 0

2 a 12% increase   **b** 23% increase   **c** 4% decrease
    d 15% decrease   **e** 45% increase   **f** 33% decrease
    g 140% increase   **h** 70% decrease

3 a $52.50   **b** €18.90   **c** £624   **d** $78.20   **e** £175
    f €480   **g** $35.28   **h** €338   **i** £3200   **j** $1125

4 a 25% increase   **b** 20% decrease   **c** 3.53% decrease
    d 12.5% increase   **e** 14.7% increase   **f** 31.4% decrease

5 26.8% increase   **6** 12.5% decrease   **7** 11.6% loss

EXERCISE 11A.2

1 a €33.60   **b** €77.50   **c** €162   **d** €416   **e** €57.60

2 $1.17   **3 a** $3.68 per kg   **b** $1.44 each   **c** $2.04 each

4 $8.25 per kg   **5** $594

EXERCISE 11A.3

1 a $56   **b** $120   **c** $156   **2** €17   **3** £697   **4** $392

5 RM 52.25   **6** $3.79 per kg   **7** €1.05   **8** ¥2950   **9** €80

EXERCISE 11A.4

1 a £282   **b** £42   **2 a** $2294.90   **b** $129.90

3 a €420   **b** €70   **4 a** HRK 76 250   **b** HRK 13 750

EXERCISE 11A.5

1 $54.56   **2** $462   **3** $2.11   **4** €47.60   **5** 12% profit

6 4% profit   **7** 60% profit

EXERCISE 11B

1 £582.06   **2** $24 569.01   **3** €37 781.73   **4** 28.3%

5 28.5%   **6** 1.38% decrease

EXERCISE 11C.1

1 a i £60   **ii** £1240

    iii
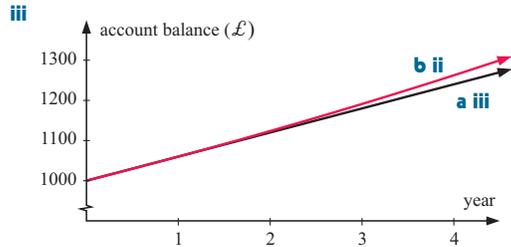
    b i

After year	Interest paid	Account Balance
0		£1000
1	£60	£1060
2	£63.60	£1123.60
3	£67.42	£1191.02
4	£71.46	£1262.48

    c The simple interest graph is a straight line, whereas the compound interest graph is curved. The account balance is higher if the bank pays compound interest.

2 a i $13 200   **ii** $13 604.89
    b compound interest, by $404.89

3 a €1223.01   **b** £7008.78   **c** $2326.78
    d 4104.54 pesos   **e** ¥109 298   **f** 9235.09 pesos

EXERCISE 11C.2

1 a £13 107.96   **b** £3107.96

2 a €6600.09   **b** €1850.09

3 a $30 866.03   **b** $10 866.03   **4** $10 570.72

5 €15 372.52   **6** £71 496.20   **7** 11.3% p.a.   **8** 13.1% p.a.

9 16.6% p.a.

EXERCISE 11D.1

1 a

Age (yrs)	Depreciation	Book Value
0		£15 000
1	15% of £15 000 = £2250	£12 750
2	15% of £12 750 = £1912.50	£10 837.50
3	15% of £10 837.50 = £1625.63	£9211.87

    b i £2250   **ii** £1912.50   **iii** £1625.63

2 a €11 200.78   **b** $459 330.89   **c** £157 959.09

EXERCISE 11D.2

1 **a** €53 393.55 **b** €171 606.45
2 **a** £16 588.80 **b** £15 811.20
3 **a** €175 702.59 **b** €64 297.41
4 **a** $52 308.81 **b** $178 949.48 **c** $126 640.68

EXERCISE 11E.1

1 €23.36 **2** $12.12 **3** £1316.27 **4** $41.98

EXERCISE 11E.2

1 **a** €1240 **b** €190 **2 a** $5500 **b** $300 **3** £314
4 If Kara is paying cash she should purchase the bed from *Store A*
($865 from *A*, $895 from *B*).
If Kara is buying on terms she should purchase the bed from
Store B ($1038.50 from *B*, $1053.10 from *A*).
5 *Dealer A*: $51 276 on terms, *Dealer B*: $53 388 on terms
If Khoa is paying cash he can choose either dealer. If buying on
terms, he should choose *Dealer A*.

EXERCISE 11E.3

1

Month	Opening Balance	Interest	Repayment	Reduction	Closing Balance
1	$3000.00	$22.50	$600	$577.50	$2422.50
2	$2422.50	$18.17	$600	$581.83	$1840.67
3	$1840.67	$13.81	$600	$586.19	$1254.48
4	$1254.48	$9.41	$600	$590.59	$663.89
5	$663.89	$4.98	$600	$595.02	$68.87
6	$68.87	$0.52	$69.39	$68.87	$0.00

5 full payments and 1 part payment are required.
2 €19 257.63 **3** £94 352.11

EXERCISE 11E.4

1 **a** €329.90 **b** €19 794 **c** €4794
2 **a** £606.50 **b** £21 834 **c** £3834
3 **a** $14 382 **b** $9198
Trina should borrow from the Cash Credit Union ($5184 less
interest) as long as she can afford the higher repayments.

REVIEW SET 11A

1 **a** 112 kg **b** $52.50 **c** 12.7%
2 £152 048.20 **3 a** $19 546.74 **b** $7546.74
4 £6273.18 **5 a** 3 full payments **b** €268.19
6 **a** €9996 **b** €29 626 **c** €4636
7 **a** ¥19 643 **b** ¥707 159 **c** ¥107 159

REVIEW SET 11B

1 **a** €4190.63 **b** $41.65 **c** 10.5% increase
2 $2643.30 **3** £124 209.01 **4** $185 765.37 **5** ¥5425
6 **a** $19.57 **b** $150 **7 a** €21 541.14 **b** €24 958.86
8 **a** 30 000 rupees **b** 644.90 rupees **c** 38 694 rupees
d 8694 rupees

EXERCISE 12A

1 **a** 4.0 m **b** 7.2 m **2** 51.3 m
3 $Z\widehat{X}Y = 57^o$, $X\widehat{Y}Z = 43^o$, $X\widehat{Z}Y = 80^o$ **4** 1072 m
5 From a scale diagram the required angle is 41° (approx). Because
of possible errors in drawing and measurement the actual answer
could be $> 41^o$ or $< 41^o$. So, a scale diagram is not accurate
enough to answer the question.

EXERCISE 12B

1 **a i** [AC] **ii** [BC] **iii** [AB]
b i [RS] **ii** [TR] **iii** [ST]
c i [AB] **ii** [BC] **iii** [AC]
2 **a** *b* units **b** *c* units **c** *c* units **d** *b* units

EXERCISE 12C.1

1 **a i** $\dfrac{a}{c}$ **ii** $\dfrac{b}{c}$ **iii** $\dfrac{a}{b}$ **iv** $\dfrac{b}{c}$ **v** $\dfrac{a}{c}$ **vi** $\dfrac{b}{a}$
b i $\dfrac{m}{k}$ **ii** $\dfrac{l}{k}$ **iii** $\dfrac{m}{l}$ **iv** $\dfrac{l}{k}$ **v** $\dfrac{m}{k}$ **vi** $\dfrac{l}{m}$
c i $\frac{4}{5}$ **ii** $\frac{3}{5}$ **iii** $\frac{4}{3}$ **iv** $\frac{3}{5}$ **v** $\frac{4}{5}$ **vi** $\frac{3}{4}$
d i $\dfrac{4}{\sqrt{41}}$ **ii** $\dfrac{5}{\sqrt{41}}$ **iii** $\dfrac{4}{5}$ **iv** $\dfrac{5}{\sqrt{41}}$ **v** $\dfrac{4}{\sqrt{41}}$ **vi** $\frac{5}{4}$
e i $\dfrac{3}{\sqrt{13}}$ **ii** $\dfrac{2}{\sqrt{13}}$ **iii** $\frac{3}{2}$ **iv** $\dfrac{2}{\sqrt{13}}$ **v** $\dfrac{3}{\sqrt{13}}$ **vi** $\frac{2}{3}$
f i $\frac{5}{6}$ **ii** $\dfrac{\sqrt{11}}{6}$ **iii** $\dfrac{5}{\sqrt{11}}$ **iv** $\dfrac{\sqrt{11}}{6}$ **v** $\frac{5}{6}$ **vi** $\dfrac{\sqrt{11}}{5}$

EXERCISE 12C.2

1 **a** $\sin 65^o = \dfrac{x}{a}$ **b** $\cos 32^o = \dfrac{x}{b}$ **c** $\tan 56^o = \dfrac{x}{c}$
d $\cos 37^o = \dfrac{d}{x}$ **e** $\tan 49^o = \dfrac{e}{x}$ **f** $\tan 73^o = \dfrac{f}{x}$
g $\sin 54^o = \dfrac{g}{x}$ **h** $\tan 27^o = \dfrac{x}{h}$ **i** $\cos 59^o = \dfrac{i}{x}$
2 **a** $x \approx 17.62$ **b** $x \approx 8.91$ **c** $x \approx 6.63$
d $x \approx 16.68$ **e** $x \approx 14.43$ **f** $x \approx 23.71$
g $x \approx 24.60$ **h** $x \approx 40.95$ **i** $x \approx 7.42$
j $x \approx 13.87$ **k** $x \approx 13.07$ **l** $x \approx 20.78$
3 **a** $\theta = 62^o$, $a \approx 12.5$, $b \approx 6.7$
b $\theta = 27^o$, $a \approx 16.4$, $b \approx 7.4$
c $\theta = 52^o$, $a \approx 54.8$, $b \approx 33.8$

EXERCISE 12C.3

1 **a** $\theta \approx 51.3^o$ **b** $\theta \approx 41.8^o$ **c** $\theta \approx 39.6^o$
d $\theta \approx 38.0^o$ **e** $\theta \approx 34.1^o$ **f** $\theta \approx 47.6^o$
g $\theta \approx 36.5^o$ **h** $\theta \approx 35.8^o$ **i** $\theta \approx 36.9^o$
2 **a** $x \approx 6.2$, $\theta \approx 38.7^o$, $\phi \approx 51.3^o$
b $x \approx 7.0$, $d \approx 60.2^o$, $\beta \approx 29.8^o$
c $x \approx 8.4$, $a \approx 41.5^o$, $b \approx 48.5^o$
3 The 3 triangles do not exist. The hypotenuse must be longer than
the other two sides.

EXERCISE 12D

1 **a** 10.3 m **b** 12.2 m **2 a** 7.34 m **b** 41.8^o
3 **a** 3.95 m **b** 2.30 m **c** 58.7^o **4** 329 m **5** 135 m
6 **a** 4.74 m **b** 5.81^o **7** 648 m
8 **a** 59.0 cm **b** 174 cm **c** $\approx$ $1950 **9** 366 m
10 **a** 5.33 m **b** 41.7 m^2
11 **a** $\theta \approx 23.20^o$ **b** 45.7 m **c** 45.7 m **12** 109°
13 10.2 cm **14 a** 13 cm **b** 22.6^o **15** 13.4 cm **16** 31.7 m
17 **a** 5.81 cm **b** 16.2 cm **18** 70.5^o **19** 412 m
20 19 600 m **21**
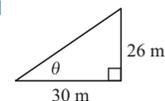
$\tan\theta = \frac{26}{30}$
$\therefore \quad \theta \approx 40.9^o$
So, Nindi should not
be concerned.

EXERCISE 12E

1 a
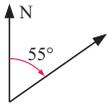
b
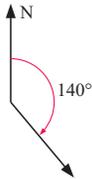
c
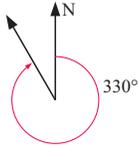
d
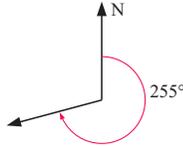
2 a 304° **b** 248° **c** 064° **d** 141°

3 a i 040° **ii** 130° **iii** 310° **iv** 090° **v** 220°
   vi 270°

b i 025° **ii** 150° **iii** 330° **iv** 105° **v** 205°
   vi 285°

4 109° **5** 9.22 km on bearing 131° **6** 14.3 km

7 95.4 km   **8** 919 km

9 a

b 2.92 km
c 159°

10
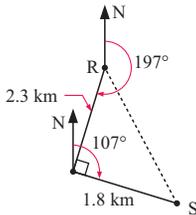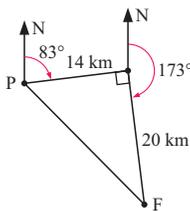
a 24.4 km
b 138°
c 318°

EXERCISE 12F

1 a $10\sqrt{2}$ cm   (14.1 cm)   **b** 54.7^o

2 a $2\sqrt{13}$ cm   (7.21 cm)   **b** 29.0^o
   c $\sqrt{73}$ cm   (8.54 cm)   **d** 25.1^o

3 a $2\sqrt{13}$ cm   (7.21 cm)   **b** 22.6^o

4 a $20\sqrt{2}$ cm   (28.3 cm)   **b** 45°

5 a [UT]   **b** [WT]   **c** [VT]   **d** [XT]

6 a [CD]   **b** [DE]   **c** [DF]   **d** [DX]

7 a [MA]   **b** [MN]   **8 a** 31.0^o **b** 25.1^o

9 a 45° **b** 35.3^o **10 a** 11.3^o **b** 15.5^o **11** 40.9^o

12 44.6^o

REVIEW SET 12A

1 a $x = \sqrt{13}$ **b i** $\frac{2}{\sqrt{13}}$ **ii** $\frac{3}{\sqrt{13}}$ **iii** $\frac{3}{2}$

2 a $a \approx 7.39$ **b** $a \approx 42.6$ **c** $a \approx 96.7$

3 a $\theta = 38.7^o$ **b** $\theta = 63.4^o$ **c** $\theta = 50.5^o$

4 a 1.7 cm   **b** 1.66 cm   **5** 35.1 m   **6** 29.4 km

7 a 28.9 m   **b** 119 m   (to nearest m above)

8 a 54.5^o **b** 32.1^o

9 a 56.3^o **b** $\sqrt{29} \approx 5.39$ cm   **c** 29.1^o

REVIEW SET 12B

1 a $\sqrt{12}$ or $2\sqrt{3}$ cm   **b i** $\frac{1}{2}$ **ii** $\frac{\sqrt{3}}{2}$ **iii** $\sqrt{3}$

2 a $b \approx 6.17$ **b** $b \approx 3.01$ **c** $b \approx 14.5$

3 a $\phi \approx 36.1^o$ **b** $\phi \approx 48.4^o$ **c** $\phi \approx 54.0^o$

4 a 107 km   **b** 306° **c** 126°

5 b 35° **c** 34.9^o **6** 303 m   **7 a** 72.1 km   **b** 252°

8 a $6\sqrt{2}$ m, $6\sqrt{3}$ m   **b** 35.3^o **c** 39.2^o **9** 43.3^o

EXERCISE 13A

1 a $8\ \text{m s}^{-1}$ **b** ≈ 87.8 seconds

2 a \$4000   **b** 6.6% p.a.

3 a i 4.9 m   **ii** 19.6 m   **iii** 44.1 m   **b** 90.6 m
   c 4.52 seconds

4 a 1600 m^3 **b** ≈ 13.2 m   **c** ≈ 45.6 m

5 a €6.40   **b** €9.40   **c** €13.80   **d** €12.20

6 a ≈ 20.3 m^2 **b** $2\sqrt{2}x^2$ units^2

7 a ≈ 1.78 m   **b** ≈ 17.5 m   **8 a** 40 km   **b** 121 m

EXERCISE 13B

1 a $y = \dfrac{13 - 3x}{4}$ **b** $y = \dfrac{18 - 5x}{7}$ **c** $y = \dfrac{3x - 6}{2}$

d $y = \dfrac{17 - 4x}{9}$ **e** $y = \dfrac{5x - 7}{2}$ **f** $y = \dfrac{6x + 10}{5}$

g $y = \dfrac{7x - 8}{12}$ **h** $y = \dfrac{4 - 8x}{3}$ **i** $y = \dfrac{b - a}{4}$

2 a $a = d - b$ **b** $a = \dfrac{c}{b}$ **c** $a = g - 2m$

d $a = b + c$ **e** $a = e - 2b$ **f** $a = \dfrac{P - m}{3}$

g $a = \dfrac{d - Q}{4}$ **h** $a = \dfrac{n - b}{x}$ **i** $a = \dfrac{c - by}{x}$

j $a = \dfrac{c - 2b}{n}$ **k** $a = \dfrac{c - K}{3}$ **l** $a = \dfrac{b}{x}$

m $a = \dfrac{b}{y - x}$ **n** $a = \dfrac{-b - c}{2}$ **o** $a = c - b$

3 a $g = aK$ **b** $a = \dfrac{g}{K}$ **c** $h = \dfrac{Ac}{b}$

d $b = \dfrac{Ac}{h}$ **e** $c = \dfrac{bh}{A}$ **f** $b = \dfrac{Vd}{l}$

g $d = \dfrac{lb}{V}$ **h** $b = \dfrac{a}{2H}$ **i** $c = \dfrac{a + b}{D}$

4 a $a = \dfrac{12}{c}$ **b** $c = \dfrac{ad}{b}$ **c** $b = \dfrac{ad}{c}$

d $d = \dfrac{bc}{a}$ **e** $b = \dfrac{z^2}{a}$ **f** $z = \pm\sqrt{ab}$

g $a = \dfrac{z^2}{b}$ **h** $b = \sqrt[3]{2a}$ **i** $b = \dfrac{a^2 n^2}{4}$

EXERCISE 13C.1

1 a $F = (70 + 60 \times 3)$ \$s   **b** $F = (70 + 60h)$ \$s

2 a $A = 250 \times 6$ \$s   **b** $A = 250m$ \$s   **c** $A = Dm$ \$s

3 a $C = (100 + 80 \times 5)$ euro   **b** $C = (100 + 80t)$ euro
   c $C = (100 + yt)$ euro   **d** $C = (F + yt)$ euro

4 a $C = (4 + 1.2 \times 8)$ dollars   **b** $C = (4 + 1.2k)$ dollars
   c $C = (4 + dk)$ dollars   **d** $C = (F + dk)$ dollars

5 **a** $M = (800\,000 - 20\,000 \times 3)$ Yen
  b $M = (800\,000 - ax)$ Yen **c** $M = (A - ax)$ Yen
6 **a** $C = 200 + 125d^{1.3}$ pounds

EXERCISE 13C.2

1 **a** $A = 2ab + \dfrac{\pi a^2}{2}$ **b** $A = ar + \dfrac{\pi r^2}{2}$ **c** $A = aw + \dfrac{\pi w^2}{4}$

  d $A = ab - \dfrac{\pi a^2}{8}$ **e** $A = 2ar - \pi r^2$

  f $A = \dfrac{a}{2}\sqrt{b^2 - a^2} + \dfrac{\pi}{8}(b^2 - a^2)$

2 **a**
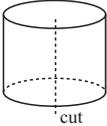
If cut along vertical line and opened out it is a rectangle of length $2\pi r$ and height h.
  $\therefore \quad A = 2\pi rh$

  b $A = 2\pi rh + \pi r^2$ **c** ≈ 4477 cm$^2 \approx 4480$ cm^2

  d ≈ 21.8 cm **e** $h = \dfrac{V - \pi r^2}{2\pi r}$, $h \approx 21.8$ ✓

EXERCISE 13D

1 The nth odd number is $2n - 1$.
2 **a** $2n + 5$ **b** $3n + 1$ **c** $4n + 1$ **d** $4n - 1$ **e** $7n$
  f $7n - 5$ **g** 2^n **h** 2^{n-1} **i** $\dfrac{1}{3^n}$ **j** $\dfrac{1}{5^{n-1}}$
3 **a** 4, 7, 10, 13, 16 matchsticks **b** 31 matchsticks
  c $3n + 1$ matchsticks
4 **a** **a** 3, 6, 9, 12, 15 matchsticks **b** 30 matchsticks
    c $3n$ matchsticks
  b **a** 6, 11, 16, 21, 26 matchsticks **b** 51 matchsticks
    c $5n + 1$ matchsticks
  c **a** 7, 10, 13, 16, 19 matchsticks **b** 34 matchsticks
    c $3n + 4$ matchsticks
5 **a** $2n - 1$ (the nth odd number)
  b **i** 4 **ii** 9 **iii** 16 **iv** 25
  c $S_1 = 1^2$, $S_2 = 2^2$, $S_3 = 3^2$, etc., so, $S_n = n^2$
6 **a** 2^{n-1} **b** **i** 3 **ii** 7 **iii** 15 **iv** 31
  c As $S_1 = 1 = 2 - 1$
    $S_2 = 3 = 2^2 - 1$
    $S_3 = 7 = 2^3 - 1$
    $S_4 = 15 = 2^4 - 1$
    $S_5 = 31 = 2^5 - 1$
  we guess that $S_n = 2^n - 1$
  i.e., the sum of the first n terms of
  $1 + 2 + 4 + 8 + 16 + \ldots$ is $2^n - 1$.
7 **a** $u_n = (n + 1)2^n$ **b** $S_4 = 4 \times 2^5$
  c $S_{20} = 20 \times 2^{21}$ $S_5 = 5 \times 2^6$
      $= 41\,943\,040$ $S_6 = 6 \times 2^7$
        So, we predict $S_n = n\,2^{n+1}$

REVIEW SET 13A

1 **a** $A = 92$ **b** $t = -4.8$
2 **a** ≈ 670 cm^2 **b** 1.26 m **c** ≈ 151 m^2
3 **a** $y = \dfrac{10 - 3x}{4}$ **b** $y = \dfrac{5x - 8}{3}$ **c** $y = \dfrac{3b}{a}$ **d** $y = \dfrac{b^4}{a^2}$
4 **a** $C = (40 + 30n)$ dollars **b** $C = (F + an)$ dollars
5 **a** $A = ab + \dfrac{ac}{2}$ units^2

---

  b $P = a + 2b + c + \sqrt{a^2 + c^2}$ units
6 **a** $3n$ **b** **i** $3n - 2$ **ii** $\dfrac{1}{3^{n-1}}$
7 **a** 5, 8, 11, 14, 17 matchsticks **b** 23 matchsticks
  c $3n + 2$ matchsticks
8 **a** $2n$ **b** **i** 6 **ii** 12 **iii** 20 **iv** 30
  c **i** As $S_1 = 2 = 1 \times 2$ **ii** $S_n = \dfrac{n(n + 1)}{2}$
      $S_2 = 6 = 2 \times 3$
      $S_3 = 12 = 3 \times 4$
      $S_4 = 20 = 4 \times 5$
      $S_5 = 30 = 5 \times 6$
    then we guess $S_n = n(n + 1)$

REVIEW SET 13B

1 **a** $M = -27$ **b** $b \approx 44.3$
2 **a** $b = \dfrac{c - 2a}{3}$ **b** $b = \dfrac{4}{a - c}$ **c** $b = \pm\sqrt{3e}$ **d** $b = \dfrac{4c}{a}$
3 **a** $7n$ **b** **i** $7n - 5$ **ii** $107 - 7n$
4 **a** $V = \frac{\pi}{3}d^3$ units^3 **b** ≈ 28.3 m^3 **c** $d \approx 2.67$ m
5 **a** $M = (A + 50V)$ euro **b** $M = (A + hV)$ euro
6 **a** $AB = \sqrt{h^2 - b^2}$ **b** $AC = a + 2\sqrt{h^2 - b^2}$
  c $A = b(a + \sqrt{h^2 - b^2})$
7 Pattern is: 8, 13, 18, 23, 28, ...
  In nth diagram, matchstick number is $5n + 3$.
8 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{2}{3}$, $S_3 = \frac{3}{4}$, $S_4 = \frac{4}{5}$ **b** $S_n = \dfrac{n}{n + 1}$

EXERCISE 14A

1 **a** A and B are virtually the same, with minor variations. Both are positively skewed.
  b The values of A are generally higher than the values of B. A is negatively skewed and B is positively skewed.
  c A is positively skewed, and B is almost symmetrical. The values of B are generally higher than the values of A.
  d A and B are virtually the same, and both are negatively skewed.
  e A and B are virtually the same, and both are positively skewed.
  f A and B are virtually the same, and both are symmetrical.
2 **a** 'Inside USA' is negatively skewed and 'Outside USA' is positively skewed.
  b 'Outside USA', for this set of sales makes Pedro happier than 'Inside USA' set of sales. This is because there are generally fewer breakages with the private company.

EXERCISE 14B

1 **a** A and B have the same median, however for B the data is bunched to the left of the median, and hence B is positively skewed. A and B have virtually the same IQR and range.
  b C has a higher median than D, but they have the same Q_1 and Q_3 values. They have the same IQR, but D has a larger range than C. Both data sets are close to being symmetrical.
  c F has a larger median than E. The data in E is bunched towards the middle, and so E has a smaller IQR than F. E and F have the same range. Both data sets are close to being symmetrical.
  d The median of G is much less than that of H. The values in H are generally larger than the values in G, and are negatively skewed. G has a larger range but a smaller IQR than H. G is almost symmetrical and H is negatively skewed.

e The median of I is much less than that of J. The range of I is slightly less than that of J. The IQR of I is much less than that for J. I is close to being symmetrical and J is negatively skewed.

f The median for K is less than that for L. The range of K is more than that for L (ignoring the outlier). K's IQR is larger than L's. Both data sets are close to symmetrical, but L has an outlier.

2 a

	Year 10	Year 12
Min_x	4	8
Q_1	6.5	10
Median	9	12
Q_3	11	16
Max_x	15	17

b i

Range	11	9
IQR	4.5	6

ii

3 a i Ind 80 cm, Aust 85 cm   **ii** Ind 35 cm, Aust 33 cm
iii Ind range = 45 cm, Aust range = 48 cm
iv Ind IQR = 15 cm, Aust IQR = 20 cm

b i 75%   **ii** 50%

c i symmetrical   **ii** positively skewed

d i Indonesia   **ii** Australia

4 a

	England	West Indies
Min_x	1	0
Q_1	8	5
Median	24.8	18.84
Q_3	40.1	27.92
Max_x	109.8	83.83
Outliers	109.8	83.83
Mean	28.97	22.98
Range	108.8	83.83
IQR	32.1	22.915

c

b
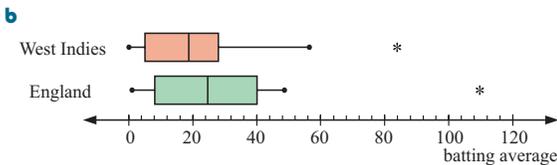
England's measure of centre are higher than for the West Indies. The spread of scores for England is also higher than for the West Indies, i.e., they are more varied.

d No, the outliers are genuine data and should be included in the analysis.

5 a

	Bay 1	Bay 2
Min_x	2.3	2.6
Q_1	2.5	2.7
Median	2.6	2.75
Q_3	2.7	2.9
Max_x	2.9	3.2
Outliers	nil	3.2
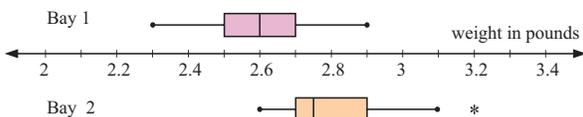
The distribution from *Bay 1* is symmetrical with no outliers. The distribution from *Bay 2* is negatively skewed with outlier 3.2 .

b Clearly *Bay 2* catches weigh more than those from *Bay 1*. Median weight 2.75 pounds compared with 2.6 pounds. The range (0.6 pounds) and IQR (0.2 pounds) are identical for each bay indicating consistency of spread.

6 a discrete

c
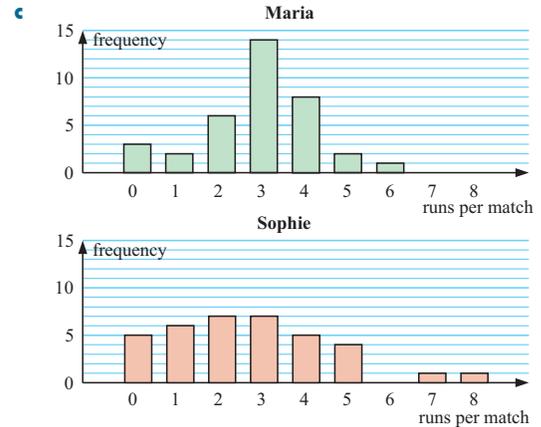
d There are no significant outliers.
e Maria's distribution is reasonably symmetrical. Sophie's distribution is slightly positively skewed.
f Maria has a higher mean (≈ 2.89 runs) compared with Sophie (≈ 2.67 runs). Maria has a higher median (3 runs) compared with Sophie (2.5 runs).
g Maria's range is 6 runs, compared with Sophie's range of 8 runs. Maria's IQR is 2 runs, compared with Sophie's IQR of 3 runs. Sophie's run making shows greater spread or variability.

h
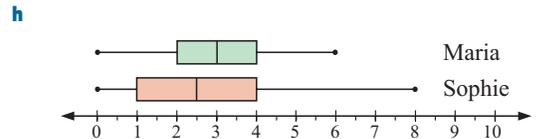
j Generally, Maria makes more runs than Sophie and is a more consistent batter.

7 b Frank's 163 and Competitor's 76
If there is no evidence for incorrect testing and/or recording, they should be included.

c

	Frank's	Competitor's
$\overline{x}$	141.9	89.2
median	142.5	89
range	35	21
IQR	10	5

Frank's have a much higher centre than his competitor's, but is not 'twice as long' as he claims. Frank's batteries have a far greater spread than do his competitor's.

d
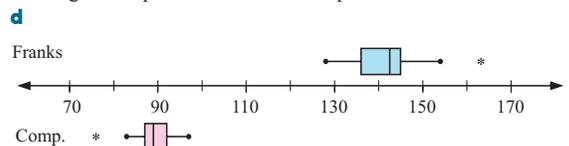
e A is slightly skewed towards higher values.
B is slightly skewed towards lower values.

f Frank's claim is not justified as 141.9 is not twice 89.2 (not even close to it).

g Frank would probably argue (with justification) that the sample sizes are far too small to truly reflect the actual 'life' of batteries.

8 a Hugh wants to show that 'machine A causes less faulty caps than machine B'.

b The column graphs are more representative of all the data, while the boxplots make it easier to obtain a visual comparison of the distribution.

c For machine A there are 4 outliers: 2, 9, 12 and 13. These are genuine data values and so should not be deleted. There are no outliers for machine B.

d

	A	B
$\overline{x}$	6.12	7.53
median	6	8

Both measures of the 'middle' support Hugh's claim that machine A causes fewer faulty caps.

e

	A	B
range	11	8
IQR	1.5	2.5

The IQR being less for A than for B suggests a smaller spread for the middle 50% of the data.
The 4 outliers affect the range dramatically. These suggest that A probably has an intermittent fault.

f Yes, it indicates greater consistency for the middle 50% of data.
We conclude that:
- machine A does cause fewer faulty caps than machine B
- machine A works with better consistency than machine B, but probably has an intermittent fault that needs attention.

REVIEW SET 14A

1 a B has higher Q_1, median and Q_3 values than A. A and B have the same IQR, but A has a larger range than B.

b The values of C are generally larger than the values of D, and are positively skewed. C and D have the same range, but C has a larger IQR than D.

2 a

	A	B
Min_x	11 s	11.2 s
Q_1	11.6 s	12 s
Median	12 s	12.6 s
Q_3	12.6 s	13.2 s
Max_x	13 s	13.8 s

b i Range: A = 2 s, B = 2.6 s

ii IQR: A = 1 s, B = 1.2 s

c i The members of squad A generally ran faster times.

ii The times in squad B were more varied.

3 a
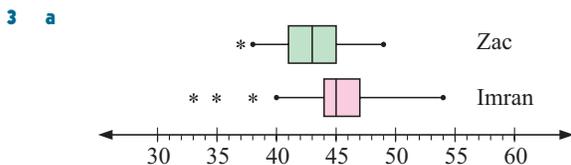
b Imran, because he has higher Q_1, median and Q_3 values than Zac.

c Imran, as indicated by the far greater range.

4 a Whether girls are generally better than boys at Mathematics.

b

c
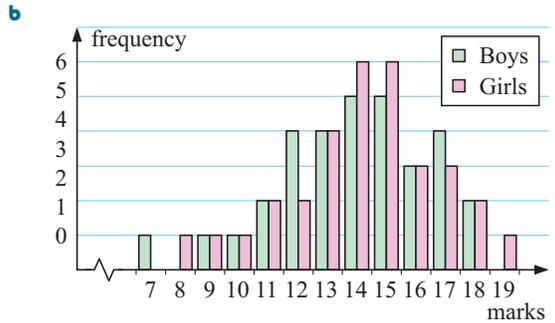
	Boys	Girls
Min_x	7	8
Q_1	12	13
Median	14	14
Q_3	16	16
Max_x	18	19

d
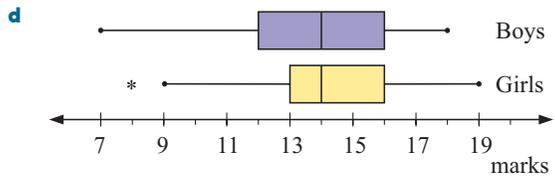
e i median: Boys = 14, Girls = 14

ii mean: Boys ≈ 13.9, Girls ≈ 14.2

iii IQR: Boys = 4, Girls = 3

f From the analysis there is no significant difference between the performance of the boys and the girls. However, for this sample the girls performed slightly better (comparing means).

REVIEW SET 14B

1 a A and B have the same median value and range, but the data in B is more centrally concentrated, as B has a smaller IQR than A.

b C has a lower median value than D, and is positively skewed. D is almost symmetrical. C and D have approximately the same range, but C has a larger IQR than D.

2 a

		start	end
i	fastest	142 s	140 s
ii	slowest	180 s	170 s
iii	median	160 s	153 s
iv	range	38 s	30 s
v	IQR	20 s	13 s

b i 50% **ii** 25%

c The median time improved by 7 seconds. The slowest and fastest times improved also.

d The 'start of the season' times show greater variability as indicated by greater range and IQR values.

3 a i mean: Before ≈ 28.3 s, After = 28.0 s

ii median: Before ≈ 28.4 s, After ≈ 28.4 s

iii IQR: Before = 10.5 s, After = 9.95 s

b The new program did not have a significant effect on the performance of the readers.

4 a Which driver, in general, has the highest daily fare total.

b

Peter		John
9 5 3	9	7
	10	
8 6	11	0 0 6
	12	7 9
2	13	9 9
7 0	14	2 7
5 4 3	15	2 3 8 9
8	16	0 1 1 2 3
	17	4
8 3 2	18	0
6 4 1 0	19	
8 5 5	20	
	21	5
3	22	3
3	23	
	24	1
	25	
	26	0
0	27	9	7 means $97

c Peter: mean $\approx$ \$169,   median = \$182
John: mean $\approx$ \$159,   median = \$158
d Peter: range = \$177,   IQR = \$64.50
John: range = \$163,   IQR = \$34.50
e Peter is the more successful driver as he has a higher mean and median daily fare total.

EXERCISE 15A

1 **a** $\binom{5}{0}$ **b** $\binom{0}{3}$ **c** $\binom{2}{-2}$ **d** $\binom{-4}{1}$ **e** $\binom{3}{-3}$ **f** $\binom{-3}{-3}$

2 **a** $\binom{-4}{2}$ **b** $\binom{6}{1}$ **c** $\binom{2}{3}$

3 **a**

b

c

d

e

f

4 **a** yes **b** no **c** no **d** no
5 **a** A(2, 4), B(3, −2), C(−2, −3), D(−5, 1)
b A′(7, 6), B′(8, 0), C′(3, −1), D′(0, 3)
6 **a**
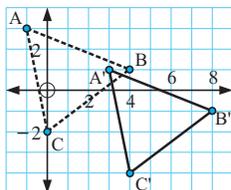
b A′(3, 1), B′(8, −1), C′(4, −4) **c** $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$
d $2\sqrt{5}$ units

EXERCISE 15B

1 **a**

b

c

d

e

f
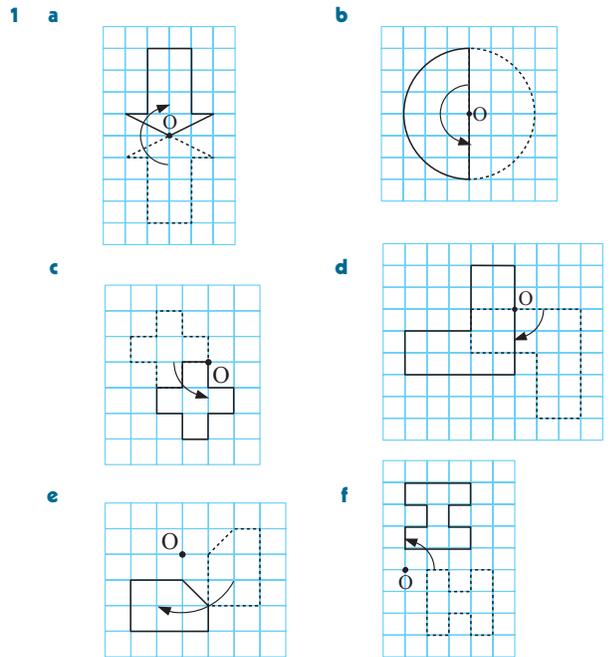
2 H, I, M, N, O, S, W, X, Z
3 **a** yes **b** yes **c** no **d** yes **4** **a** 180° **b** X
5 Rotate 75° anticlockwise about O.
6 **a** (−1, 4) **b** (2, 5) **c** (5, 0) **d** (5, −1)
e (−4, −3)
7 **a** (3, −5) **b** (−3, 0) **c** (−1, −4) **d** (3, 2)
e (−4, 2)
8 **a** (−2, 0) **b** (2, −4) **c** (−5, 1) **d** (−2, −4)
e (2, 3)
9
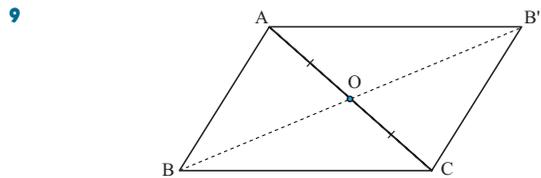
a ABCB′ is a parallelogram.
b opposite sides equal and parallel, diagonals bisect each other, opposite angles equal
10
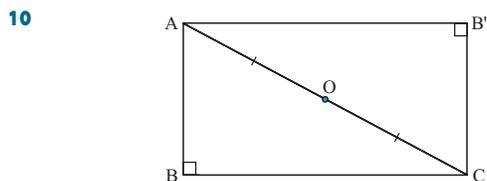
a ABCB′ is a rectangle.
b opposite sides equal and parallel, all angles are right angles, diagonals equal and bisect each other

EXERCISE 15C.1
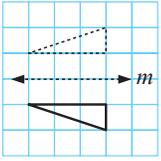
1 a **b**
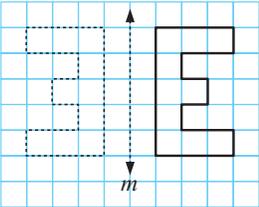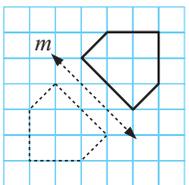
c **d**
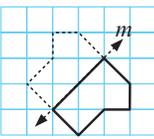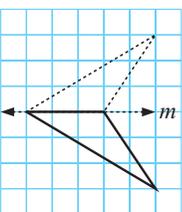
e **f**
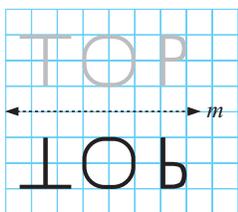
2 a yes **b** no **c** yes **d** no

3 a i no on **ii** think think
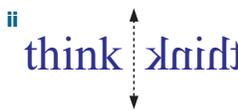
4 a

	P	P'
i	$(5, 2)$	$(5, -2)$
ii	$(-3, 4)$	$(-3, -4)$
iii	$(-2, -5)$	$(-2, 5)$
iv	$(3, -7)$	$(3, 7)$
v	$(4, 0)$	$(4, 0)$
vi	$(0, 3)$	$(0, -3)$

b $(a, -b)$

5 a

	P	P'
i	$(5, 2)$	$(-5, 2)$
ii	$(-3, 4)$	$(3, 4)$
iii	$(-2, -5)$	$(2, -5)$
iv	$(3, -7)$	$(-3, -7)$
v	$(4, 0)$	$(-4, 0)$
vi	$(0, 3)$	$(0, 3)$

b $(-a, b)$

6 a

	P	P'
i	$(5, 2)$	$(2, 5)$
ii	$(-3, 4)$	$(4, -3)$
iii	$(-2, -5)$	$(-5, -2)$
iv	$(3, -7)$	$(-7, 3)$
v	$(4, 0)$	$(0, 4)$
vi	$(0, 3)$	$(3, 0)$

b (b, a)

7 a $\triangle BXB'$ is isosceles since the pipeline perpendicularly bisects [BB'].
b The shortest distance between A and B' is through X, and since BX = B'X, AX + XB = AX + XB' must also be the shortest distance from A to B via the pipeline.

EXERCISE 15C.2

1 a **b** **c**
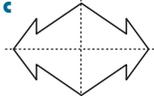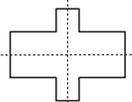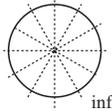
d **e** **f** infinite
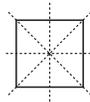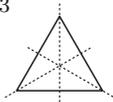
2 a 4 **b** 3 **c** 0
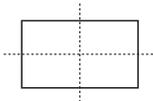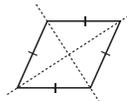
d 2 **e** 2 **f** 5
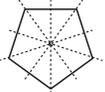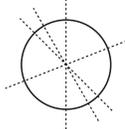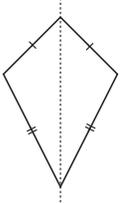
g infinite **h** 1 **i** 1
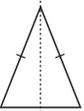
EXERCISE 15D.1

1 a 2 **b** 3 **c** 1.5 **2 a** $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{3}{4}$

EXERCISE 15D.2

1 a **b**

c

2 a **b**

c **d**

e

f

3 a **b**

c

3 a **b**

c
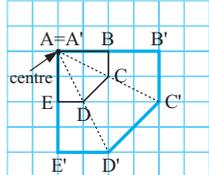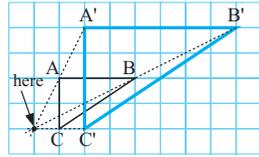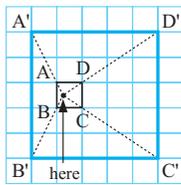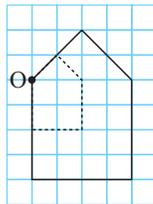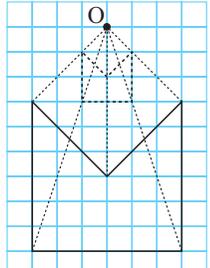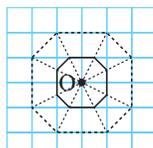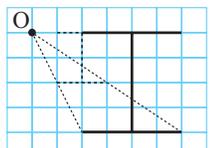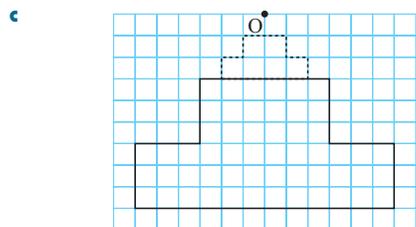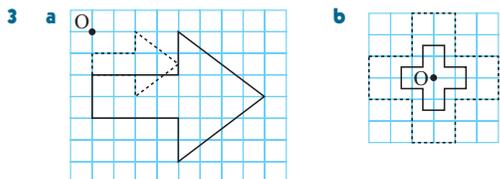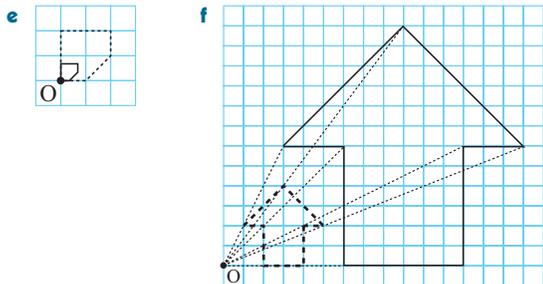
4

	Area of Object	Area of Image	k	$\dfrac{\text{Area of Image}}{\text{Area of Object}}$	k^2
a	5 units^2	20 units^2	2	4	4
b	20 units^2	5 units^2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
c	6 units^2	54 units^2	3	9	9

$\dfrac{\text{area of image}}{\text{area of object}} = k^2$ or area of image $= k^2 \times$ area of object

5 56 m^2 **6 a** $k = 2.5$ **b** 20 cm **c** \$18.75

7 $5\sqrt{2}$ m $\times 8\sqrt{2}$ m ≈ 7.07 m $\times 11.31$ m

EXERCISE 15E

1 a **b**

c

2 a yes **b** yes **c** no

3 a **b**

c
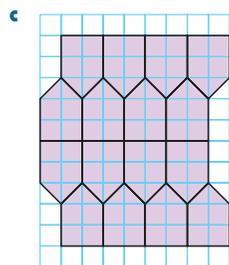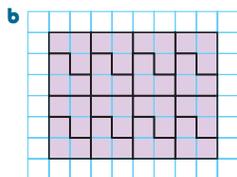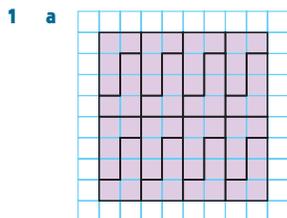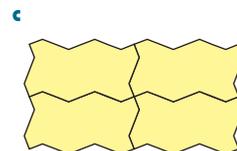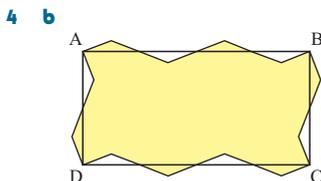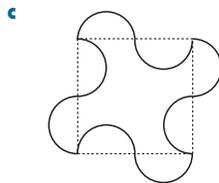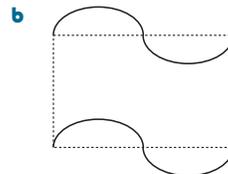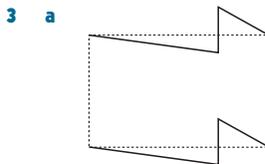
d They do not slide past one another so lock in tightly.

REVIEW SET 15A

1 a **b**

c **d**
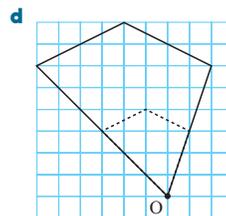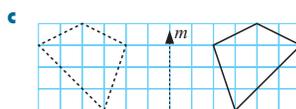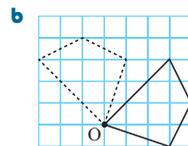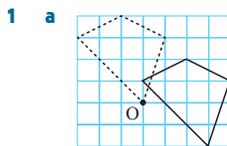
2 $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

3 a
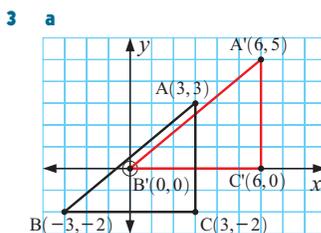
b A′(6, 5), B′(0, 0), C′(6, 0)

4

	P	P'
a	(1, 4)	(1, −4)
b	(−2, 3)	(−2, −3)
c	(−3, −1)	(−3, 1)
d	(4, 2)	(4, −2)

5 a **b** **c** no lines of symmetry

6 a

b

centre of enlargement

REVIEW SET 15B

1 a

b

c

d
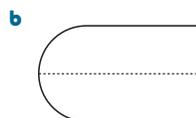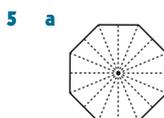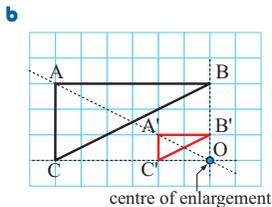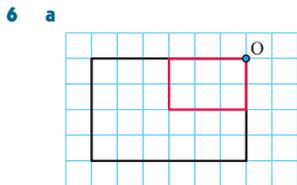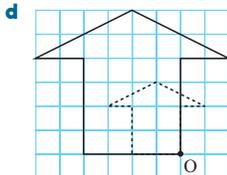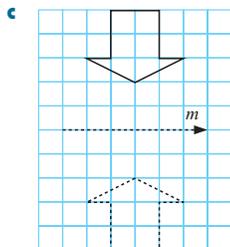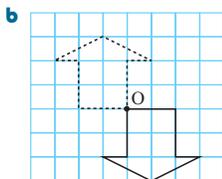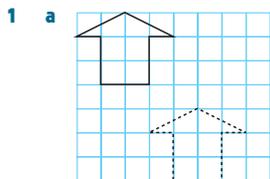
2 a $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

3 a $(-2, 0)$ **b** $(0, -5)$ **c** $(3, 1)$ **d** $(4, -2)$
e $(-2, -3)$

4

	P	P'
a	$(1, 5)$	$(-1, 5)$
b	$(-2, 4)$	$(2, 4)$
c	$(-4, -3)$	$(4, -3)$
d	$(3, 2)$	$(-3, 2)$

5
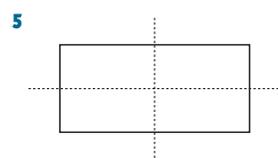
6 Area is enlarged with a scale factor of 4 (i.e., quadrupled).

7 a yes **b** yes **c** no

EXERCISE 16A

1 a $x = \pm 4$ **b** $x = \pm 3$ **c** $x = \pm 3$
d $x = \pm \sqrt{6}$ **e** no solutions **f** $x = 0$
g $x = \pm 3$ **h** no solutions **i** $x = \pm \sqrt{3}$

2 a $x = -1$ or 5 **b** $x = -9$ or 1 **c** no solutions
d $x = 4 \pm \sqrt{2}$ **e** no solutions **f** $x = 2$
g $x = -\frac{5}{2}$ **h** $x = 0$ or $\frac{4}{3}$ **i** $x = \dfrac{1 \pm \sqrt{2}}{2}$

EXERCISE 16B

1 a $a = 0$ **b** $y = 0$ **c** $p = 0$ **d** $p = 0$ or $q = 0$
e $x = 0$ or $y = 0$ **f** $m = 0$ or $n = 0$
g $a = 0$ or $c = 0$ **h** $w = 0$ or $x = 0$
i $a = 0$ or $b = c$ **j** $c = 0$ or $d = -e$
k $a = 0$ or $b = 0$ or $c = 0$
l $w = 0$ or $x = 0$ or $y = 0$ or $z = 0$

2 a $x = 0$ or -3 **b** $x = 0$ or 5 **c** $x = 1$ or 3
d $x = 0$ or 2 **e** $x = 0$ or $-\frac{1}{2}$ **f** $x = -2$ or $\frac{1}{2}$
g $x = -\frac{3}{2}$ or $-\frac{1}{2}$ **h** $x = -2$ or 7 **i** $x = 5$ or $-\frac{2}{3}$
j $x = 3$ **k** $x = -\frac{1}{2}$ **l** $x = 0$
m $x = 0$ or $\frac{5}{3}$ **n** $x = 0$ or $\frac{6}{5}$ **o** $x = 0$ or $-\frac{1}{2}$
p $x = -\frac{1}{3}$ **q** $x = \frac{3}{4}$ **r** $x = 0$ or $\frac{4}{3}$

EXERCISE 16C

1 a $x = 0$ or -3 **b** $x = 0$ or -3 **c** $x = 0$ or 4
d $x = 0$ or 7 **e** $x = 0$ or $-\frac{7}{2}$ **f** $x = 0$ or 9
g $x = 0$ or $\frac{7}{3}$ **h** $x = 0$ or $\frac{9}{4}$ **i** $x = 0$ or $-\frac{8}{3}$

2 a $x = -2$ or -4 **b** $x = -3$ or -8 **c** $x = -2$
d $x = 3$ or -5 **e** $x = 6$ or -8 **f** $x = 5$
g $x = 5$ or -3 **h** $x = 7$ or -2 **i** $x = 3$ or 4
j $x = 5$ or 12 **k** $x = 11$ or -2 **l** $x = 6$ or -3

3 a $x = -\frac{1}{2}$ or -5 **b** $x = -\frac{1}{5}$ or -4 **c** $x = -\frac{1}{3}$ or 4
d $x = \frac{2}{3}$ or -4 **e** $x = -\frac{1}{2}$ or 7 **f** $x = -\frac{3}{7}$ or 2
g $x = \frac{2}{3}$ or 5 **h** $x = \frac{5}{2}$ or 4 **i** $x = \frac{5}{3}$ or -2
j $x = \frac{15}{2}$ or -1 **k** $x = -\frac{3}{2}$ or 5 **l** $x = \frac{1}{3}$ or 2

4 a $x = -\frac{1}{3}$ or $-\frac{3}{2}$ **b** $x = -\frac{1}{6}$ or 3 **c** $x = -\frac{2}{3}$ or $\frac{1}{2}$
d $x = \frac{2}{5}$ or $-\frac{5}{2}$ **e** $x = -\frac{3}{4}$ or $-\frac{1}{3}$ **f** $x = -\frac{2}{3}$ or $\frac{7}{2}$
g $x = -\frac{4}{7}$ or $-\frac{1}{2}$ **h** $x = -\frac{2}{3}$ or $-\frac{3}{2}$ **i** $x = \frac{1}{4}$ or $-\frac{4}{3}$
j $x = \frac{2}{5}$ or $-\frac{4}{5}$ **k** $x = \frac{6}{5}$ or $\frac{1}{3}$ **l** $x = -\frac{1}{4}$ or $\frac{3}{2}$
m $x = \frac{3}{2}$ or -4 **n** $x = -\frac{8}{3}$ or 2 **o** $x = \frac{1}{5}$ or $\frac{5}{2}$

5 a $x = 2$ or -7 **b** $x = 4$ or -7 **c** $x = 14$ or -1
d $x = -\frac{1}{2}$ or 3 **e** $x = -3$ or 1 **f** $x = -\frac{5}{3}$ or -2

6 a $x = \pm 2$ **b** $x = \pm \sqrt{10}$ **c** $x = \pm 4$
d $x = -2$ or 1 **e** $x = -6$ or 2 **f** $x = 2$ or -1
g $x = 4$ or -1 **h** $x = -\frac{1}{3}$ or 1 **i** $x = \frac{1}{2}$ or -1

EXERCISE 16D

1 a 4 **b** 4 **c** 9 **d** 25 **e** 25 **f** 64
g 81 **h** 324 **i** $\frac{9}{4}$ **j** $\frac{9}{4}$ **k** $\frac{25}{4}$ **l** $\frac{25}{4}$

2 a $(x+1)^2 = 3$ **b** $(x-1)^2 = 5$ **c** $(x+3)^2 = 5$
d $(x-3)^2 = 16$ **e** $(x+5)^2 = 2$ **f** $(x+4)^2 = 3$
g $(x+3)^2 = 11$ **h** $(x+6)^2 = 35$ **i** $(x-6)^2 = 41$

3 a two solutions, $x = -3 \pm \sqrt{11}$
b no solution as $(x+3)^2 \geqslant 0$ for all x

4 a $x = 2 \pm \sqrt{6}$ **b** $x = 1 \pm \sqrt{7}$ **c** no solutions
d no solutions **e** $x = -2 \pm \sqrt{2}$ **f** $x = -2 \pm \sqrt{5}$
g no solutions **h** $x = 3 \pm 2\sqrt{2}$ **i** $x = -4 \pm \sqrt{11}$

5 a $x = \dfrac{-3 \pm \sqrt{17}}{2}$ **b** $x = 3 \pm \sqrt{10}$ **c** $x = \dfrac{5 \pm \sqrt{5}}{2}$
d $x = \dfrac{-1 \pm \sqrt{5}}{2}$ **e** $x = \dfrac{-3 \pm \sqrt{5}}{2}$ **f** $x = \dfrac{-5 \pm \sqrt{13}}{2}$

6 a We need the coefficient of x^2 to be 1.
b **i** $x = \dfrac{-4 \pm \sqrt{14}}{2}$ **ii** no solutions
iii $x = \dfrac{-5 \pm 2\sqrt{5}}{5}$

EXERCISE 16E

1 6 or -9 **2** 7 or -6 **3** 3 or 5   **4** 8 and -3

5 11 and -2 or 2 and -11

6   **a** $x = 12$ **b** $x = 5 + \sqrt{70} \approx 13.4$ **c** $x = 3$

7 7.3 cm, 12.3 cm, 14.3 cm   **8** 8 cm   **9** 120 m $\times$ 180 m

10 20 m $\times$ 12 m or 6 m $\times$ 40 m

11 9 m $\times$ 12 m or 6 m $\times$ 18 m   **12** 11 rows

13 between 200 and 500 items   **14** $40\sqrt{10} \approx 126.5$ m

REVIEW SET 16A

1   **a** $x = \pm\sqrt{11}$ **b** $x = 9$ or -1 **c** no solutions

2   **a** $x = 7$ or -3 **b** $x = \frac{5}{2}$ or $-\frac{5}{2}$ **c** $x = 6$ or -4

  d $x = -\frac{2}{3}$ or $\frac{1}{2}$ **e** $x = 8$ or 3   **f** $x = -\frac{2}{5}$ or $\frac{3}{2}$

3 no solutions   **4** 20 cm $\times$ 13 cm   **5** $x = \frac{2}{3}$ or $\frac{3}{2}$

6 $2 \pm \sqrt{3}$ **7**   **a** $x = -\frac{1}{2}$ or 2   **b** no solutions

REVIEW SET 16B

1   **a** $x = \pm 13$ **b** $x = -3 \pm \sqrt{11}$ **c** no solutions

2   **a** $x = 11$ or -3 **b** $x = -8$ or 4   **c** $x = 5$

  d $x = -8$ or 3   **e** $x = \pm 3$ **f** $x = -\frac{3}{4}$ or $\frac{1}{2}$

3 $x = 1 \pm \sqrt{101}$ **4** 9 cm $\times$ 12 cm   **5** 2 or 5

6 8 cm, 15 cm, 17 cm

7   **a** no solutions   **b** $x = \dfrac{1 \pm \sqrt{101}}{10}$ **8** 4 or -3

EXERCISE 17A.1

1   **a** $x = 2$, $y = 3$ **b** $x = 6$, $y = 10$ **c** $x = 2$, $y = -6$

  d $x = 4$, $y = 2\frac{1}{2}$ **e** $x = 0$, $y = -4$ **f** $x = 2$, $y = 3$

2   **a** $x = 3$, $y = -2$ **b** $x = \frac{2}{5}$, $y = \frac{11}{5}$ **c** $x = 3$, $y = 4$

  d $x = -1$, $y = -5$ **e** $x = -5$, $y = 1$ **f** $x = 2$, $y = -4$

3   **a** obtain $1 = 4$ **b** no solution

4   **a** obtain $2 = 2$ **b** an infinite number of solutions

EXERCISE 17A.2

1   **a** $6x = 6$ **b** $-y = 8$ **c** $5x = 7$

  d $-6x = -30$ **e** $8y = 4$ **f** $-2y = -16$

2   **a** $y = -1$, $x = 2$ **b** $x = -2$, $y = 5$

  c $x = 3$, $y = 2$ **d** $x = -2$, $y = -1$

  e $x = 5$, $y = -3$ **f** $x = 4$, $y = -3$

3   **a** $9x + 12y = 6$ **b** $-2x + 8y = -14$

  c $25x - 5y = -15$ **d** $-21x - 9y = 12$

  e $8x + 20y = -4$ **f** $-3x + y = 1$

4   **a** $x = 3$, $y = 2$ **b** $x = 8$, $y = 7$ **c** $x = -2$, $y = 3$

  d $x = 2$, $y = 1$ **e** $x = 3$, $y = 2$ **f** $x = 2$, $y = -5$

  g $x = 5$, $y = 2$ **h** $x = 4$, $y = 1$ **i** $x = 19$, $y = -17$

5   **a** infinite number of solutions   **b** no solution

EXERCISE 17B

1 $30\frac{1}{2}$ and $16\frac{1}{2}$ **2** $6\frac{1}{2}$ and $3\frac{1}{2}$ **3** 17 and 68

4 pencils €0.28, biros €0.54

5 toffees, 15 cents; chocolates 21 cents

6 16 50-cent coins, 27 \$1 coins

7 Amy, €12.60; Michelle, €16.80

8 24, 250 g packs; 34, 400 g packs

9 $a = 3$, $b = 5$ **10** length 11 cm, width 5 cm

11 current 3 km h^{-1}, boat 15 km h^{-1}

12 wind 100 km h^{-1}, plane 900 km h^{-1}

13   **a** 15 km   **b** 1 h 15 min

14   **a** 'ab' has 10s digit a and units digit b and so is really $10a + b$

  b 72

EXERCISE 17C.1

1   **a** intersect at $(2, 0)$ and $(3, 2)$

  b intersect at $(1, 4)$ and $(-8, -77)$ **c** touch at $(-3, -11)$

  d touch at $(2, -3)$

2 On substituting the resulting quadratic equation is
  $x^2 + 6x + 10 = 0$.   On completing the square it becomes
  $(x + 3)^2 = -1$ which is impossible as the LHS is never
  negative.
  $\therefore$ graphs never meet.

EXERCISE 17C.2

1   **a** They touch at $(3, 9)$.

  b They intersect at $(-4, 6)$ and $(1, 1)$.

  c They intersect at $(-2, -2)$ and $(1, 4)$.

  d They intersect at $(2, 5)$ and $(\frac{1}{3}, 0)$.

2   **a** $x = 0$, $y = 1$ or $x = -2$, $y = -1$ **b** No solutions exist.

  c $x = -2$, $y = 4$ or $x = 2$, $y = 4$

  d $x = 3$, $y = 1$ or $x = \frac{7}{9}$, $y = \frac{49}{9}$

  e $x = 4$, $y = \frac{1}{2}$ or $x = -1$, $y = -2$

EXERCISE 17C.3

1   **a** Approx $(-1.30, 1.70)$ and $(2.30, 5.30)$

  b Approx $(0.682, 1.682)$ **c** $(-2, -1)$ and $(\frac{1}{2}, 4)$

  d $(-1, 3)$

  e Approx $(-4.02, 0.124)$, $(0.226, -7.50)$, $(1.36, -3.44)$,
  $(2.44, 2.82)$

  f Approx $(-1.41, 1.41)$, $(1.41, 1.41)$, $(-1.41, -1.41)$ and
  $(1.41, -1.41)$

2   **a** $x \approx -1.79$, $y \approx 2.61$ or $x \approx 2.79$, $y \approx 1.49$

  b $x \approx -0.725$, $y \approx 0.525$ or $x \approx 1.22$, $y \approx 1.49$

REVIEW SET 17A

1 $x = 4$, $y = 1$ **2**   **a** $x = -1$, $y = -7$ **b** $x = 2$, $y = -1$

3 9 of 5 kg, 11 of 2 kg   **4**   **a** no solution   **b** $(3, -4)$

5 78 units   **6** 13, 10-pence coins; 8, 50-pence coins

7 $(\frac{7}{2}, 4)$ and $(-1, -5)$

REVIEW SET 17B

1 $x = 8$, $y = -13$ **2**   **a** $x = 4$, $y = -2$ **b** $x = 2$, $y = -1$

3 small buses 23, large buses 34

4   **a** They do not meet.   **b** $(5, -3)$

5 32 of 2 L cartons, 15 of 600 mL bottles

6 12 cm   **7** $(1, 4)$ and $(-2, 1)$

EXERCISE 18A

1   **a** 1×3 **b** 2×1 **c** 2×2 **d** 2×3

2   **a** 1   **b** 4   **c** 0

3   **a** $\begin{array}{cccc} H & P & S & D \\ (6 & 3 & 5 & 4) \end{array}$ **b** $\begin{pmatrix} 12 \\ 35 \\ 3 \\ 8 \end{pmatrix}$ $\begin{array}{l} \leftarrow \text{ H cost, \$} \\ \leftarrow \text{ P cost, \$} \\ \leftarrow \text{ S cost, \$} \\ \leftarrow \text{ D cost, \$} \end{array}$

4 $\begin{array}{ccc} Ca & Sl & Sc \\ \begin{pmatrix} 30 & 40 & 25 \\ 25 & 30 & 35 \\ 25 & 25 & 25 \end{pmatrix} & & \begin{array}{l} Sa \\ Su \\ Mo \end{array} \end{array}$

	W1	W2	W3	W4

5 a
$$\begin{matrix} 100\text{ g} \\ 250\text{ g} \\ 500\text{ g} \end{matrix} \begin{pmatrix} 2500 & 3000 & 3500 & 4500 \\ 4000 & 3500 & 3500 & 4500 \\ 6500 & 7000 & 6500 & 4500 \end{pmatrix}$$

b
$$\begin{matrix} \\ \text{W1} \\ \text{W2} \\ \text{W3} \\ \text{W4} \end{matrix} \begin{matrix} 100\text{ g} & 250\text{ g} & 500\text{ g} \\ \\ 2500 & 4000 & 6500 \\ 3000 & 3500 & 7000 \\ 3500 & 3500 & 6500 \\ 4500 & 4500 & 4500 \end{matrix}$$

EXERCISE 18B

1 a elements in row 1, column 2 are not equal
   b LHS matrix is 2×2, RHS matrix is 3×2
   Must be equal in size, i.e., same order.

2 a $a = 2$, $b = 3$ **b** $x = -2$, $y = 2$
   c $x = 2$, $y = 3$ **d** $x = 1$, $y = 3$, $z = 1$

EXERCISE 18C

1 a $\begin{pmatrix} 3 & 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 7 & 7 \\ 4 & 4 \end{pmatrix}$

   d $\begin{pmatrix} 2 & 2 & -1 \\ 3 & -1 & 4 \end{pmatrix}$ **e** cannot be done   **f** cannot be done

2 a $\begin{pmatrix} 5 & 2 \\ -1 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2 \\ -3 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

3 a $\begin{pmatrix} 33 & 28 & 36 \\ 30 & 41 & 29 \\ 35 & 27 & 42 \end{pmatrix}$ **b** $\begin{pmatrix} 28 & 33 & 22 \\ 31 & 28 & 20 \\ 9 & 43 & 6 \end{pmatrix}$

4 a $F = \begin{pmatrix} 72 \\ 84 \\ 49 \end{pmatrix}$ $S = \begin{pmatrix} 86 \\ 72 \\ 46 \end{pmatrix}$ **b** $S + F = \begin{pmatrix} 158 \\ 156 \\ 95 \end{pmatrix}$

5 a i $C = \begin{pmatrix} 1.23 \\ 22.15 \\ 0.72 \\ 3.75 \\ 4.96 \end{pmatrix}$ **ii** $S = \begin{pmatrix} 1.38 \\ 22.63 \\ 0.69 \\ 3.68 \\ 5.29 \end{pmatrix}$

   b Profit $P = S - C$ **c** $P = \begin{pmatrix} 0.15 \\ 0.48 \\ -0.03 \\ -0.07 \\ 0.33 \end{pmatrix}$

6 $B = \begin{pmatrix} -1 & -2 \\ 0 & -3 \end{pmatrix}$

EXERCISE 18D

1 a $\begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ **c** $\begin{pmatrix} -3 & -1 \\ 2 & 3 \end{pmatrix}$

   d $\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$

2 a $2H = \begin{pmatrix} 20 \\ 40 \\ 200 \\ 80 \end{pmatrix}$ **b** $\frac{1}{2}H = \begin{pmatrix} 5 \\ 10 \\ 50 \\ 20 \end{pmatrix}$ **c** $1.5H = \begin{pmatrix} 15 \\ 30 \\ 150 \\ 60 \end{pmatrix}$

3 a $\begin{pmatrix} 48 & 36 & 36 & 72 \\ 60 & 48 & 36 & 72 \\ 48 & 72 & 60 & 48 \\ 24 & 24 & 24 & 24 \end{pmatrix}$ **b** $\begin{pmatrix} 32 & 24 & 24 & 48 \\ 40 & 32 & 24 & 48 \\ 32 & 48 & 40 & 32 \\ 16 & 16 & 16 & 16 \end{pmatrix}$

4 C

EXERCISE 18E

1 a It can be found because the number of columns in **A** equals the number of rows in **B**.
   b 3×1
   c The number of columns in **B** does not equal the number of rows in **A**.

2 $AB = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$ **3 a** $AB = \begin{pmatrix} 5 \end{pmatrix}$ **b** $BA = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$

4 a $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} 8 & 1 \\ 3 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ **d** $\begin{pmatrix} 5 & 9 & 7 \\ 4 & 0 & 2 \end{pmatrix}$

5 a $C = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \begin{matrix} \text{P} \\ \text{G} \end{matrix}$ **b** $N = \begin{pmatrix} 43 & 87 \\ 48 & 66 \end{pmatrix} \begin{matrix} \text{D}_1 \\ \text{D}_2 \end{matrix}$

   c $NC = \begin{pmatrix} 564 \\ 474 \end{pmatrix}$ The total revenue from pony and go-cart rides was \$564 on day 1 and \$474 on day 2.

   d \$1038

6 a $R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$ **b** $P = \begin{pmatrix} 43 & 7 & 3 \\ 39 & 8 & 4 \end{pmatrix}$

   c $PR = \begin{pmatrix} 56 & 66 \\ 55 & 67 \end{pmatrix}$

   d My costs at A are \$56. My friend's costs at B are \$67.
   e Store B

7 Real estate matrix is:
$$\begin{matrix} \text{No.} \begin{pmatrix} 10 & 5 & 3 \end{pmatrix} \\ \text{A} \quad \text{U} \quad \text{H} \\ 1 \times 3 \end{matrix} \quad \text{or} \quad \begin{matrix} \begin{pmatrix} 10 \\ 5 \\ 3 \end{pmatrix} \begin{matrix} \text{A} \\ \text{U} \\ \text{H} \end{matrix} \\ 3 \times 1 \end{matrix}$$
with heading "No." over the first.

Quantities matrix is:
$$\begin{pmatrix} 4 & 1 & 2 \\ 6 & 1 & 3 \\ 8 & 2 & 4 \end{pmatrix} \begin{matrix} \text{A} \\ \text{U} \\ \text{H} \end{matrix} \quad \text{or} \quad \begin{pmatrix} 4 & 6 & 8 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} \begin{matrix} \text{C} \\ \text{T} \\ \text{B} \end{matrix}$$
with column labels C T B (left) and A U H (right).

Cost matrix is:
$$\begin{matrix} \begin{pmatrix} 35 & 175 & 235 \end{pmatrix} € \\ \text{C} \quad \text{T} \quad \text{B} \end{matrix} \quad \text{or} \quad \begin{matrix} € \\ \begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix} \begin{matrix} \text{C} \\ \text{T} \\ \text{B} \end{matrix} \end{matrix}$$

We want one final cost (a 1×1 matrix).
So,
$$\begin{matrix} \text{No.} \begin{pmatrix} 10 & 5 & 3 \end{pmatrix} \\ \text{A} \quad \text{U} \quad \text{H} \\ 1 \times 3 \end{matrix} \begin{matrix} \begin{pmatrix} 4 & 1 & 2 \\ 6 & 1 & 3 \\ 8 & 2 & 4 \end{pmatrix} \begin{matrix} \text{A} \\ \text{U} \\ \text{H} \end{matrix} \\ \text{C} \ \text{T} \ \text{B} \\ 3 \times 3 \end{matrix} \begin{matrix} € \\ \begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix} \begin{matrix} \text{C} \\ \text{T} \\ \text{B} \end{matrix} \\ 3 \times 1 \end{matrix}$$

$$= \begin{pmatrix} 94 & 21 & 47 \end{pmatrix} \begin{pmatrix} 35 \\ 175 \\ 235 \end{pmatrix}$$

$$= €18\,010$$

8 $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9 a the matrix $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ **b i** $\overline{x} = \frac{1}{3} \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

   ii $\overline{x} = \frac{1}{4} \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

10 Either $a = b = 1$, $x = 1$ or $a = b = -1$, $x = 1$.

EXERCISE 18F

1 a $\begin{pmatrix} 37 & 111 & 148 \\ 111 & 0 & 185 \\ 74 & 37 & 222 \end{pmatrix}$ **b** $\begin{pmatrix} 138 & 46 & 414 \\ 0 & 92 & 322 \\ 46 & 230 & 184 \end{pmatrix}$

c $\begin{pmatrix} 314 & 166 & 965 \\ 69 & 194 & 794 \\ 143 & 508 & 526 \end{pmatrix}$ **d** $\begin{pmatrix} -12 & 100 & 3 \\ 117 & -34 & 76 \\ 61 & -46 & 166 \end{pmatrix}$

e does not exist **f** $\begin{pmatrix} 28 & 9 & 65 \end{pmatrix}$ **g** $\begin{pmatrix} 41 \\ 37 \\ 54 \end{pmatrix}$ **h** does not exist

2 a $\begin{pmatrix} 74 \\ -13 \\ 131 \\ 75 \\ 72 \end{pmatrix}$ **3** $\begin{pmatrix} 1618 \\ 2441 \\ 3198 \\ 2976 \\ 2016 \\ 2751 \end{pmatrix}$

b $N = \begin{pmatrix} 48 & 67 & 103 & 89 & 114 \end{pmatrix}$

c $31\,057$

4 a $C = \begin{pmatrix} 2.00 \\ 1.80 \\ 2.10 \\ 2.25 \end{pmatrix}$ $S = \begin{pmatrix} 3.00 \\ 2.90 \\ 3.15 \\ 2.95 \end{pmatrix}$ **b** $P = S - C$ $= \begin{pmatrix} 1.00 \\ 1.10 \\ 1.05 \\ 0.70 \end{pmatrix}$

c $\begin{pmatrix} 367 & 413 & 519 & 846 \end{pmatrix} \begin{pmatrix} 1.00 \\ 1.10 \\ 1.05 \\ 0.70 \end{pmatrix} = \$1958.45$

5 a 6×1 because to find the total profit for each store she will need to multiply the profits matrix by **S** so that the orders are:
$$5 \times 6 \quad \times \quad 6 \times 1 \quad = \quad 5 \times 1$$
$$\mathbf{S} \qquad \qquad \mathbf{P} \qquad \quad \text{result}$$

b $\begin{pmatrix} 38\,951 \\ 43\,335 \\ 44\,375 \\ 41\,711 \\ 50\,573 \end{pmatrix}$

REVIEW SET 18A

1 a 3×4 **b** 2 **2** $a = 1$, $b = 0$, $c = 0$

3 a $\begin{pmatrix} 7 & 1 \\ 4 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$ **c** $\begin{pmatrix} 10 & 14 \\ 20 & -14 \end{pmatrix}$

d $\begin{pmatrix} 2 & 17 \\ 24 & -6 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 9 \\ 12 & -3 \end{pmatrix}$ **f** $\begin{pmatrix} 2\frac{1}{2} & -1 \\ 0 & 3 \end{pmatrix}$

g $\begin{pmatrix} -10 & 4 \\ 0 & -12 \end{pmatrix}$ **h** $\begin{pmatrix} 4 & -3\frac{1}{2} \\ -2 & 6\frac{1}{2} \end{pmatrix}$

4 a $n = r$ **b** $s = m$

5 a impossible **b** $\begin{pmatrix} -1 & 4 \\ 3 & 4 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 6 & 2 \\ 6 & -3 & -5 \end{pmatrix}$

6 a $N = \begin{pmatrix} 15 & 14 & 8 & 18 \end{pmatrix}$ **b** $C = \begin{pmatrix} 45 \\ 67 \\ 315 \\ 56 \end{pmatrix}$

c $\$5141$

7 a Jack: £6.05, Angus: £5.90, Karaline: £7.20, Joan: £9.65
b £28.80

REVIEW SET 18B

1 a 2×3 **b** 3×1 **c** 3×4 **2** $a = 2$, $b = -4$, $c = -3$

3 a $\begin{pmatrix} -3 & -4 \\ 4 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 22 & -1 \\ -6 & 8 \end{pmatrix}$

d $\begin{pmatrix} -10 & -1 \\ 0 & -2 \end{pmatrix}$

4 a $\begin{pmatrix} 5 & -8 \\ 9 & -4 \end{pmatrix}$ **b** cannot be done **c** $\begin{pmatrix} 10 & -2 & 8 \\ -15 & 3 & -12 \\ 20 & -4 & 16 \end{pmatrix}$

5 $S = \begin{pmatrix} 15 & 23 \\ 8 & 12 \\ 6 & 8 \end{pmatrix}$, $P = \begin{pmatrix} 0.75 & 0.55 & 1.20 \end{pmatrix}$

$PS = \begin{pmatrix} 22.85 & 33.45 \end{pmatrix}$ Total $= \$56.30$

6 a if $n = 7$ **b** 3×4 **c** no

7 a $\begin{pmatrix} 12 & 21 & 3 & 37 & 6 \\ 3 & 11 & 3 & 6 & 5 \\ 3 & 14 & 14 & 6 & 10 \\ 13 & 27 & 10 & 4 & 9 \end{pmatrix}$

b $\begin{pmatrix} 34 & 43 & 25 & 59 & 28 \\ 27 & 37 & 31 & 34 & 35 \\ 18 & 34 & 29 & 26 & 25 \\ 30 & 48 & 26 & 23 & 39 \end{pmatrix}$ **c** $\$29\,848$

EXERCISE 19A

1 a yes, $a = 2$, $b = 1$, $c = 4$ **b** no, there is no x^2 term
c yes, $a = -1$, $b = -1$, $c = -1$
d yes, $a = 1$, $b = 0$, $c = 0$
e yes, $a = \frac{1}{2}$, $b = 0$, $c = -6$ **f** no, there is an x^3 term
g yes, $a = -3$, $b = 0$, $c = 0$ **h** no, there is no x^2 term
i yes, $a = -2$, $b = 0$, $c = -\frac{5}{2}$

2 a $y = 10$ **b** $y = -5$ **c** $y = -5$

3 a $y = 4$ **b** $y = 14$ **c** $y = 34$

4 a $y = 1$ **b** $y = 1$ **c** $y = \frac{9}{2}$

5 a A does not, B does **b** P and R do, Q does not

6 a $k = 3$ **b** $d = -39$ **c** $m = 2$ or -5
d $n = -\frac{7}{2}$ or 3

7 a no solutions **b** $x = -1$

8 a $x = 2$ **b** no solutions **c** $x = -1$ or 5

9 a $x = 0$ or -6 **b** $x = 6$ or -2 **c** $x = \frac{2}{3}$ or 1

10 a **i** $H = 0$ m **ii** $H = 55$ m **iii** $H = 135$ m
   iv $H = 160$ m
b **i** $t = 0$ sec or 12 sec **ii** $t = 2$ sec or 10 sec
   iii $t = 4$ sec or 8 sec
c
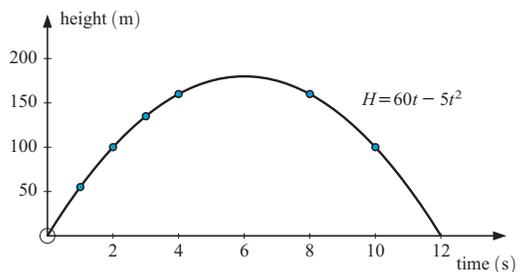
11 a 2 m **b** 47 m
c **i** $t = 1$ sec or 5 sec **ii** $t = 2$ sec or 4 sec

d

$H = -5t^2 + 30t + 2$
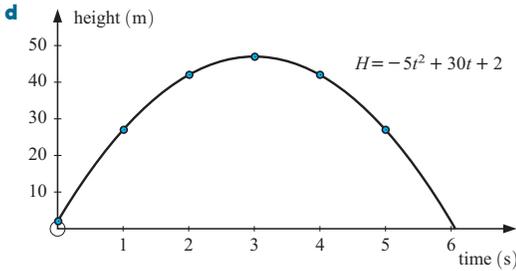
12 a i €320 loss **ii** €640 **iii** €880
  b approximately 3 or 13 cabinets per day

EXERCISE 19B

1 a

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

$y = x^2$
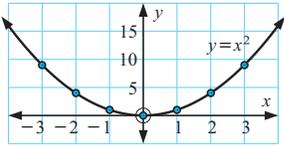
b

x	−3	−2	−1	0	1	2	3
y	15	8	3	0	−1	0	3

$y = x^2 - 2x$
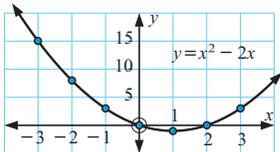
c

x	−3	−2	−1	0	1	2	3
y	13	6	1	−2	−3	−2	1

$y = x^2 - 2x - 2$
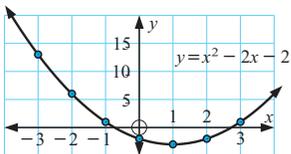
d

x	−3	−2	−1	0	1	2	3
y	18	8	2	0	2	8	18

$y = 2x^2$
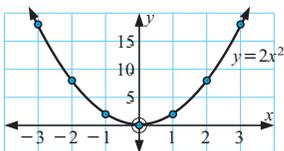
e

x	−3	−2	−1	0	1	2	3
y	19	9	3	1	3	9	19

$y = 2x^2 + 1$
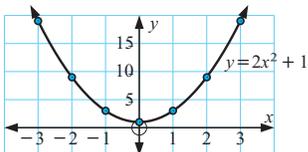
f

x	−3	−2	−1	0	1	2	3
y	$\frac{5}{2}$	0	$-\frac{3}{2}$	−2	$-\frac{3}{2}$	0	$\frac{5}{2}$

$y = \frac{1}{2}x^2 - 2$
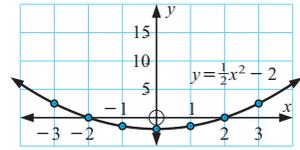
g

x	−3	−2	−1	0	1	2	3
y	−9	−4	−1	0	−1	−4	−9

$y = -x^2$
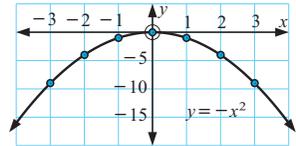
h

x	−3	−2	−1	0	1	2	3
y	−12	−6	−2	0	0	−2	−6

$y = -x^2 + x$
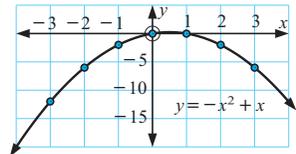
i

x	−3	−2	−1	0	1	2	3
y	−14	−7	−2	1	2	1	−2

$y = -x^2 + 2x + 1$
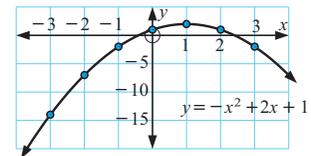
3 a the sign of a, if $a > 0$, if $a < 0$
  b the magnitude of a, if a is larger the graph is thinner
  c the y-intercept is c

EXERCISE 19C

1 a

$y = x^2$
$y = x^2 + 1$

b

$y = x^2$
$y = x^2 + 3$

c

$y = x^2$
$y = x^2 - 1$

d

$y = x^2$
$y = 2x^2$

e

$y = -3x^2$
$y = -x^2$

f

$y = x^2$ $y = \frac{1}{2}x^2$

g

$y = -x^2 + 2$
$y = -x^2$

h

$y = -x^2 - 1$
$y = -x^2$

i

$y = -4x^2$
$y = -x^2$

i

$y = -(x-2)^2 + 1$

vertex at $(2, 1)$

j

$y = -(x+4)^2 + 2$

vertex at $(-4, 2)$

k

$y = -(x-3)^2 - 2$

vertex at $(3, -2)$

l

$y = -(x+4)^2 + 10$

vertex at $(-4, 10)$

EXERCISE 19D

1 a $y = (x+1)^2 + 1$

$y = x^2 + 2x + 2$

vertex at $(-1, 1)$

b $y = (x-2)^2 - 2$

$y = x^2 - 4x + 2$

vertex at $(2, -2)$

c $y = (x+1)^2 - 4$

$y = x^2 + 2x - 3$

vertex at $(-1, -4)$

d $y = (x-3)^2 - 8$

$y = x^2 - 6x + 1$

vertex at $(3, -8)$

e $y = (x+1)^2 - 1$

$y = x^2 + 2x$

vertex at $(-1, -1)$

f $y = (x - \frac{3}{2})^2 - \frac{9}{4}$

$y = x^2 - 3x$

vertex at $(\frac{3}{2}, -\frac{9}{4})$

g $y = (x + \frac{1}{2})^2 - \frac{5}{4}$

$y = x^2 + x - 1$

vertex at $(-\frac{1}{2}, -\frac{5}{4})$

h $y = (x - \frac{3}{2})^2 - \frac{1}{4}$

$y = x^2 - 3x + 2$

vertex at $(\frac{3}{2}, -\frac{1}{4})$

2 a

$y = (x-1)^2 + 2$

vertex at $(1, 2)$

b

$y = (x+1)^2 + 3$

vertex at $(-1, 3)$

c

$y = (x-2)^2 + 1$

vertex at $(2, 1)$

d

$y = (x+2)^2 - 1$

vertex at $(-2, -1)$

e

$y = (x+3)^2$

vertex at $(-3, 0)$

f

$y = (x-4)^2$

vertex at $(4, 0)$

g

$y = -(x-1)^2 + 3$

vertex at $(1, 3)$

h

$y = -(x+1)^2 - 1$

vertex at $(-1, -1)$

i $y = (x + \frac{5}{2})^2 - \frac{45}{4}$

$y = x^2 + 5x - 5$

vertex at $(-\frac{5}{2}, -\frac{45}{4})$

2 a i $y = -(x-1)^2 + 3$
   ii $(1, 3)$
   iii
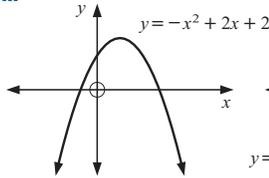
$y = -x^2 + 2x + 2$

b i $y = -(x+2)^2 + 2$
   ii $(-2, 2)$
   iii

$y = -x^2 - 4x - 2$

c i $y = -(x-3)^2 - 1$
   ii $(3, -1)$
   iii

$y = -x^2 + 6x - 10$

d i $y = -(x+4)^2 + 4$
   ii $(-4, 4)$
   iii

$y = -x^2 - 8x - 12$

e i $y = -(x-5)^2 - 3$
   ii $(5, -3)$
   iii

$y = -x^2 + 10x - 28$

f i $y = -(x-0)^2 + 2$
   ii $(0, 2)$
   iii

$y = -x^2 + 2$
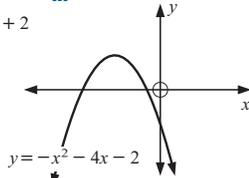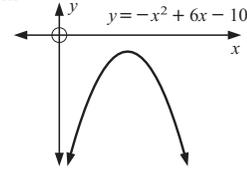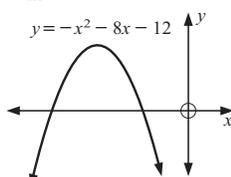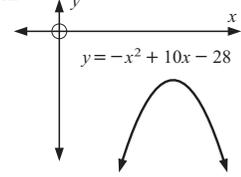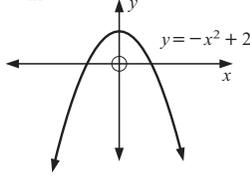
g i $y = -(x - \frac{3}{2})^2 + \frac{1}{4}$
   ii $(\frac{3}{2}, \frac{1}{4})$
   iii

$y = -x^2 + 3x - 2$
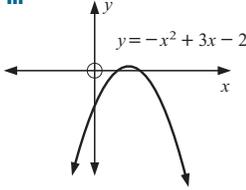
h i $y = -(x + \frac{1}{2})^2 - \frac{3}{4}$
   ii $(-\frac{1}{2}, -\frac{3}{4})$
   iii

$y = -x^2 - x - 1$
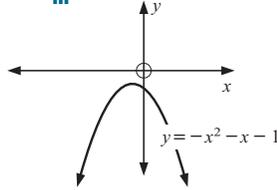
i i $y = -(x + \frac{3}{2})^2 + \frac{1}{4}$
   ii $(-\frac{3}{2}, \frac{1}{4})$
   iii

$y = -x^2 - 3x - 2$
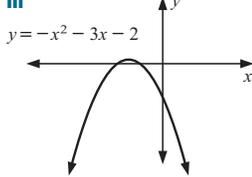
j i $y = -(x - \frac{1}{2})^2 + \frac{1}{4}$
   ii $(\frac{1}{2}, \frac{1}{4})$
   iii

$y = -x^2 + x$
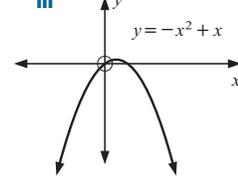
k i $y = -(x + \frac{5}{2})^2 - \frac{3}{4}$
   ii $(-\frac{5}{2}, -\frac{3}{4})$
   iii

$y = -x^2 - 5x - 7$
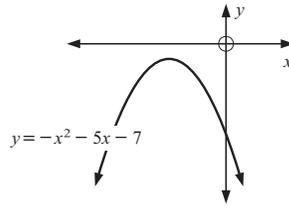
l i $y = -(x - \frac{7}{2})^2 - \frac{3}{4}$
   ii $(\frac{7}{2}, -\frac{3}{4})$
   iii

$y = -x^2 + 7x - 13$
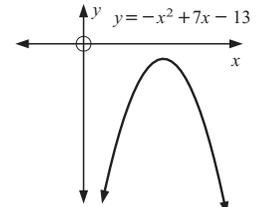
EXERCISE 19E

1 a 1   **b** 3   **c** -2 **d** 4   **e** 6   **f** 1
  g 4   **h** 3   **i** 7

2 a -2 **b** -3 **c** 2   **d** 0   **e** 4   **f** -2

3 a 1 and -3 **b** -2 and 3   **c** 3 and -7 **d** 2 and 5
  e -5 **f** 7

4 a 2 and -2 **b** 3 and -3 **c** 4 and 3   **d** 3 and -2
  e 0 and 3   **f** 5 and 2   **g** -1 **h** 3
  i -6 and 1   **j** -5 and 1   **k** no x-intercepts
  l -7 and 3

EXERCISE 19F

1 a $x = 2$ **b** $x = -1$ **c** $x = 3$ **d** $x = -\frac{1}{2}$
  e $x = -\frac{7}{2}$ **f** $x = \frac{3}{2}$

2 a i 1 and 3   **ii** $x = 2$
   iii $(2, -1)$ **iv** 3

$y = (x - 1)(x - 3)$
$(2, -1)$

b i 1 and 5   **ii** $x = 3$
   iii $(3, 4)$ **iv** -5

$(3, 4)$
$y = -(x - 1)(x - 5)$

c i 0 and 4   **ii** $x = 2$
   iii $(2, -4)$ **iv** 0

$y = x(x - 4)$
$(2, -4)$

d i 0 and -2 **ii** $x = -1$
   iii $(-1, 1)$ **iv** 0

$(-1, 1)$
$y = -x(x + 2)$

e i -2 and 4   **ii** $x = 1$
   iii $(1, -9)$ **iv** -8

$y = (x + 2)(x - 4)$
$(1, -9)$

f i 1   **ii** $x = 1$
   iii $(1, 0)$ **iv** -1

$y = -(x - 1)^2$
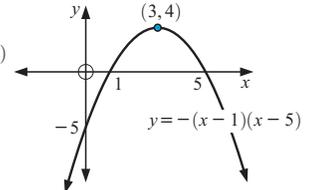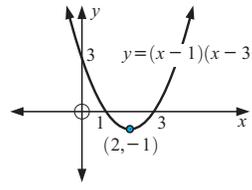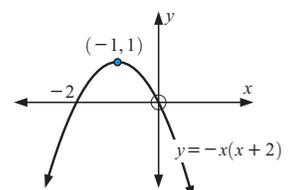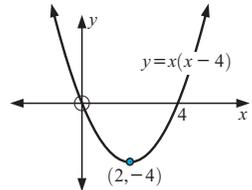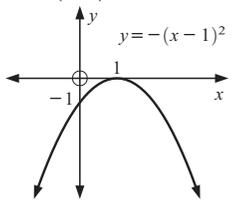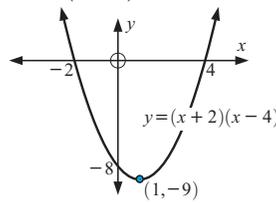
g **i** -4 and 2 **ii** $x = -1$ **h** **i** -4 and 2 **ii** $x = -1$
iii $(-1, -9)$ **iv** -8 **iii** $(-1, 9)$ **iv** 8

$y = (x+4)(x-2)$
$(-1, -9)$

$(-1, 9)$
$y = -(x+4)(x-2)$
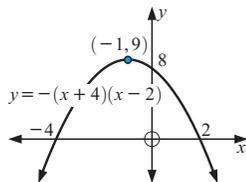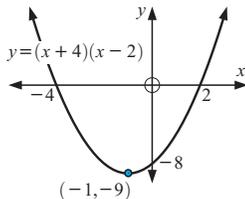
i **i** -3 **ii** $x = -3$ **j** **i** -1 and 2 **ii** $x = \frac{1}{2}$
iii $(-3, 0)$ **iv** 9 **iii** $(\frac{1}{2}, \frac{9}{4})$ **iv** 2

$y = (x+3)^2$

$(\frac{1}{2}, \frac{9}{4})$
$y = -(x+1)(x-2)$
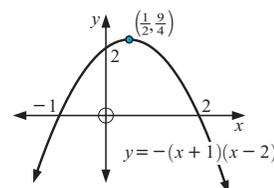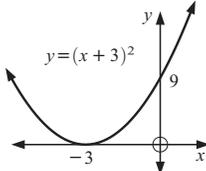
k **i** 0 and -2 **ii** $x = -1$ **l** **i** -2 **ii** $x = -2$
iii $(-1, -1)$ **iv** 0 **iii** $(-2, 0)$ **iv** -4

$y = x^2 + 2x$
$(-1, -1)$

$y = -x^2 - 4x - 4$
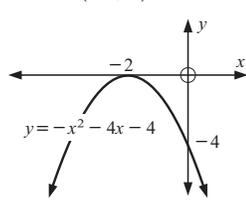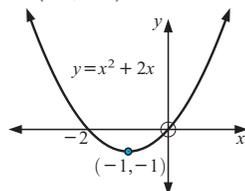
m **i** 0 and -3 **ii** $x = -\frac{3}{2}$ **n** **i** -2 and 5 **ii** $x = \frac{3}{2}$
iii $(-\frac{3}{2}, -\frac{9}{4})$ **iv** 0 **iii** $(\frac{3}{2}, -\frac{49}{4})$ **iv** -10

$y = x^2 + 3x$
$(-\frac{3}{2}, -\frac{9}{4})$

$y = x^2 - 3x - 10$
$(\frac{3}{2}, -\frac{49}{4})$
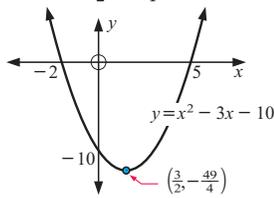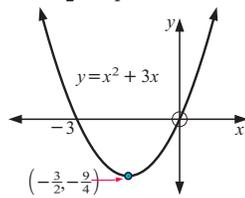
o **i** -6 and -1 **ii** $x = -\frac{7}{2}$ **iii** $(-\frac{7}{2}, \frac{25}{4})$ **iv** -6

$(-\frac{7}{2}, \frac{25}{4})$
$y = x^2 - 7x - 6$
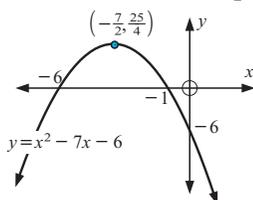
3 • This method cannot be used if the functions do not have x-intercepts.
• Not all quadratics are in or can be converted simply into factored form.

4 **a** 2 and 6 **b** -1 and -5 **c** 3

EXERCISE 19G

1 **a** min. value is 6, when $x = 1$
b max. value is 8, when $x = -1$
c min. value is 2, when $x = -1$
d min. value is -6, when $x = 1$
e max. value is 2, when $x = 1$
f max. value is -3, when $x = -1$
g min. value is -3, when $x = -2$

h max. value is 5, when $x = 2$
i max. value is 5, when $x = -2$
j min. value is $\frac{15}{4}$, when $x = -\frac{3}{2}$
k max. value is $\frac{1}{4}$, when $x = -\frac{3}{2}$
l min. value is -16, when $x = -4$
m max. value is 4, when $x = 2$
n min. value is $-\frac{25}{4}$, when $x = -\frac{5}{2}$
o max. value is $\frac{49}{4}$, when $x = -\frac{7}{2}$

2 **a** -9 **b** $\frac{9}{4}$

3 **a** 8 tables per month **b** $300 **c** $1200

4 **a** 40 km h^{-1} **b** 1 second **c** 5 seconds
d 49 km h^{-1}, at time $t = 3$ seconds

5 **a** 80 stalls **b** $4880 **c** $1520 loss

REVIEW SET 19A

1 $a = -2$ or -1

2

x	-3	-2	-1	0	1	2	3
y	-4	0	2	2	0	-4	-10

$y = 2 - x - x^2$

3
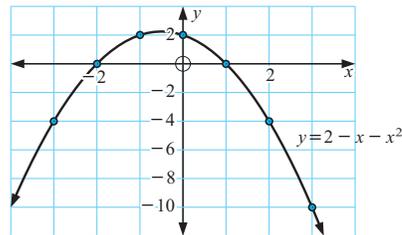
$y = x^2$
$y = (x-1)^2 + 3$
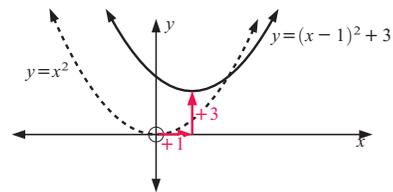
vertex is $(1, 3)$

4 **a** $y = (x-2)^2 - 1$ **c**
b $(2, -1)$

$y = x^2$
$y = x^2 - 4x + 3$
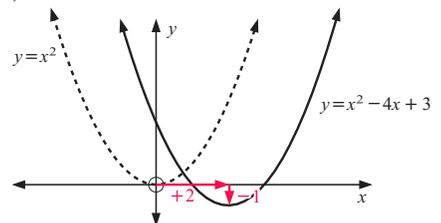
5 **a** $y = -(x+1)^2 + 2$ **c**
b $(-1, 2)$

$y = -x^2 - 2x + 1$
$y = -x^2$
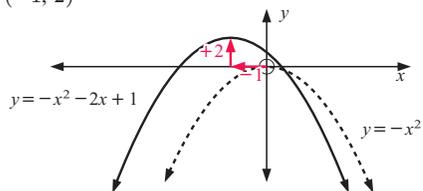
6 **a** -2 and 3 **b** -3 and 4

7 **a** -3 and 2 **b** $x = -\frac{1}{2}$ **c** $(-\frac{1}{2}, -\frac{25}{4})$ **d** -6
e
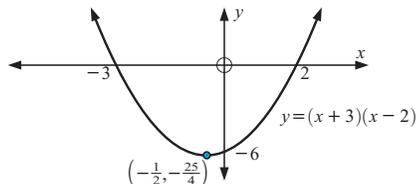
8 **a** max. value is -2, when $x = 1$
 b min. value is $-\frac{17}{4}$, when $x = \frac{5}{2}$

9 -4

REVIEW SET 19B

1 $t = 0$ or 3

2

x	-3	-2	-1	0	1	2	3
y	18	11	6	3	2	3	6

3

vertex is $(-1, 3)$
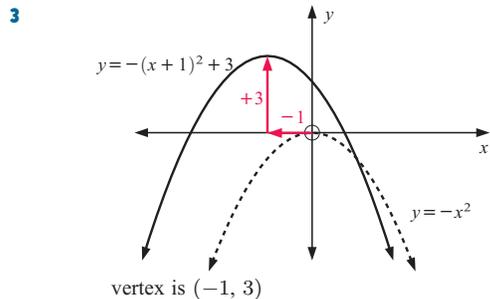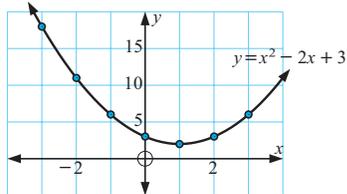
4 **a** $y = (x + 1)^2 + 0$ **c**
 b $(-1, 0)$
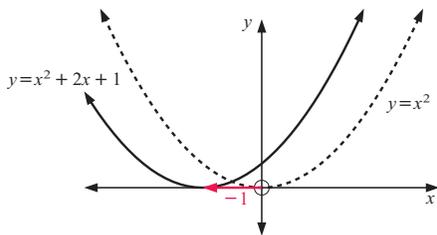
5 **a** $y = -(x - 2)^2 + 4$ **c**
 b $(2, 4)$
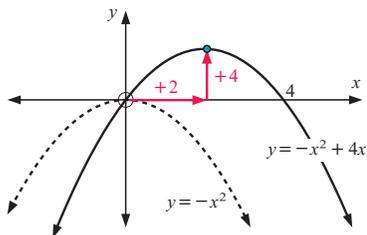
6 **a** -2 and 5 **b** -2 (touches)

7 **a** 2 and -4 **b** $x = -1$ **c** $(-1, 9)$ **d** 8
e
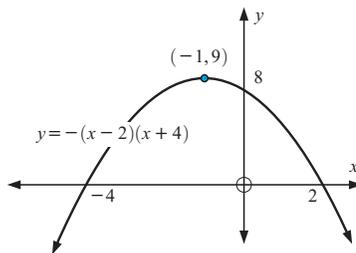
8 **a** min. value is 0, when $x = 3$
 b max. value is $\frac{13}{4}$, when $x = -\frac{3}{2}$

9 **a** $(20 - x)$ m **b** $A = -x^2 + 20x$ m^2
 c 100 m^2, when $x = 10$ (the rectangle is a square)

EXERCISE 20A

1 **a**

b

c

d
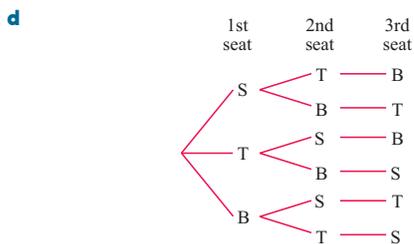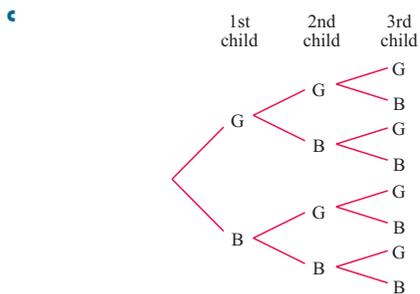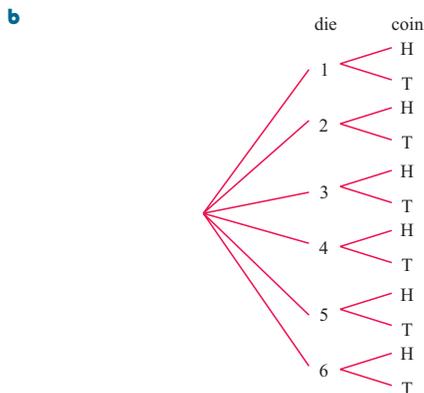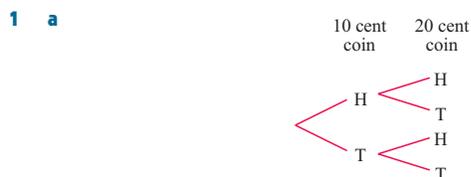
2 a

	1st child	2nd child	3rd child	4th child

(tree diagram with branches G and B)

b

bag — marble

A — R, W, Bl
B — Y
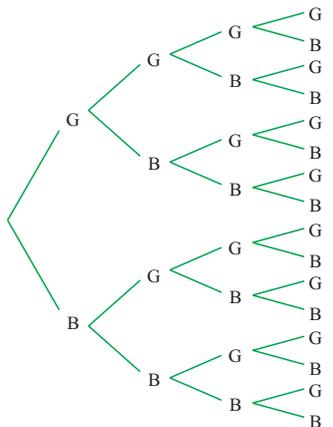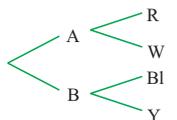
c

hat	1st ticket	2nd ticket

A — P_i (P_i, P_u), P_u (P_i, P_u)
B — P_i (P_i, P_u), P_u (P_i, P_u)
C — P_i (P_i, P_u), P_u (P_i, P_u)
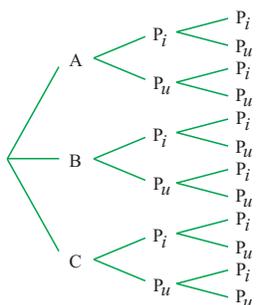
d

1st goal	2nd goal	3rd goal	4th goal	5th goal

(tree diagram with branches X and Y)
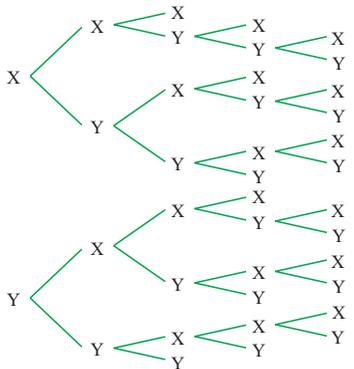
e

1st set	2nd set	3rd set	4th set	5th set

(tree diagram with branches J and P)
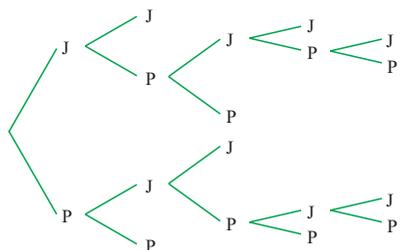
3

1st marble	2nd marble

B — R, G
R — B, G
G — B, R, G
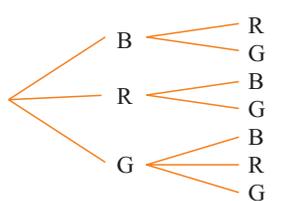
EXERCISE 20B.1

1 a

1st spin	2nd spin	

$\frac{1}{3}$ — B $\frac{1}{3}$ — B $\frac{1}{3}\times\frac{1}{3}$
B $\frac{2}{3}$ — G $\frac{1}{3}\times\frac{2}{3}$
$\frac{2}{3}$ — G $\frac{1}{3}$ — B $\frac{2}{3}\times\frac{1}{3}$
$\frac{2}{3}$ — G $\frac{2}{3}\times\frac{2}{3}$
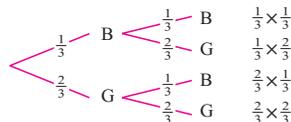
b $\frac{4}{9}$

2 a

100 m sprint	200 m sprint	

$\frac{1}{5}$ — W $\frac{1}{3}$ — W $\frac{1}{5}\times\frac{1}{3}$
$\frac{2}{3}$ — L $\frac{1}{5}\times\frac{2}{3}$
$\frac{4}{5}$ — L $\frac{1}{3}$ — W $\frac{4}{5}\times\frac{1}{3}$
$\frac{2}{3}$ — L $\frac{4}{5}\times\frac{2}{3}$
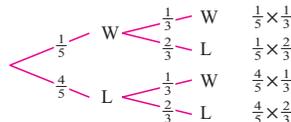
b $\frac{1}{15}$
c $\frac{6}{15}=\frac{2}{5}$
d $\frac{8}{15}$

e $\frac{1}{15}+\frac{6}{15}+\frac{8}{15}=1$, one of these events is certain to happen.

EXERCISE 20B.2

1 a $\frac{7}{15}$ **b** $\frac{7}{30}$ **c** $\frac{7}{15}$

2 a

1st player	1st player	1st player

C — V (O), O (V), C (O)
V — C (O), O (C), O (V)
O — C (O), V (C), O (V)
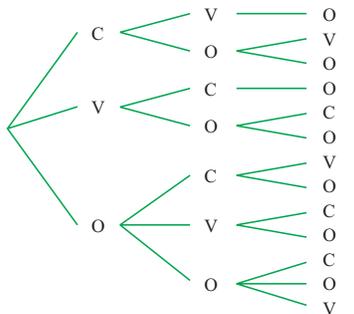
b **i** $\frac{4}{7}$ **ii** $\frac{2}{7}$ **iii** $\frac{4}{7}$

3 a $\frac{2}{15}$ **b** $\frac{1}{3}$ **c** $\frac{4}{15}$ **d** $\frac{4}{15}$ **e** $\frac{8}{15}$

4 a

can 1	can 2

$\frac{4}{10}$ — T $\frac{3}{9}$ — T, $\frac{6}{9}$ — P
$\frac{6}{10}$ — P $\frac{4}{9}$ — T, $\frac{5}{9}$ — P
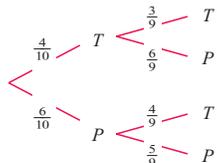
b $\frac{2}{15}$
c $\frac{8}{15}$

EXERCISE 20B.3

1 a $\frac{2}{100}=0.02$ **b** $\frac{1}{4950}\approx 0.000\,202$
c $\frac{49}{2475}\approx 0.0198$ **d** $\frac{4753}{4950}\approx 0.960$

2 a $\frac{14}{55}$ **b** $\frac{1}{55}$ **3 a** $\frac{2}{7}$ **b** $\frac{4}{7}$ **c** $\frac{6}{7}$

EXERCISE 20B.4

1 $\frac{5}{12}$ **2 a** $\frac{20}{49}$ **b** $\frac{10}{21}$ **3 a** $\frac{3}{10}$ **b** $\frac{1}{10}$ **c** $\frac{3}{5}$

4 a $\frac{2}{9}$ **b** $\frac{5}{9}$ **5 a** $\frac{1}{5}$ **b** $\frac{3}{5}$ **c** $\frac{4}{5}$ **6** $\frac{22}{63}$

EXERCISE 20C.1

1 a binomial **b** not binomial **c** binomial **d** not binomial
e binomial **f** not binomial **g** not binomial

EXERCISE 20C.2

1 a i $\frac{5}{6}$ **b** **i** $\frac{25}{36}$

ii $\frac{1}{6}$ **ii** $\frac{5}{18}$

iii $\frac{1}{36}$
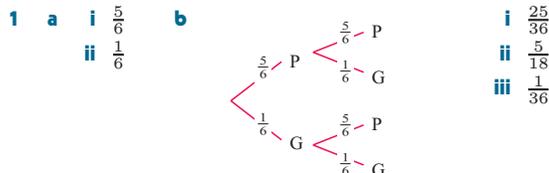
c $\left(\frac{5}{6}+\frac{1}{6}\right)^2 = \left(\frac{5}{6}\right)^2 + 2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2$

d The expansion of $\left(\frac{5}{6}+\frac{1}{6}\right)^2$ generates the same probabilities as from the tree diagram.

2 a {MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF}

i $\frac{1}{8}$ **ii** $\frac{3}{8}$ **iii** $\frac{3}{8}$ **iv** $\frac{1}{8}$

b
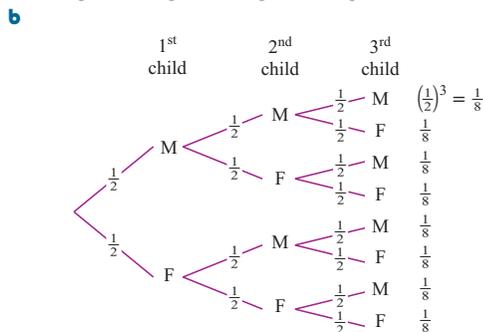
$P(3 \text{ boys}) = P(MMM) = \frac{1}{8}$

$P(2 \text{ boys}) = P(MMF) + P(MFM) + P(FMM) = \frac{3}{8}$

$P(1 \text{ boy}) = P(MFF) + P(FMF) + P(FFM) = \frac{3}{8}$

$P(0 \text{ boys}) = P(FFF) = \frac{1}{8}$

c $\left(\frac{1}{2}+\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$
$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$

d The expansion of $(\frac{1}{2} + \frac{1}{2})^3$ generates the same probabilities as from the tree diagram.

3 a

i $\frac{8}{27}$
ii $\frac{12}{27}$
iii $\frac{6}{27}$
iv $\frac{1}{27}$
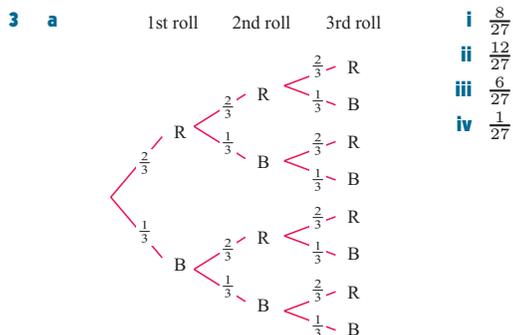
b $\left(\frac{2}{3}+\frac{1}{3}\right)^3 = \left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) + 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3$
$= \frac{8}{27} + \frac{12}{27} + \frac{6}{27} + \frac{1}{27}$

c The expansion of $\left(\frac{2}{3}+\frac{1}{3}\right)^3$ generates the same probabilites as from the tree diagram.

4 a A tree diagram with 4 trials has $2^4 = 16$ end branches, which is less practical to use than the expansion of $(0.03 + 0.97)^4$.

b $(0.03 + 0.97)^4 = (0.03)^4 + 4(0.03)^3(0.97)$
$\qquad + 6(0.03)^2(0.97)^2 + 4(0.03)(0.97)^3 + (0.97)^4$

c i $\approx 0.000\,000\,81$ **ii** $\approx 0.000\,105$
iii $\approx 0.005\,08$ **iv** ≈ 0.110 **v** ≈ 0.885

5 0.0036

REVIEW SET 20A

1

2 a 0.02 **b** 0.72 **c** 0.28 **d** 0.18

3

4 a Yes **b** No, probability of success differs in each trial.

5 a ≈ 0.240 **b** ≈ 0.265

REVIEW SET 20B

1

2 a $\frac{53}{140}$ **b** $\frac{3}{56}$

3 a

Mathematics   Biology   **b** 0.34

4 a No, trials are independent.   **b** Yes

5 a

b i $\frac{9}{64}$ **ii** $\frac{27}{64}$

EXERCISE 21A

1 a 3   **b** 2   **c** $\frac{1}{2}$ **d** -6 **e** $4\frac{1}{2}$ **f** 1

g $4\frac{1}{2}$ **h** 12   **i** 2   **j** 6

2 a -2 **b** $-\frac{1}{2}$ **c** $\frac{1}{3}$ **d** 8   **e** 4   **f** -1

g $\frac{1}{2}$ **h** -3 **i** $-\frac{2}{3}$ **j** -4

EXERCISE 21B.1

1 a $5 \neq 14$, $\therefore$ false

b $3 = 3$, $5 = 5$, $7 = 7$ $\therefore$ likely to be correct

c $7\frac{1}{2} \neq 9$, $\therefore$ false   **d** $3 \neq 7$, $\therefore$ false

2 a $\frac{a}{2}$ **b** $2m$ **c** 6   **d** 3   **e** $2a$ **f** x^2

g $2x$ **h** 2   **i** $\frac{1}{2a}$ **j** $2m$ **k** 4   **l** $\frac{2}{t}$

m $2d$ **n** $\frac{b}{2}$ **o** $\frac{2b}{3a}$

3 a t **b** not possible   **c** y **d** not possible

e not possible   **f** $\frac{a}{b}$ **g** not possible   **h** $\frac{a}{2}$

i not possible   **j** a

4 a $\frac{a}{2} + 2$ **b** $a + 2$ **c** $x + \frac{3}{2}$ **d** $x + 4$ **e** $2x + \frac{3}{2}$

f $x^2 + 2x$ **g** $\frac{x^2}{2} + x$ **h** $1 - \frac{6}{x}$ **i** $x - 6$ **j** $\frac{x^2}{6} - x$

5 a 4   **b** $2n$ **c** a **d** 1   **e** a^2 **f** $\frac{3n}{2}$ **g** $x + y$

h $\frac{2}{x + 2}$

6 a $a + 3$ **b** $2(x + 2)$ **c** $2(c + 3)$ **d** $\frac{d - 3}{3}$

e $\frac{2}{x + 1}$ **f** $\frac{3}{2 - x}$ **g** $\frac{1}{3}$ **h** 2   **i** $\frac{x + y}{3}$

j $\frac{y}{3}$ **k** $\frac{y - 3}{3}$ **l** $\frac{x + 1}{3}$

7 a $\frac{x + 2}{2}$ **b** $2(x + 2)$ **c** cannot be simplified

d cannot be simplified   **e** $\frac{x - 1}{2(x + 1)}$

f cannot be simplified   **g** $\frac{x + 3}{5}$ **h** $\frac{1}{a + 3}$

EXERCISE 21B.2

1 a $\frac{2}{x + 1}$ **b** $\frac{3}{2 - x}$ **c** $x + 2$ **d** $x + 2$

e $\frac{x + 2}{2}$ **f** $\frac{x + 5}{2}$ **g** $\frac{y + 3}{3}$ **h** $a + b$

i $\frac{a + b}{c + d}$ **j** $\frac{a + 2}{2}$ **k** $\frac{1}{2(b - 4)}$ **l** $\frac{2(p + q)}{3}$

2 a $\frac{3}{4}$ **b** $\frac{a + b}{2}$ **c** x **d** $\frac{1}{a + b}$ **e** $\frac{1}{y}$ **f** $\frac{x}{y}$ **g** $4x$

h $3x$

3 a -2 **b** $-\frac{3}{2}$ **c** not possible   **d** -1 **e** $-\frac{1}{2}$ **f** -3

g $-\frac{2}{x}$ **h** $-\frac{ab}{2}$ **i** $-2x$ **j** $2x + 3$ **k** not possible

l $\frac{2x + 3}{2}$ **m** not possible   **n** $\frac{2x + 3}{3}$ **o** not possible

p not possible   **q** $\frac{3a + 1}{2}$ **r** not possible   **s** $\frac{b + 3}{2}$

t not possible

4 a $x + 1$ **b** $x - 1$ **c** $-x - 1$ **d** $\frac{1}{x - 2}$

e $a - b$ **f** $-a - b$ **g** $\frac{2}{x - 1}$ **h** $\frac{3 + x}{x}$

i $\frac{3(x + y)}{2y}$ **j** $-\frac{2(b + a)}{a}$ **k** $\frac{y}{4x + y}$ **l** $-\frac{4x}{4 + x}$

5 a $x + 1$ **b** $\frac{1}{x - 5}$ **c** $\frac{2x}{x - 5}$ **d** $\frac{x - 2}{x + 2}$

e $\frac{x + 3}{x - 1}$ **f** $\frac{x + 1}{1 - x}$ **g** $\frac{x - 5}{x + 3}$ **h** $\frac{x + 2}{x + 3}$

i $\frac{x + 2}{2x - 1}$ **j** $\frac{2x + 1}{x - 1}$ **k** $\frac{4x - 3}{2x + 1}$ **l** $\frac{3x + 4}{x + 2}$

EXERCISE 21C

1 a $\frac{xy}{10}$ **b** $\frac{3}{2}$ **c** $\frac{a^2}{2}$ **d** $\frac{1}{6}$ **e** $\frac{1}{5}$ **f** $\frac{c^2}{10}$ **g** $\frac{ac}{bd}$

h 1   **i** $\frac{1}{2m}$ **j** 2   **k** $\frac{a}{b}$ **l** 4   **m** $\frac{3}{m}$ **n** $\frac{a^2}{b^2}$

o $\frac{4}{x^2}$ **p** $\frac{1}{c}$

2 a $\frac{3}{2}$ **b** $\frac{3}{a}$ **c** $\frac{3x}{16}$ **d** $\frac{3}{4}$ **e** 2   **f** $\frac{c}{25}$

g $\frac{1}{5}$ **h** $\frac{m^2}{2}$ **i** 2   **j** $\frac{n}{m}$ **k** $\frac{3}{4g}$ **l** $\frac{g}{3}$

EXERCISE 21D

1 a $\frac{5a}{6}$ **b** $\frac{b}{10}$ **c** $\frac{7c}{4}$ **d** $\frac{-5x}{14}$ **e** $\frac{4a + 3b}{12}$ **f** $\frac{-2t}{9}$

g $\frac{5m}{21}$ **h** $\frac{d}{2}$ **i** $\frac{11p}{35}$ **j** m **k** $\frac{5a}{12}$ **l** $\frac{x}{12}$

2 a $\frac{7b + 3a}{ab}$ **b** $\frac{3c + 2a}{ac}$ **c** $\frac{4d + 5a}{ad}$ **d** $\frac{a}{m}$

e $\frac{2a + b}{2x}$ **f** $\frac{5}{2a}$ **g** $\frac{4y - 1}{xy}$ **h** $\frac{ad + bc}{bd}$

i $\frac{ay - bx}{by}$ **j** $\frac{4 + 3a}{6}$ **k** $\frac{4x + 9}{12}$ **l** $\frac{3x + 2y}{3y}$

3 **a** $\dfrac{x+2}{2}$ **b** $\dfrac{y-3}{3}$ **c** $\dfrac{3a}{2}$ **d** $\dfrac{b-12}{4}$

e $\dfrac{x-8}{2}$ **f** $\dfrac{6+a}{3}$ **g** $\dfrac{15-x}{5}$ **h** $\dfrac{2x+1}{x}$

i $\dfrac{5x-2}{x}$ **j** $\dfrac{a^2+2}{a}$ **k** $\dfrac{3+b^2}{b}$ **l** $\dfrac{-5x}{3}$

4 **a** $\dfrac{9x}{10}$ **b** $\dfrac{-7x}{10}$ **c** $\dfrac{11}{3a}$ **d** $\dfrac{5}{2y}$ **e** $\dfrac{5b+3a}{ab}$

f $\dfrac{4b-15a}{3ab}$ **g** $\dfrac{x+14}{7}$ **h** $\dfrac{12-x}{4}$

EXERCISE 21E

1 **a** $\dfrac{5x+2}{6}$ **b** $\dfrac{-x-1}{4}$ **c** $\dfrac{11x+9}{12}$ **d** $\dfrac{5x+1}{6}$

e $\dfrac{-2x-1}{12}$ **f** $\dfrac{10x+3}{6}$

2 **a** $\dfrac{7x+3}{12}$ **b** $\dfrac{5x+2}{4}$ **c** $\dfrac{23x-6}{30}$ **d** $\dfrac{5b-a}{6}$

e $\dfrac{14x-1}{20}$ **f** $\dfrac{23-5x}{14}$ **g** $\dfrac{11x-12}{30}$ **h** $\dfrac{3x-4}{20}$

i $\dfrac{3x-2}{8}$ **j** $\dfrac{3x-2}{10}$ **k** $\dfrac{32-7x}{15}$ **l** $\dfrac{-17x-1}{12}$

m $\dfrac{3-11x}{30}$ **n** $\dfrac{7x+3}{x(x+1)}$ **o** $\dfrac{2x-6}{x(x+2)}$

p $\dfrac{x-7}{(x+1)(x-1)}$ **q** $\dfrac{3x+7}{x+2}$ **r** $\dfrac{5x-4}{x(x-4)}$

s $\dfrac{-4x-10}{x+3}$ **t** $\dfrac{2x^2+x+1}{(x-1)(x+1)}$ **u** $\dfrac{11x-10}{x(x-2)}$

REVIEW SET 21A

1 **a** -2 **b** 4 **c** $\frac{1}{3}$ **d** $-1\frac{2}{11}$

2 **a** $\dfrac{2t}{3}$ **b** $2\frac{2}{3}$ **c** $\dfrac{x}{3}$ **d** $\dfrac{2}{x+2}$

3 **a** -2 **b** $\dfrac{2}{x-3}$ **c** $\dfrac{x+2}{x}$ **d** $\dfrac{3x}{3x+1}$

4 **a** $\dfrac{a}{3}$ **b** $\dfrac{3a}{b^2}$ **c** $\dfrac{3a+b^2}{3b}$ **d** $\dfrac{3a-b^2}{3b}$

5 **a** $\dfrac{11x}{12}$ **b** $\dfrac{14+x}{7}$ **c** $\dfrac{x-4}{4}$ **d** $\dfrac{5x}{12}$

6 **a** $\dfrac{7x-3}{12}$ **b** $\dfrac{3x+2}{6}$ **c** $\dfrac{3x}{(x+1)(x-2)}$

d $\dfrac{3x+3}{10}$ **e** $\dfrac{x+9}{(x-1)(x+1)}$ **f** $\dfrac{x^2+x+1}{x^2(x+1)}$

REVIEW SET 21B

1 **a** -6 **b** 2 **c** 0 **d** undefined

2 **a** $\dfrac{3}{2x}$ **b** $a+2b$ **c** $\dfrac{x+2}{x}$ **d** $\dfrac{3}{x+2y}$

3 **a** $\frac{1}{3}$ **b** $2(x-2)$ **c** $\dfrac{x-3}{4}$ **d** $\dfrac{2x-1}{3x+1}$

4 **a** $\dfrac{2m}{n^2}$ **b** $\dfrac{m}{2}$ **c** $\dfrac{11}{2x}$ **d** $\dfrac{6b-ay}{by}$

5 **a** $\dfrac{5x}{14}$ **b** $\dfrac{10+x}{2}$ **c** $\dfrac{3x-y}{x}$ **d** $\dfrac{6+3x+2y}{6}$

6 **a** $\dfrac{3x-2}{8}$ **b** $\dfrac{9x+27}{10}$ **c** $\dfrac{7-x}{(x-1)(x+2)}$

d $\dfrac{7x-3}{4}$ **e** $\dfrac{15-x}{(x-3)(x+3)}$ **f** $\dfrac{x^2-2x+2}{x^2(x-1)}$

EXERCISE 22A

1 **a** 4 **b** 5 **c** 7 **d** $3\frac{1}{2}$ **e** $3\frac{1}{4}$

2 **a** 5 **b** 10 **c** 20 **d** $2\frac{1}{2}$ **e** $1\frac{1}{4}$

3 **a** 2 **b** 4 **c** 16 **d** 1 **e** $\frac{1}{4}$

4 **a** 1 **b** $\frac{1}{4}$ **c** $\frac{1}{16}$ **d** 4 **e** 16

5 **a** 6 **b** 2 **c** 18 **d** $\frac{2}{3}$ **e** 54

EXERCISE 22B

1 **a** **i** 1.4 **ii** 3.0 **iii** 4.3

   b **i** 1.41 **ii** 3.03 **iii** 4.29

2 **a** **i** 1 **ii** 3 **iii** 1.7 **iv** 3.7

   b **i** 1 **ii** 3 **iii** 1.73 **iv** 3.74

3 **a** **i** $x=2$ **ii** $x\approx1.6$ **iii** $x=0$ **iv** $x\approx-0.3$

4 **a** **i** $x=1$ **ii** $x=1.3$ **iii** $x=0$ **iv** $x=-0.8$

5 **a**

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

b
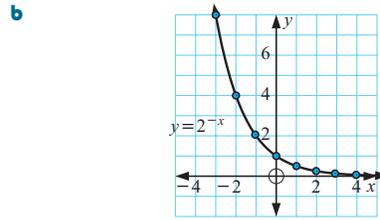
c Yes, as x gets large and positive, y-values get closer and closer to 0.

6 **a** **i** 0.6 **ii** 4.7 **iii** 0.1 **iv** 1

   b **i** 0.577 **ii** 4.656 **iii** 0.892 **iv** 1

EXERCISE 22C

1 **a** 500 ants **b** **i** 1550 ants **ii** 4820 ants

   c
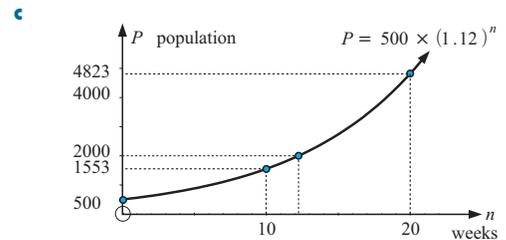
   d ≈ 12 weeks

2 **a** 20 grams **b** ≈ 23.6 g

   c
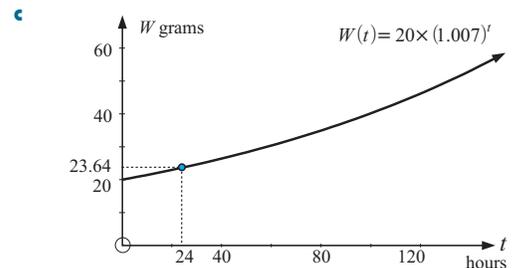
   d ≈ 231 hours

3 **a** 250 wasps **b** **i** 1070 wasps **ii** 6530 wasps

c

Graph showing $P = 250 \times (1.06)^n$ with axis values 8000, 6532, 6000, 4000, 2000, 1073, 250 on P-axis and 20, 25, 40, 56, 60 on n (days) axis.

d ≈ 12 days

4 a

Graph showing $P = 23 \times (1.018)^x$ with P-axis values 50, 40, 30, 20 and x-axis values 10, 20, 30, 40.
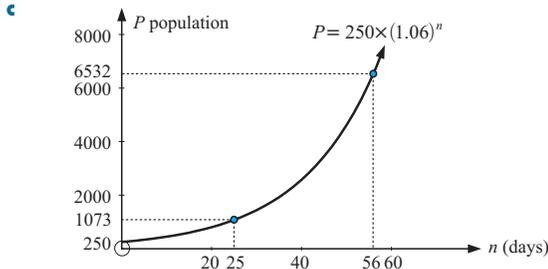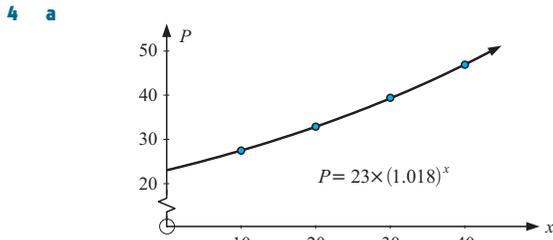
b   i $a = 23\,000$ **ii** $b = 1.018$
c 1970, 27 500; 1980, 32 900; 1990, 39 300
d   i 56 100   **ii** 115 000

EXERCISE 22D

1   a 100°C   **b   i** 50.0°C   **ii** 25.0°C   **iii** 12.5°C
c

Graph showing $T = 100 \times (0.993)^t$ with T-axis values 100, 80, 60, 40, 20 and t-axis values 10, 20, 30, 40, 50.
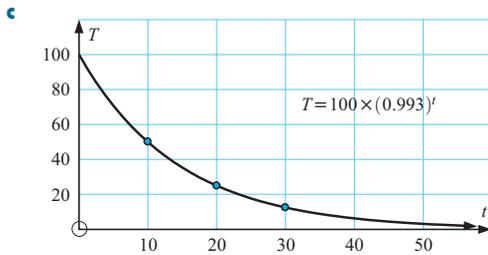
d   i ≈ 13 mins   **ii** ≈ 33 mins
2   a 150 grams   **b   i** ≈ 111 g   **ii** ≈ 82.2 g   **iii** ≈ 45.1 g
c

Graph showing $W = 150 \times (0.997)^t$ with W (grams) axis values 150, 100, 50 and t-axis values 200, 400, 600.
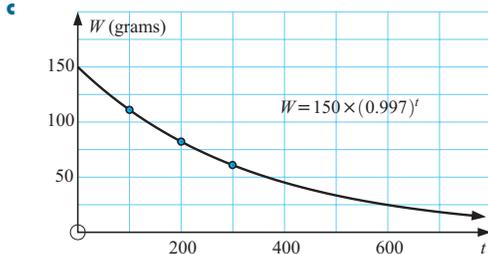
d ≈ 596 years

3   a

n	0	5	10	15	20	25
P	255	204	163	131	104	84

Graph showing P-axis values 250, 200, 150, 100, 50 and x-axis values 5, 10, 15, 20, 25.
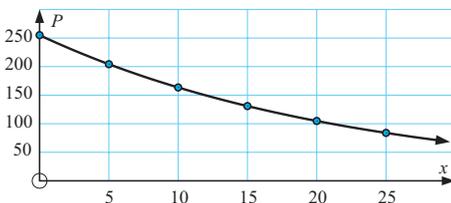
b   i $a = 255$ **ii** $b \approx 0.957$
c So, $P = 255 \times (0.9566)^n$ **d** 2016
    1985 $\approx$ 204;
    1990 $\approx$ 164;
    1995 $\approx$ 131;
    2000 $\approx$ 105

EXERCISE 22E.1

1   a when x or $y = 0$, $xy = 0 \neq 5$
    b vertical asymptote $x = 0$,   horizontal asymptote $y = 0$
    c   i $y = 0.01$ **ii** $y = -0.01$
    d   i $x = 0.01$ **ii** $x = -0.01$
    e $y = \dfrac{5}{x}$

Graph with points $(1,5)$, $(5,1)$, $(-5,-1)$, $(-1,-5)$.

    f $y = -\dfrac{5}{x}$

Graph with points $(-5,1)$, $(5,-1)$.
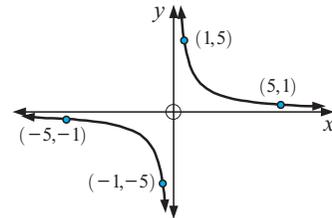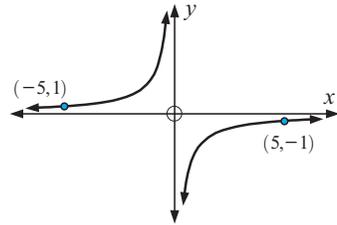
2   a

n	10	20	30	40	50
t	40	20	$13\frac{1}{3}$	10	8

b

Graph with t-axis values 40, 30, 20, 10 and n-axis values 10, 20, 30, 40.
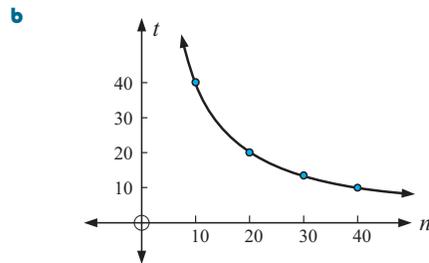
c Yes, one part of a hyperbola.   **d** $t = \dfrac{400}{n}$

3   a $y = \dfrac{8}{x}$ **b** $y = \dfrac{3}{x}$ **c** $y = -\dfrac{12}{x}$
4 $y = -x$ and $y = x$

EXERCISE 22E.2

1   a 35   **b** 20 weeds per hectare   **c** 16 days
    d

Graph with N-axis value 35, points $(6, 20)$ and $(16, 15)$, t-axis.
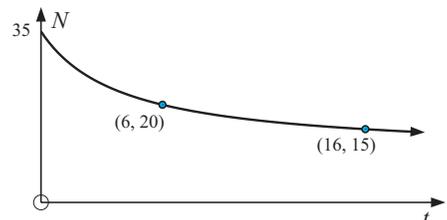
e No, as t gets very large, N approaches 10.

2 **a** 240 amps **b** **i** 192 amps **ii** 96 amps **iii** 40 amps
c 92 milliseconds
d
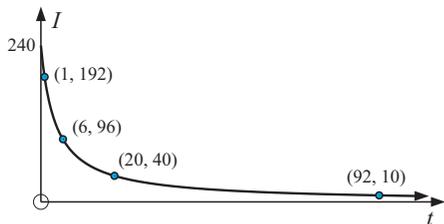
3 **b** **i** 3600 m **ii** 3800 m
c
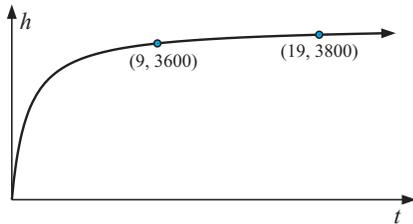
d as t gets very large, h approaches 4000 m
∴ helicopter cannot go beyond 4000 m

4 **a** 21 m s^{-1} **b** $\frac{11}{13}$ seconds **c** 7 seconds
d
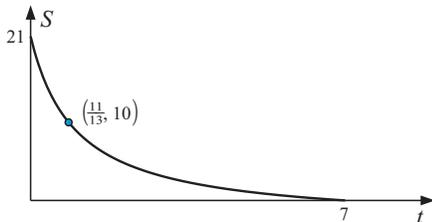
e $0 \leqslant t \leqslant 7$

EXERCISE 22F

1 **a** $\dfrac{500}{x}$ m **b** $L = 2x + \dfrac{1000}{x}$
c
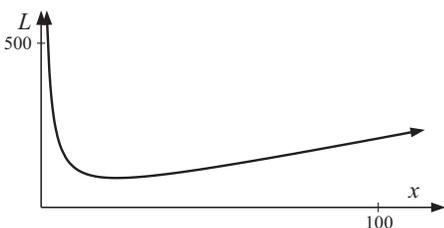
d (22.36, 89.44) **e** garden is square with sides 22.36 m

2 **a** $y = \dfrac{600}{x}$ **b** $C = 85\left(4x + \dfrac{1800}{x}\right)$
c
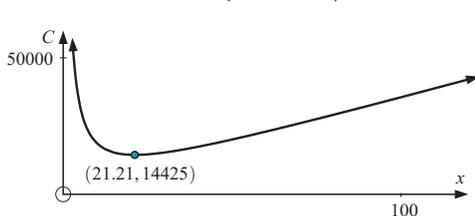
d (21.21, 14 425) **e** 21.21 m × 28.28 m
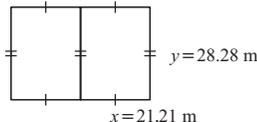
3 **a** Hint: Equate volumes
b Hint: Find the surface area of each face.

c $A = x^2 + \dfrac{1\,000\,000}{x}$
d
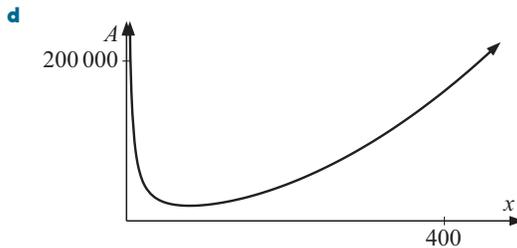
e (79.37, 18 900) **f** Base sides 79.4 cm, height 39.7 cm
4 radius 31.7 cm, height 31.7 cm

EXERCISE 22G

1 **a** **i** (0.910, 2.99) **ii** $x = 0$ **iii** no axes intercepts
iv
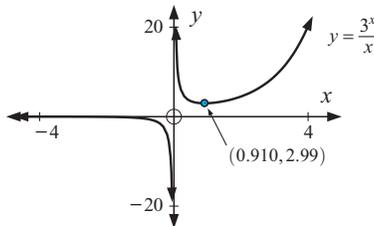
b **i** (−2.89, 1.13) and (0, 0) **ii** no vertical asymptotes
iii x-intercept 0, y-intercept 0
iv
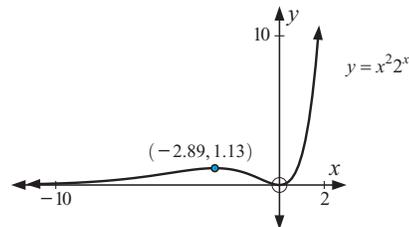
c **i** (0.794, 1.89) **ii** $x = 0$
iii no y-intercept, x-intercept −1
iv
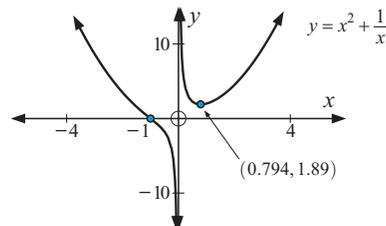
d **i** (0, −4) and (2.89, −2.87)
ii no vertical asymptotes
iii x-intercept −1.28, y-intercept −4

iv
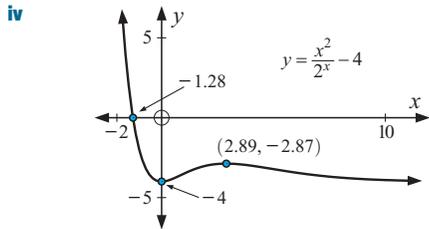
e **i** $(3.73, 0.134)$ and $(0.268, 1.87)$

ii $x = -1$ and $x = 1$

iii x-intercept 2, y-intercept 2

iv
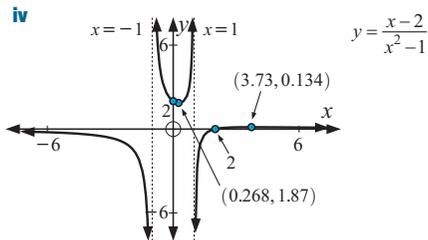
f **i** $(0, 0.25)$ **ii** $x = -2$ and $x = 2$

iii x-intercepts -1 and 1, y-intercept $\frac{1}{4}$

iv
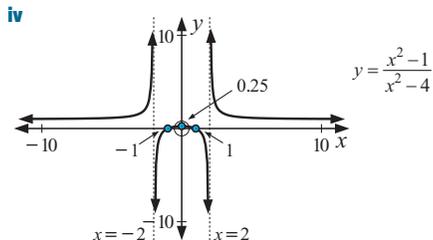
g **i** $(-3.83, -0.0858)$ and $(1.83, -2.91)$

ii $x = 1$ and $x = 3$

iii x-intercept -1, y-intercept $\frac{1}{3}$

iv
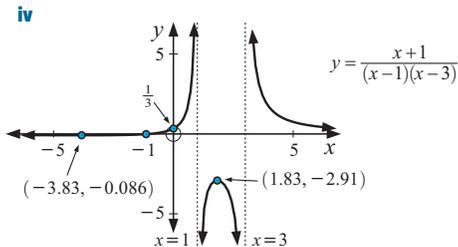
h **i** $(-12, 1.92)$ and $(0, 0)$ **ii** $x = -2$ and $x = 3$

iii x-intercept 0, y-intercept 0

iv

2 **a**
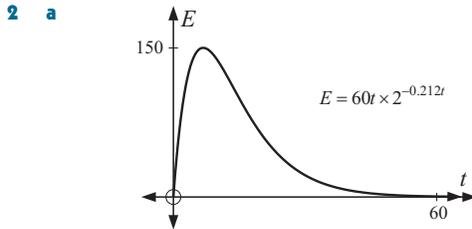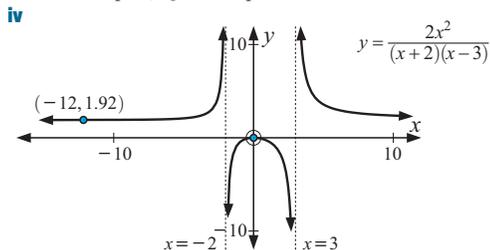
b The injection takes effect very quickly, and then steadily wears off over time.

c 6 hours 48 minutes after the injection is given.

d Between 2 hrs 21 min and 14 hrs 55 min after the injection is given, i.e., an interval of 12 hours 34 min.

3 **a**
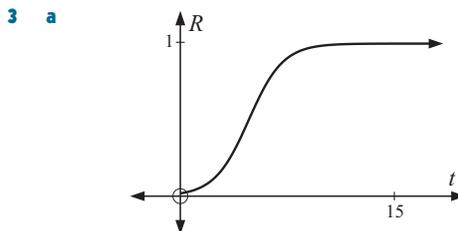
b Inititally the rumour spreads quickly, but then slows down as there are less people yet to hear the rumour.

c 4 hours 50 minutes after the rumour is started.

d 1.78% **e** 5 people

REVIEW SET 22A

1 **a** $y = -1$ **b** $y = 7$ **c** $y = -\frac{17}{9}$

2
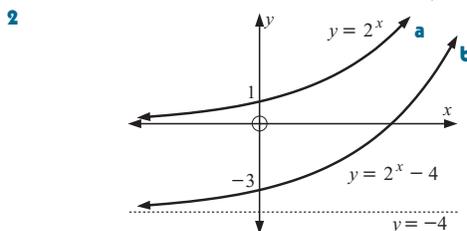
a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$.

b $y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$.

3 **a** $80^\circ C$ **b** **i** $26.8^\circ C$ **ii** $9.00^\circ C$ **iii** $3.02^\circ C$

c
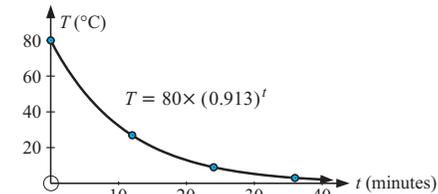
d ≈ 12.8 minutes

4 **a** 1500 ants **b** **i** 1930 ants **ii** 13 100 ants

c
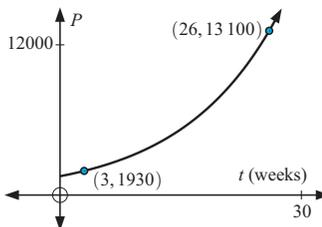
d ≈ 42 weeks

5 **a** 0 cm s^{-1} **b** 2.98 cm s^{-1} **c** after 0.697 seconds

d
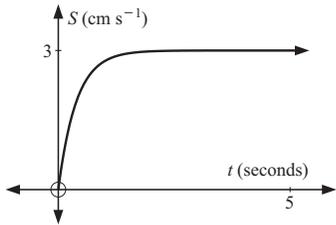
e The object accelerates quickly at first, but as time goes by its speed approaches, but never reaches, 3 cm s^{-1}.

6 **a** x-intercepts -1.30 and 2.30, y-intercept $-\frac{3}{2}$

b $(-3.73, -8.46)$ and $(-0.268, -1.54)$ **c** $x = -2$

d
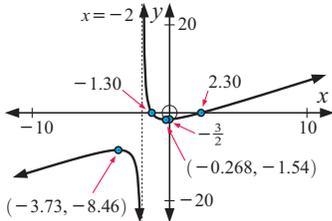
REVIEW SET 22B

1 **a** $\frac{1}{2}$ **b** $\frac{1}{8}$ **c** 4

2
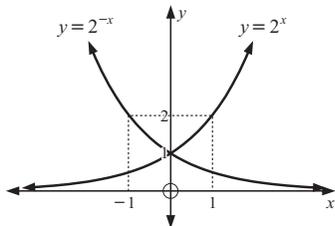
a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$

b $y = 2^{-x}$ has y-intercept 1 and horizontal asymptote $y = 0$

3 **a** 1000 zebra

b **i** ≈ 4000 **ii** $\approx 16\,000$ **iii** $\approx 64\,100$

c
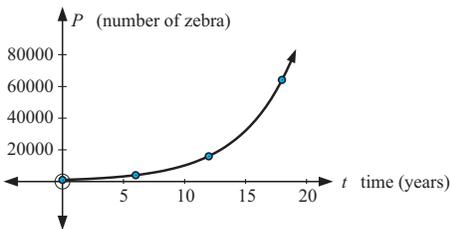
d ≈ 7 years

4 **a** 1500 g **b** **i** 90.3 g **ii** 5.44 g

c
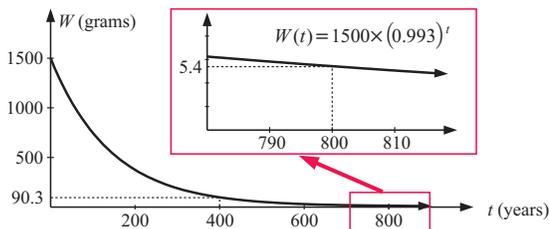
d ≈ 385.5 years

5 **a** 26 marsupials **b** 46 marsupials **c** 12.6 years

d
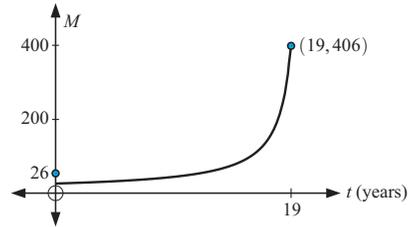
6 **a** no x-intercepts, y-intercept $-\frac{1}{2}$

b $(0.5, -0.444)$ **c** $y = 0$

d
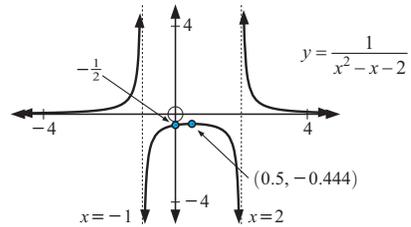
EXERCISE 23A

1 **a**

b

c

d

e

f

g

h

2 $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$, $\mathbf{g} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
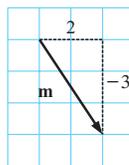
3 **a** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} -6 \\ -1 \end{pmatrix}$ **d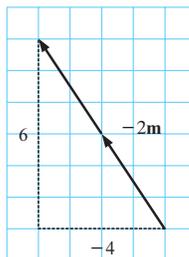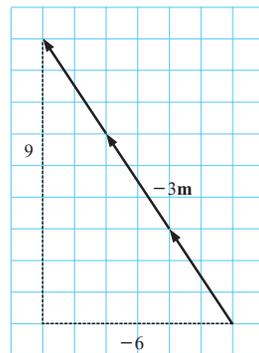** $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ **e** $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ **f** $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ **g** $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ **h** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

EXERCISE 23B

1 **a** $\sqrt{29}$ units   **b** $\sqrt{13}$ units   **c** $2\sqrt{5}$ units   **d** 3 units

2 **a** 5 units   **b** 5 units   **c** 5 units   **d** 5 units
Regardless of a vector's direction, if its components involve ± 3 and ± 4, its length is 5.

3 **a** $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ **b** $\sqrt{52}$ units   or $2\sqrt{13}$ units

4 **a** $\sqrt{34}$ units   **b** 3 units   **c** 13 units   **d** $3\sqrt{13}$ units
e 4 units   **f** $3\sqrt{5}$ units   **g** $2\sqrt{17}$ units   **h** $2\sqrt{17}$ units

5 **a** $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$, $\left|\overrightarrow{AB}\right| = \sqrt{13}$ units
b $\overrightarrow{BC} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\left|\overrightarrow{BC}\right| = 2\sqrt{5}$ units

6 **a** They are parallel (in the same direction) and equal in length.
b **i** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **ii** $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$
c $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right| = \sqrt{17}$ units

7 **a** $\overrightarrow{PQ} = \overrightarrow{SR} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{SR}\right| = 2\sqrt{2}$ units
b $\overrightarrow{RQ} = \overrightarrow{SP} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\left|\overrightarrow{RQ}\right| = \left|\overrightarrow{SP}\right| = \sqrt{13}$ units
b PQRS is a parallelogram

8 **a**
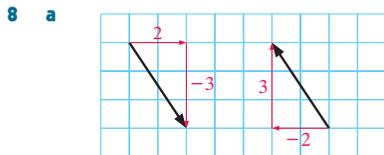
b They are parallel, have equal length, but have opposite direction.
c True

EXERCISE 23C

1 **a**
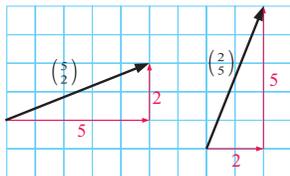
As the vectors do not have the same direction $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.
They do however have the same length of $\sqrt{29}$ units.
b $a = 4$ and $b = -3$ **c** $a = 0$ or 1 and $b = 0$

2 **a** $\overrightarrow{DC} = \mathbf{a}$ {as DC and AB are opposite sides of a parallelogram and so these sides are parallel and equal in length.}
Likewise $\overrightarrow{BC} = \mathbf{b}$.

3 **a** **i** a   **ii** b   **iii** b   **iv** c
b $\mathbf{a} \neq \mathbf{b}$ as the vectors are not parallel
c Yes, as $\triangle ABC$ is equilateral.

EXERCISE 23D

1 **a** $\begin{pmatrix} 6 \\ 10 \end{pmatrix}$ **b** $\begin{pmatrix} 11 \\ 11 \end{pmatrix}$ **c** $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ **d** $\begin{pmatrix} -8 \\ -2 \end{pmatrix}$ **e** $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$
f $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

2 **a** $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ **d** $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **e** $\begin{pmatrix} 10 \\ 4 \end{pmatrix}$
f $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ **g** $\begin{pmatrix} -12 \\ 3 \end{pmatrix}$ **h** $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

3 **a** $\mathbf{a} = \mathbf{b} + \mathbf{c}$ **b** $\mathbf{q} = \mathbf{p} + \mathbf{r}$ **c** $\mathbf{p} = \mathbf{m} + \mathbf{n}$
d $\mathbf{b} = \mathbf{a} + \mathbf{d} + \mathbf{c}$ **e** $\mathbf{p} = \mathbf{q} + \mathbf{r} + \mathbf{s}$ **f** $\mathbf{a} = \mathbf{b} + \mathbf{d} + \mathbf{c}$

4 **a** $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} + \mathbf{c}$ **c** $\mathbf{c} + \mathbf{d}$ **d** $\mathbf{a} + \mathbf{c} + \mathbf{d}$

5 **a** c   **b** f   **c** g   **d** g   **e** g

6 **a** $\overrightarrow{AE}$ **b** $\overrightarrow{BE}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{AE}$

7 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{LD}$ **d** $\overrightarrow{SN}$ **e** $\overrightarrow{AE}$ **f** $\overrightarrow{CP}$

8 **a** $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
d **i** $2\sqrt{29} \times 40 \approx 431$ m   **ii** $2\sqrt{13} \times 40 \approx 288$ m
iii $\sqrt{13} \times 40 \approx 144$ m   **iv** $\sqrt{5} \times 40 \approx 89$ m
e $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ This vector describes the position of the hole from the tee.
Its length $\sqrt{116}$ multiplied by 40 gives the length of the hole, ≈ 431 m.

9 **a** $\overrightarrow{SA} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$,
$\overrightarrow{DE} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$, $\overrightarrow{FG} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\overrightarrow{GS} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$
b $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
c The finishing point is the same as the starting point, i.e., we are back where we started.

EXERCISE 23E

1 **a** $\begin{pmatrix} 6 \\ 9 \end{pmatrix}$ **b** $3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$

2 **a** $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ **c** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$ **e** $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$
f $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **g** $\begin{pmatrix} -\frac{2}{3} \\ -1 \end{pmatrix}$ **h** $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ **i** $\begin{pmatrix} 12 \\ 18 \end{pmatrix}$ **j** $\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

3 **a**

b
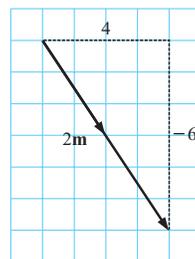
c
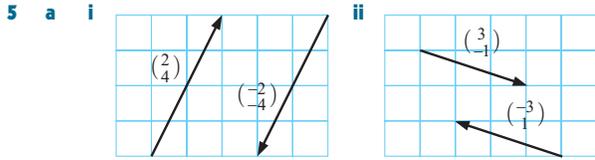
d

e

f

4 **a** $\begin{pmatrix} 9 \\ -15 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$
e $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **f** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
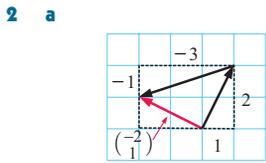
5 a i

$\binom{2}{4}$ $\binom{-2}{-4}$

ii

$\binom{3}{-1}$ $\binom{-3}{1}$

iii

$\binom{-1}{-4}$ $\binom{1}{4}$
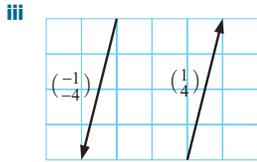
b "Geometrically, **a** and −**a** are *parallel* and *equal* in length, but *opposite* in direction."

c ...... $\overrightarrow{BA}$ is
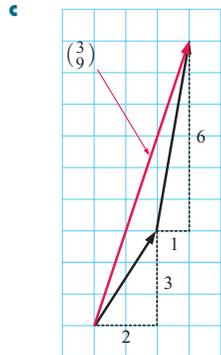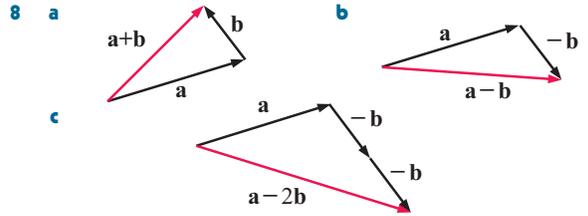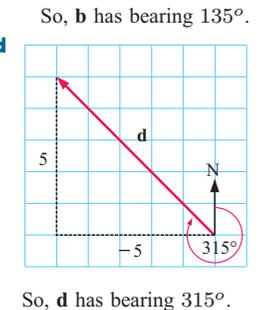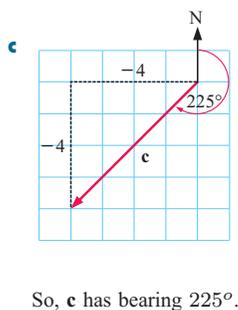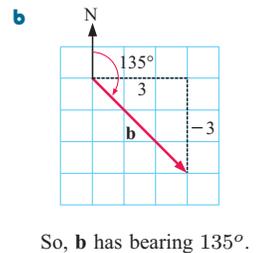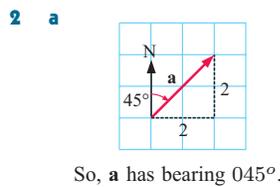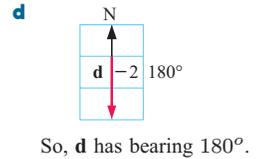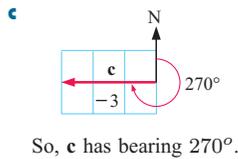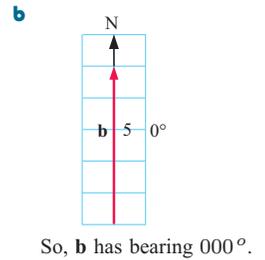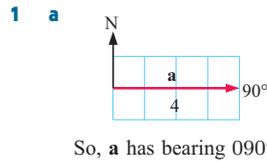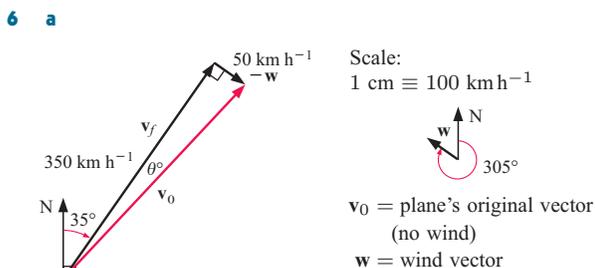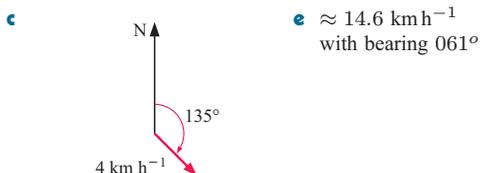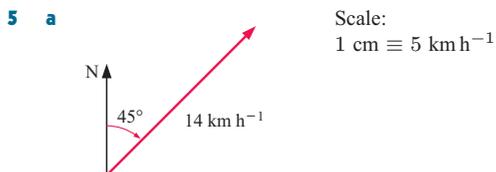

EXERCISE 23F

1 a $\binom{-3}{-4}$ **b** $\binom{1}{-3}$ **c** $\binom{2}{4}$ **d** $\binom{0}{5}$

2 a

So, $\binom{1}{2} - \binom{3}{1} = \binom{-2}{1}$

b

So, $\binom{2}{0} - \binom{4}{1} = \binom{-2}{-1}$

c

So, $\binom{2}{3} - \binom{-1}{-6} = \binom{3}{9}$

3 a $\binom{-1}{1}$ **b** $\binom{2}{4}$ **c** $\binom{9}{-1}$ **d** $\binom{8}{-2}$ **e** $\binom{3}{3}$ **f** $\binom{9}{0}$

4 a $\overrightarrow{AC}$ **b** $\overrightarrow{QR}$ **c** $\overrightarrow{AD}$ **d** $\overrightarrow{PV}$

5 a $\binom{-2}{4}$ **b** $\binom{9}{-1}$ **c** $\binom{9}{6}$ **d** $\binom{11}{2}$

e $\binom{-8}{-4}$ **f** $\binom{0}{7}$ **g** $\binom{-3}{-\frac{1}{2}}$ **h** $\binom{23}{5}$

6 a $\binom{6}{1}$ **b** $\binom{-2}{-4}$ **c** $\binom{8}{5}$ **d** $\binom{-4}{3}$

7 a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{c}$ **c** $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **d** $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

8 a

b

c

9 a **a**   **b** **a**   **c** **b**   **d** **a** − **b**   **e** **a** − **b**   **f** 2**a**

10 a **d**   **b** **g**   **c** **d**   **d** **a**   **e** −**c**   **f** −**g**

EXERCISE 23G

1 a

So, **a** has bearing 090º.

b

So, **b** has bearing 000º.

c

So, **c** has bearing 270º.

d

So, **d** has bearing 180º.

2 a

So, **a** has bearing 045º.

b

So, **b** has bearing 135º.

c

So, **c** has bearing 225º.

d

So, **d** has bearing 315º.

3 a 076º   **b** 146º   **c** 307º   **d** 207º

4 a $|\mathbf{a}| = \sqrt{10}$, bearing is 342º
b $|\mathbf{b}| = \sqrt{29}$, bearing is 158º
c $|\mathbf{c}| = \sqrt{10}$, bearing is 252º
d $|\mathbf{d}| = \sqrt{13}$, bearing is 124º

5 a

Scale:
1 cm ≡ 2 km h^{-1}

b velocity vector is $\binom{4.19}{2.72}$

6

Scale:
$1 \text{ cm} \equiv 6 \text{ km h}^{-1}$

$\mathbf{v} = \begin{pmatrix} 6.71 \\ -9.95 \end{pmatrix}$

7 a

Scale:
$1 \text{ cm} \equiv 4 \text{ km}$

**b** $\mathbf{v} = \begin{pmatrix} -6.71 \\ 4.36 \end{pmatrix}$
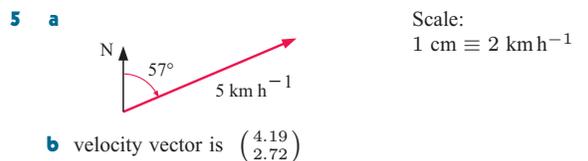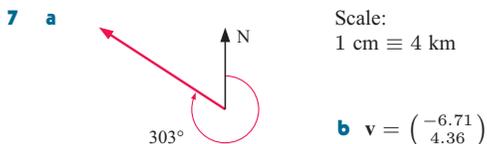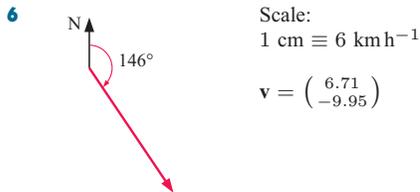
EXERCISE 23H

1 a $\begin{pmatrix} 0 \\ 10 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ 10 \end{pmatrix}$ **c** $5\sqrt{5}$ km **d** 333°

2 a
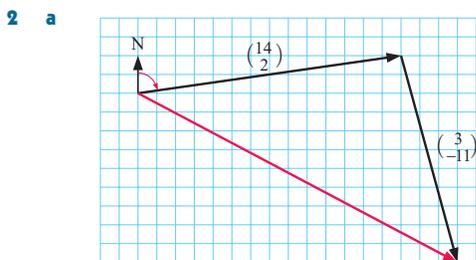
b $\begin{pmatrix} 17 \\ -9 \end{pmatrix}$ **c** $\sqrt{370} \approx 19.2$ units **d** 118°

3 a $\sqrt{5} \approx 2.24 \text{ m s}^{-1}$ **b** 27° **c** 117°

4 15 km h^{-1}

Yes, if she walks from the stern to the bow her relative speed is greater than the ferry's speed.

5 a

Scale:
$1 \text{ cm} \equiv 5 \text{ km h}^{-1}$

c

e $\approx 14.6 \text{ km h}^{-1}$
with bearing 061°

6 a

Scale:
$1 \text{ cm} \equiv 100 \text{ km h}^{-1}$

$\mathbf{v_0}$ = plane's original vector
(no wind)
$\mathbf{w}$ = wind vector

b $\theta \approx 8.13^o$ and $|\mathbf{v_0}| \approx 353.6$, $\mathbf{v_0} \approx \begin{pmatrix} 242 \\ 258 \end{pmatrix}$

REVIEW SET 23A

1 a

b

c
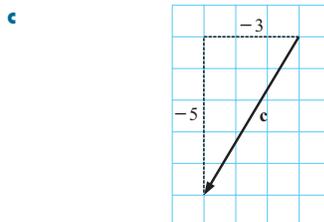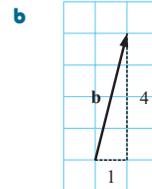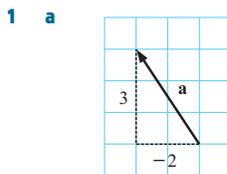
2 $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

3 a
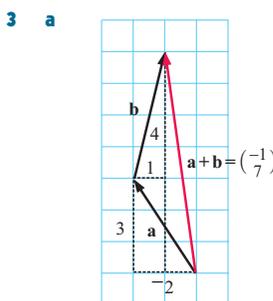
b i $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

ii $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

iii $\begin{pmatrix} 2 \\ 12 \end{pmatrix}$

4 a
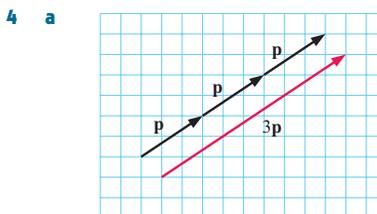
b i $\begin{pmatrix} -6 \\ -10 \end{pmatrix}$ **ii** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **iii** $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$

c

$\mathbf{q} + 2\mathbf{p} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
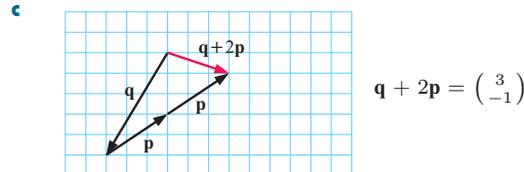
5 a

b $\sqrt{29}$ units $(\approx 5.4 \text{ units})$
c 158°
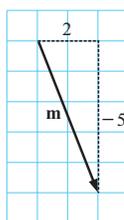
6 a i $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ **b** $\overrightarrow{AC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ **c** $\sqrt{34}$ units

7
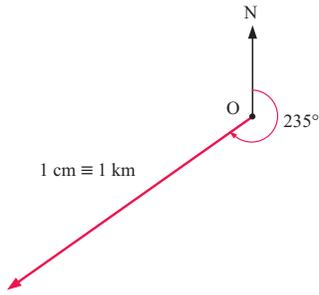
8 a
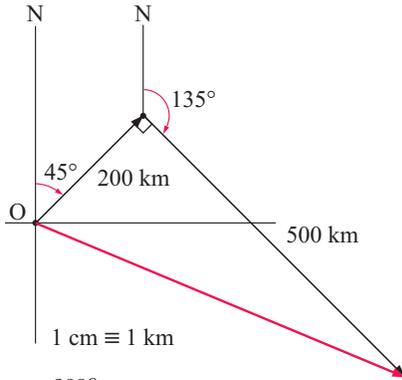
b 539 km **c** 293°

REVIEW SET 23B

1
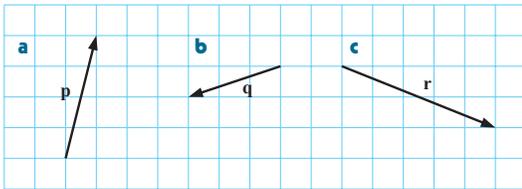
2 a $m = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **b** $n = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

3 a
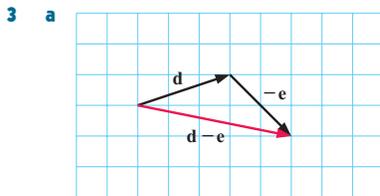
b $d - e = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ **c** **i** $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ **ii** $\begin{pmatrix} 18 \\ -2 \end{pmatrix}$

4 We get the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

5 a $-p$ **b** $\frac{1}{2}q$ **c** $\frac{1}{2}q - p$ **d** q

6 Scale:
1 cm ≡ 2 km h⁻¹
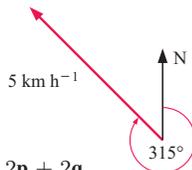
7 a **i** p **ii** q **iii** $p + q$ **iv** $2p + 2q$
  b $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ which means that [PQ] and [AC] are parallel
  and [PQ] is half as long as [AC].
  c ...... triangle is *parallel* to the third side and *half* its length.

8 a
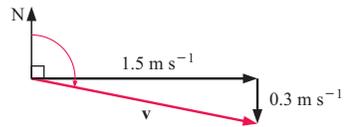
b speed ≈ 1.53 m s⁻¹ with bearing 101.3°
c 567 sec (9 min 27 sec) **d** 170 m

EXERCISE 24A

1 a $x = 100$ **b** $x = 40$ **c** $x = 70$ **d** $x = 35$
  e $x = 36$ **f** $x = 90, \ y = 110$
  g $a = 35, \ b = 70, \ c = 70$ **h** $x = 33\frac{1}{3}$ **i** $x = 107\frac{1}{2}$

2 a $a = 85$ **b** $a = 61$

3 a $\widehat{PSQ} = a^o, \ \widehat{QSR} = b^o$ {base angles of isosceles $\triangle$}
  b 180°

4 a 360° {sum of interior angles of a quadrilateral}
  b 180° **c** co-interior angles are supplementary
  d The quadrilateral is a parallelogram.

5 a $\widehat{TPQ} = \alpha^o, \ \widehat{TQP} = \beta^o$
  b 180° {co-interior angles between parallel lines}
  c angle sum of a triangle

EXERCISE 24B

1 a $x = 90$ **b** $x = 38$ **c** $x = 30$ **d** $x = 8$
  e $x = 3\sqrt{2}$ **f** $x = 2\sqrt{5}$

2 a $x \approx 5.8$ **b** $x \approx 3.6$ **c** $x \approx 9.8$

3 6.3 cm **4** 5.0 cm **5** 9.2 cm

6 a $x \approx 8.7$ **b** $x \approx 6.40$ **c** $x \approx 5.9$
  d $x = 8, \ y = 6$ **e** $x = y = 50$ **f** $x = 3$

7 $PQ \approx 9.75$ cm **8 a** It is a square. **b** 21 cm

EXERCISE 24D.1

1 Show all pairs are equiangular.
  Note: **a** $\triangle$FGH is similar to $\triangle$FDE
  b $\triangle$CDE is similar to $\triangle$CAB
  c $\triangle$WUV is similar to $\triangle$WYX
  d $\triangle$PQR is similar to $\triangle$TSR
  e $\triangle$ABC is similar to $\triangle$EDC
  f $\triangle$KLM is similar to $\triangle$NLJ

EXERCISE 24D.2

1 a $x = 3\frac{3}{4}$ **b** $x = 2\frac{4}{5}$ **c** $x = 4\frac{4}{5}$ **d** $x = 4\frac{4}{5}$
  e $x = 7\frac{5}{7}$ **f** $x = 6$ **g** $x = 1\frac{5}{8}$ **h** $x = 7\frac{2}{3}$
  i $x = 13\frac{1}{3}$

EXERCISE 24E

1 9.8 m **2 a** 1650 m **b** 1677 m **3** 53.75 m

4 a As [AD] ∥ [BC], $\triangle$s are equiangular and ∴ similar.
  b 5.5 m wide by 4 m high

5 a As [MN] ∥ [AB], $\triangle$s are equiangular and ∴ similar.
  c $V = \frac{2}{3}\pi x^3$

6 a As [CD] ∥ [TA], $\triangle$s are equiangular and ∴ similar.
  b $13\frac{1}{3}$ m

EXERCISE 24F

1 a $\theta = 60, \ x = 10$ {midpoint theorem}
  b $x = 8$ {converse midpoint theorem}

6 (1) given (2) {alternate angles} (3) $\widehat{AQP} = \widehat{CQR}$
  (4) AAcorS (5) BP (6) parallelogram (7) BC

EXERCISE 24G

1 a 7 edges   **b** 6 vertices   **c** 4 regions

2 a   **b**

c

3
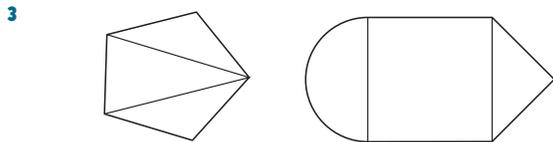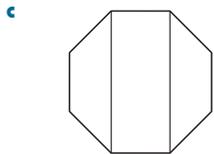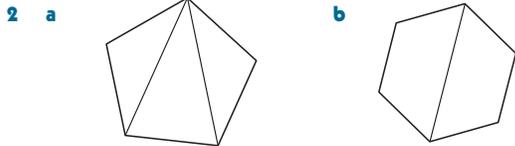
REVIEW SET 24A

1 a $x = 35$ {vertically opposite angles}
  b $x = 30$ {angles on a line}
  c $x = 70$ {angle sum of a triangle}
  d $x = 26$ {exterior angle of a triangle}
  e $x = 75$ **f** $x = 55$ {diagonal of a rhombus}

2 a $x = \sqrt{5}$ **b** $x = 2.4$

3 $\triangle$RQS is similar to $\triangle$RPT; $x = 12$

5 a As   [MN] || [AB], $\triangle$s are equiangular and $\therefore$ similar.
  c 6.4 cm

6 a α^o **7** 16 cm

REVIEW SET 24B

1 a $x = 75$ **b** $x = 60$ **c** $x = 102.5$
  d $x = 40$ **e** $x = 66$ **f** $x = 69$

2 a $x = 8$ **b** $y = 8$, $z = 4\sqrt{5}$

4 a $x = 3$ **b** $x = 4$

5 a As   [MN] || [BC], $\triangle$s are equiangular and $\therefore$ similar.
  c 10.8 cm

6 66.7 m

EXERCISE 25A.1

1 a 0   **b** 0.26   **c** 0.42   **d** 0.50   **e** 0.71
  f 0.87   **g** 0.97   **h** 1

2 a 1   **b** 0.97   **c** 0.91   **d** 0.87   **e** 0.71
  f 0.50   **g** 0.26   **h** 0

3 (0.57, 0.82)   *Check:* $\approx (0.5736, 0.8192)$ {GDC}

4

$\cos 60^o = \dfrac{\frac{1}{2}}{1} = \frac{1}{2}$
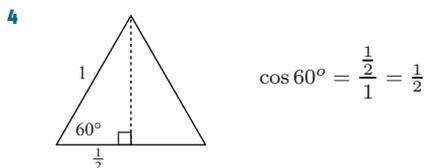
EXERCISE 25A.2

1 a 0   **b** 0.18   **c** 0.36   **d** 0.70   **e** 0.84
  f 1.00   **g** 1.19   **h** 1.43

2

So, $\tan 45^o = 1$.
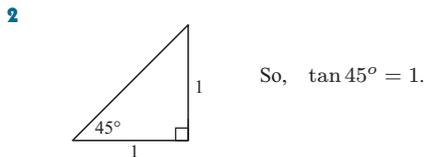
3 We would need a much higher graph.
Can only use the one given for angles up to about 60°.
$\tan 80^o \approx 5.67$

4 a $(\cos \theta, \sin \theta)$

  b **i** OM $= \cos \theta$ **ii** PM $= \sin \theta$ **iii** TN $= \tan \theta$

  c $\dfrac{\text{TN}}{1} = \dfrac{\text{PM}}{\text{OM}}$ {similar $\triangle$s}

    $\therefore \quad \dfrac{\tan \theta}{1} = \dfrac{\sin \theta}{\cos \theta},$ etc.

EXERCISE 25B

1 a 0.94   **b** 0.94   **c** 0.87   **d** 0.87   **e** 0.77
  f 0.77   **g** 0   **h** 0

2 a $\sin(180^o - \theta) = \sin \theta$

  b
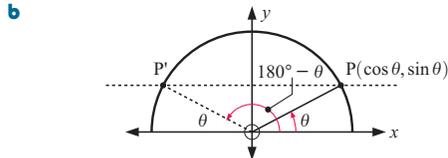
P′ is $(-\cos \theta, \sin \theta)$ {symmetry}
and P′ is $(\cos(180^o - \theta), \sin(180^o - \theta))$
So, $\sin(180^o - \theta) = \sin \theta$

3 a -0.17 **b** 0.17   **c** -0.5 **d** 0.5   **e** -0.64
  f 0.64   **g** -1 **h** 1

4 a $\cos(180^o - \theta) = -\cos \theta$
  b Use same argument as **2 b** above only compare x-coordinates.

5 a 154° **b** 135° **c** 111° **d** 94°

6 a 82° **b** 53° **c** 24° **d** 12°

7 a ≈ 0.9272 **b** ≈ 0.4384 **c** ≈ -0.9781
  d ≈ 0.6561 **e** ≈ 0.5736 **f** ≈ -0.1392

EXERCISE 25C

1 a 25.7 m^2 **b** 57.6 km^2 **c** 30.9 cm^2
  d 33.0 m^2 **e** 6.49 m^2 **f** 1.52 km^2

2 a Area (1) 85 m^2,   Area (2) 85 m^2
  b   Area (1) $= \frac{1}{2} \times 17 \times 20 \times \sin 30^o$
       Area (2) $= \frac{1}{2} \times 17 \times 20 \times \sin 150^o$
   and $\sin 150^o = \sin 30^o$

3 $x \approx 20.0$

EXERCISE 25D.1

1 a $x \approx 9.74$ **b** $x \approx 104$ **c** $x \approx 7.04$
  d $x \approx 177$ **e** $x \approx 6.02$ **f** $x \approx 386$

2 a $\theta = 50^o$, $x \approx 104$, $y \approx 93.2$
  b $\phi = 49^o$, $x \approx 161$, $z \approx 163$
  c $\theta = \phi = 39^o$, $x \approx 2.49$

**EXERCISE 25D.2**

1   **a** $\theta \approx 30.0^o$ **b** $\theta \approx 46.8^o$ **c** $\theta \approx 62.0^o$
     d $\theta \approx 111.8^o$ **e** $\theta \approx 122.1^o$ **f** $\theta \approx 130.8^o$

2   **a** $\approx 46.1^o$ or 133.9^o **b** $\approx 77.0^o$ or 103.0^o **c** $\approx 31.3^o$
     d $\widehat{B}$ cannot be found, the triangle is impossible.   **e** $\approx 49.4^o$
     f $\approx 67.1^o$ or 112.9^o **g** $\approx 43.7^o$

EXERCISE 25E

1   **a** ≈ 17.3 m   **b** ≈ 6.29 m   **c** ≈ 25.4 m
     d ≈ 12.6 m   **e** ≈ 195 m   **f** ≈ 15.8 km

2   **a** $\theta \approx 48.2^o$ **b** $\theta = 90^o$ **c** $\phi \approx 72.0^o$ **d** $\phi \approx 101.5^o$
     e α cannot be found. No such triangle can exist. Why?
     f $\beta \approx 99.8^o$

3   **a** $\theta = 180^o$ **b** $9.6 + 7.2 = 16.8$
                    So, A, B and C are on a straight line.

4   **a** $x \approx 6.27$, $\alpha \approx 41.7^o$, $\beta \approx 94.3^o$
     b $\alpha \approx 36.3^o$, $\theta \approx 26.4^o$, $\beta \approx 117.3^o$

EXERCISE 25F

1   **a** $\approx 38.6^o$ **b** $\approx 113.8^o$

2   **a**

     b   B as they are closer
     c   69.4 km
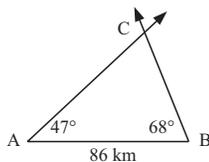
3   ≈ 100 m   4   88.7 km   5 13.4^o 6   1.64 ha

7   19.6 km in direction $\approx 106^o$

8   **a** $\theta = 10^o$ **b**   BD ≈ 88.7 km   **c**   56.0 km

9   AB ≈ 208 m   (to 3 s.f.)

REVIEW SET 25A

1   $a^2 + a^2 = 1$ {right angled isos. Δ}
    $\therefore$ $a^2 = \frac{1}{2}$
    $\therefore$ $a = \frac{1}{\sqrt{2}}$ $\{a > 0\}$
      etc.

2   **a** $\sin(180^o - \theta) = a$ **b** $\cos(180^o - \theta) = -b$

3   **a**   77 m^2 **b**   BC ≈ 15.9 m

4   RQ ≈ 14.6 m   5   **a** $x \approx 223$ **b** $x \approx 99.4$

6   81.3^o or 98.7^o 7   **a**   185 m in direction $\approx 184^o$

REVIEW SET 25B

1   ON $= \frac{1}{2}$ $\therefore$ PN $= \frac{\sqrt{3}}{2}$ {Pythagoras}
    So $\cos 60^o = \frac{\frac{1}{2}}{1} = \frac{1}{2}$ and $\sin 60^o = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

2   **a** $\theta \approx 109.5^o$ **b** $\theta = 30^o$ or 150°

3   **a** $\widehat{B} \approx 104.37^o$ **b** 69.3^o 4 $\widehat{P} \approx 59.0^o$

5   $\theta = 99^o$, $x \approx 16.4$, $y \approx 14.8$

6   **a**   Use the Cosine Rule; $x^2 = 240$,   etc.   **b** $k = \frac{3}{16}$

7   **a** $\theta \approx 56.6^o$ or 123.4^o
     b $60^o + 123.4^o > 180^o$, $\therefore$ 123.4^o is not possible

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